EITF75 Systems and Signals

Lecture 7
The DTFT and LTI systems

Fredrik Rusek

EITF75 Systems and Signals

Recap

In general, a signal $y(n)$ generated from $x(n)$ via $\oplus$ $\boxed{z^{-1}}$ $\downarrow$

can be mathematically described by

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Difference equation

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For systems at rest: z-transform

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

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For systems at rest: z-transform

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \qquad Y(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \cdot X(z) = H(z)X(z)$$

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System describes a convolution

$$y(n) = h(n) * x(n) = \sum_{k} h(k)x(n-k)$$

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For systems at rest: z-transform

$$Y(z)+\sum_{k=1}^{N} a_k z^{-k} Y(z)=\sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z)=\frac{b_0+b_1 z^{-1}+\cdots+b_M z^{-M}}{1+a_1 z^{-1}+\cdots+a_N z^{-N}} \cdot X(z)=H(z)X(z)$$

System describes a convolution

$$y(n)=h(n)*x(n)=\sum_{k} h(k)x(n-k)$$

FIR filter: $a_k=0,\ k>0$

IIR filter: otherwise

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$$y(n) = h(n) * x(n) = \sum_{k} h(k)x(n-k)$$

Alternative way to reach the convolution

The difference equation describes an LTI system

EITF75 Systems and Signals

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In general, a signal $y(n)$ generated from $x(n)$ via $\oplus$, z^{-1}, ↓

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System describes a convolution

$$y(n) = h(n) * x(n) = \sum_{k} h(k) x(n-k)$$

Alternative way to reach the convolution

The difference equation describes an LTI system

For an LTI system, the input-output relation is a convolution (see lecture 2)

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System describes a convolution

$$y(n) = h(n) * x(n) = \sum_{k} h(k) x(n-k)$$

Systems not at rest: one-sided z-transform

Output is a convolution between h(n) and x(n) plus a transient part that depends on the initial conditions

EITF75 Systems and Signals

Recap

For stable h(n) $H(f) = H\left(e^{i2\pi f}\right)$

This is the z-transform

This is the DTFT

For systems at rest: z-transform

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

System describes a convolution

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EITF75 Systems and Signals

Recap

For stable h(n) $H(f) = H\left(e^{i2\pi f}\right)$

→ ROC includes unit circle

For systems at rest: z-transform

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EITF75 Systems and Signals

Recap

For stable h(n) $H(f) = H\left(e^{i2\pi f}\right)$

→ ROC includes unit circle

→ All poles inside the unit circle

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EITF75 Systems and Signals

Recap

For stable h(n) $H(f) = H\left(e^{i2\pi f}\right)$

→ ROC includes unit circle

→ All poles inside the unit circle

→ Videos claim that h(n) must be causal which is false

For systems at rest: z-transform

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

System describes a convolution

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Systems not at rest: one-sided z-transform

Output is a convolution between h(n) and x(n) plus a transient part that depends on the initial conditions

EITF75 Systems and Signals

Recap

For stable h(n) $H(f) = H\left(\mathrm{e}^{i2\pi f}\right)$

For input $x(n) = \exp(i2\pi f_0 n)$

We get the output $y(n) = H(f_0)\exp(i2\pi f_0 n)$

For systems at rest: z-transform

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

System describes a convolution

$$y(n) = h(n) * x(n) = \sum_{k} h(k)x(n-k)$$

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EITF75 Systems and Signals

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We get the output $y(n) = H(f_0)\exp(i2\pi f_0 n)$

Today:

1. $x(n) = \cos(2\pi f_0 n)$ $x(n) = \sin(2\pi f_0 n)$

2. $x(n) = \cos(2\pi f_0 n)u(n)$

3. Phase/group delay

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Input

$$x(n) = \cos(2\pi f_0 n)$$
$$= \frac{1}{2}\left[\exp(i2\pi f_0 n) + \exp(-i2\pi f_0 n)\right]$$

Output (LTI system)

$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

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Input

$$x(n) = \cos(2\pi f_0 n)$$
$$= \frac{1}{2}\left[\exp(i2\pi f_0 n) + \exp(-i2\pi f_0 n)\right]$$

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$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

To continue, we need to investigate the relation between $H(f_0)$ and $H(-f_0)$

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Relation between $H(f_0)$ and $H(-f_0)$

Output (LTI system)

$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

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EITF75 Systems and Signals

Relation between $H(f_0)$ and $H(-f_0)$

$$H(f) = \sum_k h(k)\exp(i2\pi fk) = \sum_k h(k)\cos(2\pi fk) + i\sum_k h(k)\sin(2\pi fk)$$

Output (LTI system)

$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

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EITF75 Systems and Signals

Relation between $H(f_0)$ **and** $H(-f_0)$

$$H(f) = \sum_k h(k)\exp(i2\pi f k) = \underbrace{\sum_k h(k)\cos(2\pi f k)}_{H_{\mathrm{R}}(f)} + i\underbrace{\sum_k h(k)\sin(2\pi f k)}_{H_{\mathrm{I}}(f)}$$

Output (LTI system)

$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

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Assume h(n) real-valued

Output (LTI system)

$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

To continue, we need to investigate the relation between $H(f_0)$ and $H(-f_0)$

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Relation between $H(f_0)$ and $H(-f_0)$

$$H(f) = \sum_k h(k)\exp(i2\pi fk) = \underbrace{\sum_k h(k)\cos(2\pi fk)}_{\text{real-valued}} + i\underbrace{\sum_k h(k)\sin(2\pi fk)}_{\text{real-valued}}$$

Assume h(n) **real-valued**

Output (LTI system)

$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

To continue, we need to investigate the relation between $H(f_0)$ and $H(-f_0)$

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$$H(f) = \sum_k h(k)\exp(i2\pi fk) = \sum_k h(k)\cos(2\pi fk) + i\sum_k h(k)\sin(2\pi fk)$$

$$= H_\mathrm{R}(f) + iH_\mathrm{I}(f)$$

Output (LTI system)

$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

To continue, we need to investigate the relation between $H(f_0)$ **and** $H(-f_0)$

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$$= H_{\mathrm{R}}(f) + iH_{\mathrm{I}}(f)$$

$$H_{\mathrm{R}}(f) = \sum_k h(k)\cos(2\pi fk)$$

$$H_{\mathrm{I}}(f) = \sum_k h(k)\sin 2\pi fk)$$

To continue, we need to investigate the relation between $H(f_0)$ **and** $H(-f_0)$

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$$H(f) = \sum_k h(k)\exp(i2\pi fk) = \sum_k h(k)\cos(2\pi fk) + i\sum_k h(k)\sin(2\pi fk)$$

$$= H_{\mathrm{R}}(f) + iH_{\mathrm{I}}(f)$$

$$H_{\mathrm{R}}(f) = \sum_k h(k)\cos(2\pi fk) = \sum_k h(k)\cos(-2\pi fk)$$

$$H_{\mathrm{I}}(f) = \sum_k h(k)\sin 2\pi fk) = -\sum_k h(k)\sin(-2\pi fk)$$

To continue, we need to investigate the relation between $H(f_0)$ **and** $H(-f_0)$

EITF75 Systems and Signals

Relation between $H(f_0)$ **and** $H(-f_0)$

$$H(f) = \sum_k h(k)\exp(i2\pi fk) = \sum_k h(k)\cos(2\pi fk) + i\sum_k h(k)\sin(2\pi fk)$$

$$= H_\mathrm{R}(f) + iH_\mathrm{I}(f)$$

$$H_\mathrm{R}(f) = \sum_k h(k)\cos(2\pi fk) = \sum_k h(k)\cos(-2\pi fk) = H_\mathrm{R}(-f)$$

$$H_\mathrm{I}(f) = \sum_k h(k)\sin 2\pi fk) = -\sum_k h(k)\sin(-2\pi fk) = -H_\mathrm{I}(-f)$$

To continue, we need to investigate the relation between $H(f_0)$ **and** $H(-f_0)$

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Relation between $H(f_0)$ **and** $H(-f_0)$

$$H(f) = \sum_k h(k)\exp(i2\pi fk) \; = \sum_k h(k)\cos(2\pi fk) + i\sum_k h(k)\sin(2\pi fk)$$

$$= H_{\mathrm{R}}(f) + iH_{\mathrm{I}}(f)$$

$$H_{\mathrm{R}}(f) = \sum_k h(k)\cos(2\pi fk) = \sum_k h(k)\cos(-2\pi fk) = H_{\mathrm{R}}(-f)$$

$$H_{\mathrm{I}}(f) = \sum_k h(k)\sin 2\pi fk) = -\sum_k h(k)\sin(-2\pi fk) = -H_{\mathrm{I}}(-f)$$

Collect in a single equation: $H(f) = H^*(-f)$

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Relation between $H(f_0)$ **and** $H(-f_0)$**:** $H(f) = H^*(-f)$

Output (LTI system)

$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

Concentrate on this

EITF75 Systems and Signals

Relation between $H(f_0)$ **and** $H(-f_0)$: $H(f) = H^*(-f)$

$$H(f) = |H(f)| \exp(i2\pi\Theta(f))$$

Always possible to write a complex number in this way

Output (LTI system)

$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

Concentrate on this

EITF75 Systems and Signals

Relation between $H(f_0)$ **and** $H(-f_0)$**:** $H(f) = H^*(-f)$

$$H(f) = |H(f)| \exp(i2\pi\Theta(f))$$

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Output (LTI system)

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EITF75 Systems and Signals

Relation between $H(f_0)$ **and** $H(-f_0)$**:** $H(f) = H^*(-f)$

$$H(f) = |H(f)| \exp(i2\pi\Theta(f))$$

$$H(-f) = |H(-f)| \exp(i2\pi\Theta(-f)) = |H(f)| \exp(-i2\pi\Theta(f))$$

Output (LTI system)

$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

Concentrate on this

EITF75 Systems and Signals

Relation between $H(f_0)$ **and** $H(-f_0)$**:** $H(f) = H^*(-f)$

$$H(f) = |H(f)| \exp(i2\pi\Theta(f))$$

$$H(-f) = |H(-f)| \exp(i2\pi\Theta(-f)) = |H(f)| \exp(-i2\pi\Theta(f))$$

Output (LTI system)

$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

$$= \frac{|H(f_0)|}{2}\left[\exp(i2\pi\Theta(f_0))\exp(i2\pi f_0 n) + \exp(-i2\pi\Theta(f_0))\exp(-i2\pi f_0 n)\right]$$

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Relation between $H(f_0)$ **and** $H(-f_0)$**:** $H(f) = H^*(-f)$

$$H(f) = |H(f)| \exp(i2\pi\Theta(f))$$

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$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

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Relation between $H(f_0)$ **and** $H(-f_0)$**:** $H(f) = H^*(-f)$

$$H(f) = |H(f)| \exp(i2\pi\Theta(f))$$

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$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

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EITF75 Systems and Signals

Input

$$x(n) = \cos(2\pi f_0 n)$$
$$= \frac{1}{2}\left[\exp(i2\pi f_0 n) + \exp(-i2\pi f_0 n)\right]$$

Output (LTI system)

$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$
$$= \frac{|H(f_0)|}{2}\left[\exp(i2\pi\Theta(f_0))\exp(i2\pi f_0 n) + \exp(-i2\pi\Theta(f_0))\exp(-i2\pi f_0 n)\right]$$

$$y(n) = |H(f_0)|\cos(2\pi f_0 n + \Theta(f_0))$$

EITF75 Systems and Signals

Input *Changes ?*

$$x(n) = \sin(2\pi f_0 n)$$

$$= \frac{1}{2}\left[\exp(i2\pi f_0 n) + \exp(-i2\pi f_0 n)\right]$$

Output (LTI system)

$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

$$= \frac{|H(f_0)|}{2}\left[\exp(i2\pi\Theta(f_0))\exp(i2\pi f_0 n) + \exp(-i2\pi\Theta(f_0))\exp(-i2\pi f_0 n)\right]$$

$$y(n) = |H(f_0)|\cos(2\pi f_0 n + \Theta(f_0))$$

EITF75 Systems and Signals

Input Changes ?

$$x(n) = \sin(2\pi f_0 n)$$

$$= \frac{1}{2i}[\exp(i2\pi f_0 n) - \exp(-i2\pi f_0 n)]$$

Output (LTI system)

$$y(n) = \frac{1}{2}\left[H(f_0)\exp(i2\pi f_0 n) + H(-f_0)\exp(-i2\pi f_0 n)\right]$$

$$= \frac{|H(f_0)|}{2}\left[\exp(i2\pi\Theta(f_0))\exp(i2\pi f_0 n) + \exp(-i2\pi\Theta(f_0))\exp(-i2\pi f_0 n)\right]$$

$$y(n) = |H(f_0)|\cos(2\pi f_0 n + \Theta(f_0))$$

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Input *Changes ?*

$$x(n) = \sin(2\pi f_0 n)$$

$$= \frac{1}{2i}[\exp(i2\pi f_0 n) - \exp(-i2\pi f_0 n)]$$

Output (LTI system) *Changes ?*

$$y(n) = \frac{1}{2i}[H(f_0)\exp(i2\pi f_0 n) - H(-f_0)\exp(-i2\pi f_0 n)]$$

$$= \frac{|H(f_0)|}{2}[\exp(i2\pi\Theta(f_0))\exp(i2\pi f_0 n) + \exp(-i2\pi\Theta(f_0))\exp(-i2\pi f_0 n)]$$

$$y(n) = |H(f_0)|\cos(2\pi f_0 n + \Theta(f_0))$$

EITF75 Systems and Signals

Input *Changes ?*

$$x(n) = \sin(2\pi f_0 n)$$

$$= \frac{1}{2i}[\exp(i2\pi f_0 n) - \exp(-i2\pi f_0 n)]$$

Output (LTI system) *Changes ?*

$$y(n) = \frac{1}{2i}[H(f_0)\exp(i2\pi f_0 n) - H(-f_0)\exp(-i2\pi f_0 n)]$$

$$= \frac{|H(f_0)|}{2i}[\exp(i2\pi\Theta(f_0))\exp(i2\pi f_0 n) - \exp(-i2\pi\Theta(f_0))\exp(-i2\pi f_0 n)]$$

$$y(n) = |H(f_0)|\cos(2\pi f_0 n + \Theta(f_0))$$

EITF75 Systems and Signals

Input *Changes ?*

$$x(n) = \sin(2\pi f_0 n)$$

$$= \frac{1}{2i}[\exp(i2\pi f_0 n) - \exp(-i2\pi f_0 n)]$$

Output (LTI system) *Changes ?*

$$y(n) = \frac{1}{2i}[H(f_0)\exp(i2\pi f_0 n) - H(-f_0)\exp(-i2\pi f_0 n)]$$

$$= \frac{|H(f_0)|}{2i}[\exp(i2\pi\Theta(f_0))\exp(i2\pi f_0 n) - \exp(-i2\pi\Theta(f_0))\exp(-i2\pi f_0 n)]$$

$$y(n) = |H(f_0)|\sin(2\pi f_0 n + \Theta(f_0))$$

EITF75 Systems and Signals

Inputs

$$x(n) = \cos(2\pi f_0 n)$$

$$x(n) = \sin(2\pi f_0 n)$$

Outputs (LTI system)

$$y(n) = |H(f_0)| \cos(2\pi f_0 + \Theta(f_0))$$

$$y(n) = |H(f_0)| \sin(2\pi f_0 + \Theta(f_0))$$

EITF75 Systems and Signals

EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$

Find $H(f)$

Find b such that $\max|H(f)|=1$

Find $y(n)$ for $x(n)=5 + 12\sin(\pi n/2) - 20\cos(\pi n + \pi/4)$

EITF75 Systems and Signals

EXAMPLE $y(n) = a\,y(n-1) + b\,x(n)$

Find $H(f)$

$Y(z) = a z^{-1} Y(z) + b\,X(z)$

Start by z-transform

EITF75 Systems and Signals

EXAMPLE $y(n) = a\,y(n-1) + b\,x(n)$ Find H(f)

$Y(z) = az^{-1}Y(z) + b\,X(z)$ Start by z-transform

$Y(z)(1 - az) = b\,X(z)$ Minor (standard) manipulation

EITF75 Systems and Signals

EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$ Find H(f)

$Y(z)=az^{-1}Y(z)+b\,X(z)$ Start by z-transform

$Y(z)(1-az^{-1}) = b\,X(z)$ Minor (standard) manipulation

$$Y(z) = \frac{b}{(1-az^{-1})}X(z)$$

Minor (standard) manipulation

EITF75 Systems and Signals

EXAMPLE $y(n) = a\,y(n-1) + b\,x(n)$ — Find $H(f)$

$Y(z) = az^{-1}Y(z) + b\,X(z)$ — Start by z-transform

$Y(z)(1 - az^{-1}) = b\,X(z)$ — Minor (standard) manipulation

$$Y(z) = \frac{b}{(1 - az^{-1})} X(z)$$

Minor (standard) manipulation

$$H(z) = \frac{b}{(1 - az^{-1})}$$

Identify system

EITF75 Systems and Signals

EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$ Find H(f)

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Identify system

Stable if $|a|<1$

EITF75 Systems and Signals

EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$ Find H(f)

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Minor (standard) manipulation

$$H(z) = \frac{b}{(1-az^{-1})}$$

Identify system

Stable if $|a|<1$

$$H(f) = \frac{b}{(1-ae^{-i2\pi f})}$$

Find DTFT

EITF75 Systems and Signals

EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$ Find H(f)

$$1 - ae^{-i2\pi f} = [1 - a\cos(2\pi f)] + i\sin(2\pi f)$$

$$H(f) = \frac{b}{(1 - ae^{-i2\pi f})}$$

Find DTFT

EITF75 Systems and Signals

EXAMPLE $y(n) = a\,y(n-1) + b\,x(n)$ Find H(f)

$$1 - ae^{-i2\pi f} = [1 - a\cos(2\pi f)] + ia\sin(2\pi f)$$

$$|1 - ae^{-i2\pi f}| = \sqrt{[1 - a\cos(2\pi f)]^2 + [a\sin(2\pi f)]^2}$$

$$H(f) = \frac{b}{(1 - ae^{-i2\pi f})}$$

Find DTFT

EITF75 Systems and Signals

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$$|H(f)| = \frac{|b|}{\sqrt{[1 - a\cos(2\pi f)]^2 + [a\sin(2\pi f)]^2}}$$

$$\theta(f) = -\tan^{-1}\frac{a\sin(2\pi f)}{1 - a\cos(2\pi f)}$$

(assume b > 0)

EITF75 Systems and Signals

EXAMPLE $y(n) = a\,y(n-1) + b\,x(n)$

Find b such that $\max|H(f)| = 1$

Assume $a > 0$

$$|H(f)| = \frac{|b|}{\sqrt{[1 - a\cos(2\pi f)]^2 + [a\sin(2\pi f)]^2}}$$

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EITF75 Systems and Signals

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$$|H(f)| = \frac{|b|}{\sqrt{[1 - a\cos(2\pi f)]^2 + [a\sin(2\pi f)]^2}}$$

EITF75 Systems and Signals

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$$[1 - a\cos(2\pi f)]^2 + [a\sin(2\pi f)]^2 = 1 + a^2 - 2a\cos(2\pi f)$$

maximized

$$|H(f)| = \frac{|b|}{\sqrt{1 + a^2 - 2a\cos(2\pi f)}}$$

← When minimized

EITF75 Systems and Signals

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Find b such that $\max|H(f)|=1$

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Minimized when f=0

EITF75 Systems and Signals

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maximized

$$|H(f)| = \frac{|b|}{\sqrt{1 + a^2 - 2a\cos(2\pi f)}}$$

← Minimized when f=0

EITF75 Systems and Signals

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$$\max_f |H(f)| = |H(0)| = \frac{|b|}{\sqrt{1 + a^2 - 2a}} = \frac{|b|}{1 - a}$$

maximized

$$|H(f)| = \frac{|b|}{\sqrt{1 + a^2 - 2a\cos(2\pi f)}}$$

← Minimized when f=0

EITF75 Systems and Signals

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$$\max_f |H(f)| = |H(0)| = \frac{|b|}{\sqrt{1 + a^2 - 2a}} = \frac{|b|}{1 - a}$$

$$\max_f |H(f)| = 1 \rightarrow b = 1 - a$$

maximized

$$|H(f)| = \frac{|b|}{\sqrt{1 + a^2 - 2a\cos(2\pi f)}}$$

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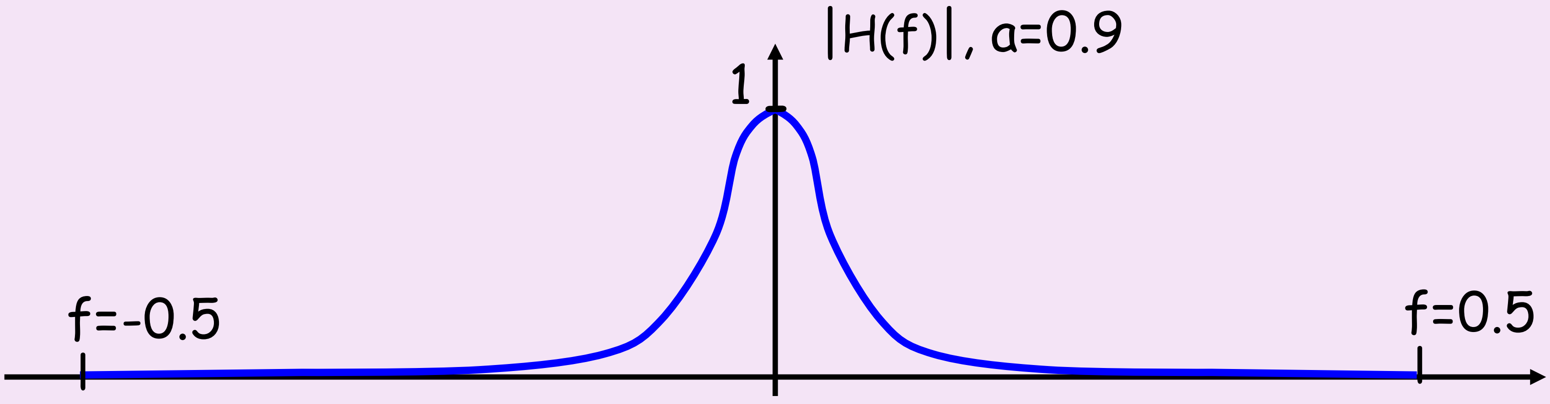
EITF75 Systems and Signals

EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$ Find b such that $\max|H(f)|=1$



$$|H(f)| = \frac{1-a}{\sqrt{1 + a^2 - 2a\cos(2\pi f)}}$$

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EITF75 Systems and Signals

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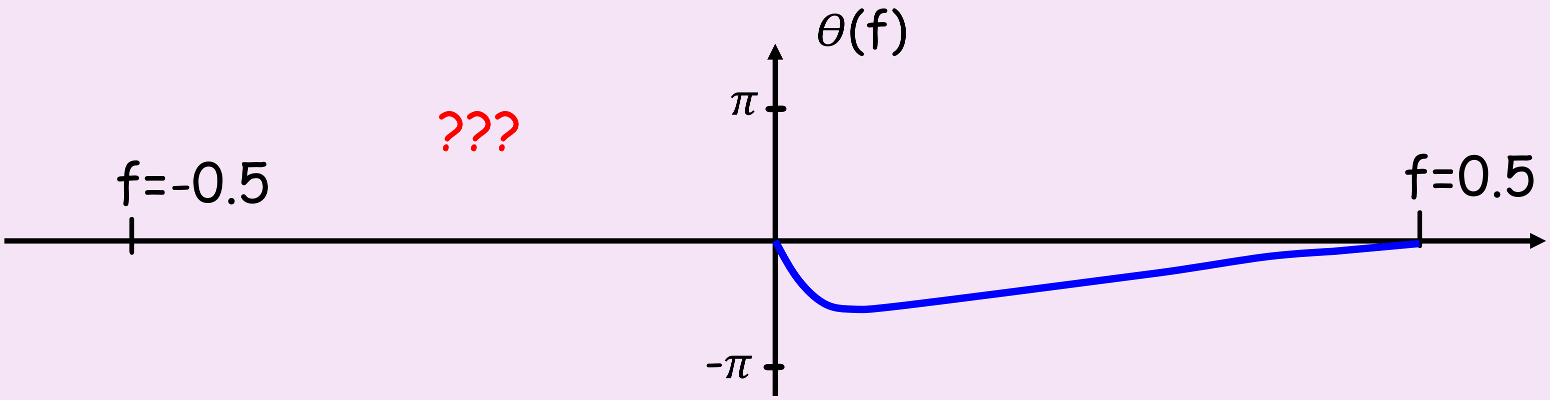

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EITF75 Systems and Signals

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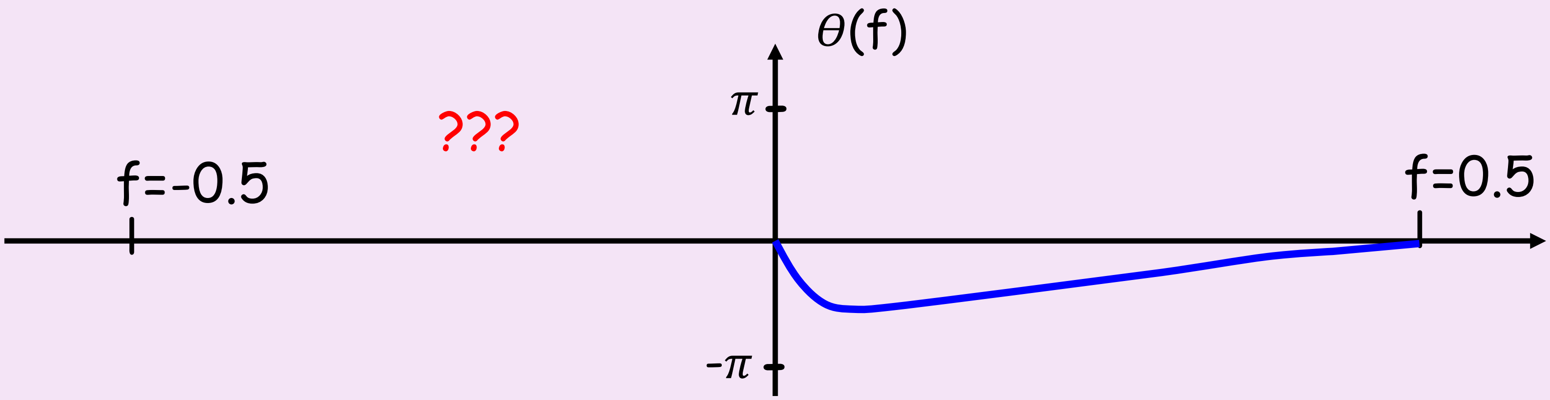

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Relation between $H(f_0)$ **and** $H(-f_0)$: $H(f) = H^*(-f)$

EITF75 Systems and Signals

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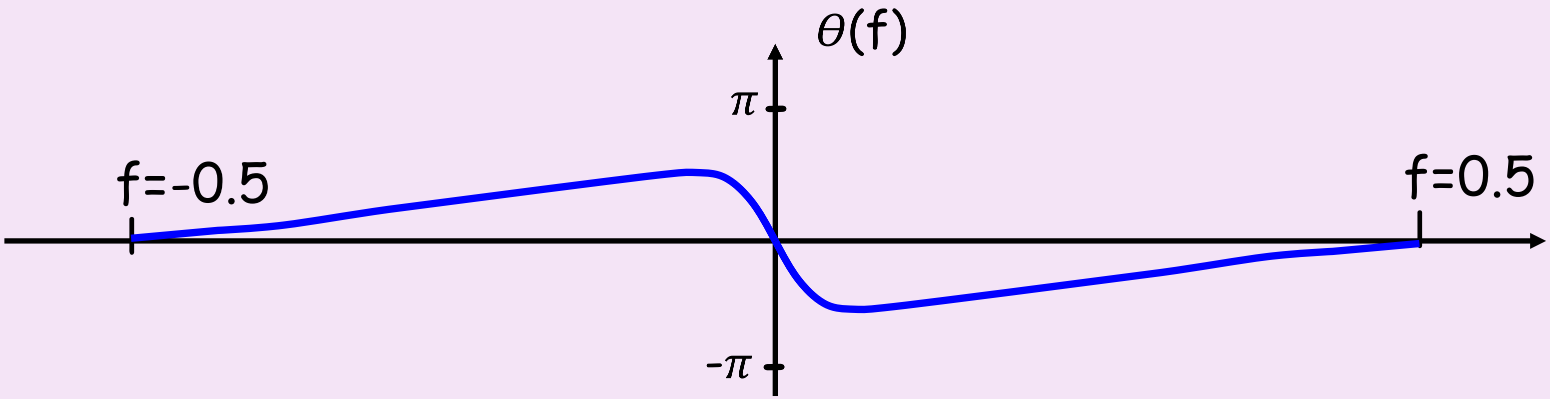

$$|H(f)| = \frac{1-a}{\sqrt{1+a^2-2a\cos(2\pi f)}}$$

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EITF75 Systems and Signals

EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$

Find $y(n)$ for $x(n)=5 + 12\sin(\pi n/2) - 20\cos(\pi n + \pi/4)$

Step 1: Identify the frequencies

$$|H(f)| = \frac{1-a}{\sqrt{1 + a^2 - 2a\cos(2\pi f)}}$$

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EITF75 Systems and Signals

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f=0 f=1/4 f=1/2

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EITF75 Systems and Signals

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f=0 f=1/4 f=1/2

Step 2: How to handle delay in input ?

$$|H(f)| = \frac{1-a}{\sqrt{1 + a^2 - 2a\cos(2\pi f)}}$$

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EITF75 Systems and Signals

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f=0 f=1/4 f=1/2

Step 2: How to handle delay in input ? LTI systems, so delays are remain in output

$$|H(f)| = \frac{1-a}{\sqrt{1 + a^2 - 2a\cos(2\pi f)}}$$

$$\theta(f) = -\tan^{-1}\frac{a\sin(2\pi f)}{1 - a\cos(2\pi f)}$$

EITF75 Systems and Signals

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f=0 f=1/4 f=1/2

Step 3: Compute $|H(f)|$ for above frequencies. (assume a=0.9)

$|H(0)| = \ldots = 1$ $|H(0.25)| = \ldots = 0.074$ $|H(0.5)| = \ldots = 0.053$

$$|H(f)| = \frac{1-a}{\sqrt{1 + a^2 - 2a\cos(2\pi f)}}$$

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EITF75 Systems and Signals

EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$

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$f=0$ $\quad f=1/4$ $\quad f=1/2$

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Step 3: Compute $\theta(f)$ for above frequencies. (assume a=0.9)

$\theta(0) = 0$ $\quad \theta(0.25) = \ldots = -42^\circ$ $\quad \theta(0.5) = \ldots = 0$

$$|H(f)| = \frac{1-a}{\sqrt{1 + a^2 - 2a\cos(2\pi f)}}$$

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EITF75 Systems and Signals

EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$

Find $y(n)$ for $x(n)=5 + 12\sin(\pi n/2) - 20\cos(\pi n + \pi/4)$

$f=0$	$f=1/4$	$f=1/2$
$\lvert H(0)\rvert = \ldots = 1$	$\lvert H(0.25)\rvert = \ldots = 0.074$	$\lvert H(0.5)\rvert = \ldots = 0.053$
$\theta(0) = 0$	$\theta(0.25) = \ldots = -42^\circ$	$\theta(0.5) = \ldots = 0$

Step 4: "Modify" $x(n)$ to obtain $y(n)$

$y(n) = 5 +$

EITF75 Systems and Signals

EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$

Find y(n) for $x(n)=5 + 12\sin(\pi n/2) - 20\cos(\pi n + \pi/4)$

f=0 f=1/4 f=1/2

$|H(0)| = \ldots = 1$ $|H(0.25)| = \ldots = 0.074$ $|H(0.5)| = \ldots = 0.053$

$\theta(0) = 0$ $\theta(0.25) = \ldots = -42^\circ$ $\theta(0.5) = \ldots = 0$

Step 4: "Modify" x(n) to obtain y(n)

$y(n) = 5 + 0.888\sin(\pi n/2 - 42^\circ) -$

$12 \times 0.074 = 0.888$

EITF75 Systems and Signals

EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$

Find y(n) for $x(n)=5 + 12\sin(\pi n/2) - 20\cos(\pi n + \pi/4)$

f=0	f=1/4	f=1/2
$\lvert H(0)\rvert = \ldots = 1$	$\lvert H(0.25)\rvert = \ldots = 0.074$	$\lvert H(0.5)\rvert = \ldots = 0.053$
$\theta(0) = 0$	$\theta(0.25) = \ldots = -42^{o}$	$\theta(0.5) = \ldots = 0$

Step 4: "Modify" x(n) to obtain y(n)

$$y(n) = 5 + 0.888\sin(\pi n/2 - 42^{o}) - 1.06\cos(\pi n + \pi/4)$$

$20 \times 0.053 = 1.06$

EITF75 Systems and Signals

EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$

Find y(n) for $x(n)=5 + 12\sin(\pi n/2) - 20\cos(\pi n + \pi/4)$

	f=0	f=1/4	f=1/2
	$\lvert H(0)\rvert = \ldots = 1$	$\lvert H(0.25)\rvert = \ldots = 0.074$	$\lvert H(0.5)\rvert = \ldots = 0.053$
	$\theta(0) = 0$	$\theta(0.25) = \ldots = -42^{\circ}$	$\theta(0.5) = \ldots = 0$

Step 4: "Modify" x(n) to obtain y(n)

$$y(n) = 5 + 0.888\sin(\pi n/2 - 42^{\circ}) - 1.06\cos(\pi n + \pi/4)$$

Note: this is due to LTI

EITF75 Systems and Signals

Next case to study

$$x(n) = \cos(2\pi f_0 n)u(n)$$

What can we say before starting ?

EITF75 Systems and Signals

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$$x(n) = \cos(2\pi f_0 n)u(n)$$

What can we say before starting ?

1. More realistic, so important
2. For a causal h(n), y(n) also causal

EITF75 Systems and Signals

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$$x(n) = \cos(2\pi f_0 n)u(n)$$

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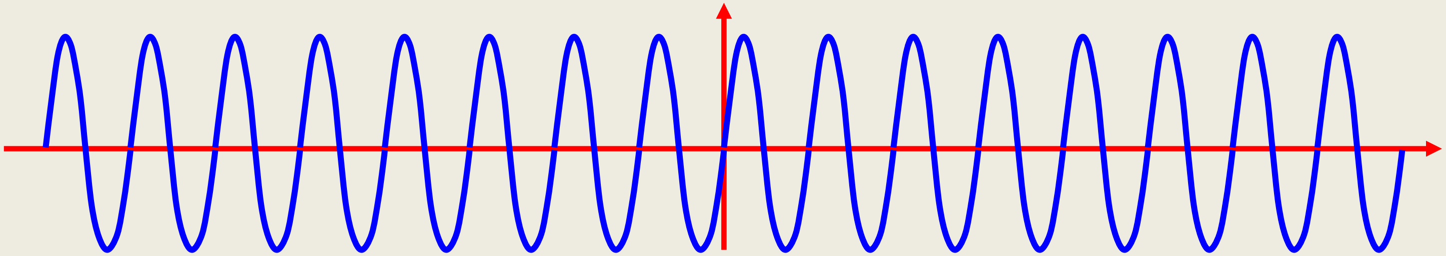
3. For a stable h(n), let us think about why the y(n) from before was periodic.

Input



EITF75 Systems and Signals

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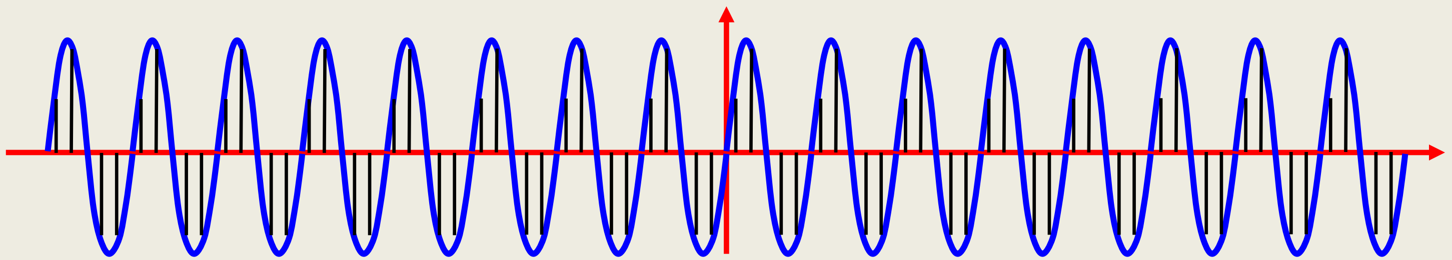

(Should be seen as a discrete time signal)

EITF75 Systems and Signals

Next case to study

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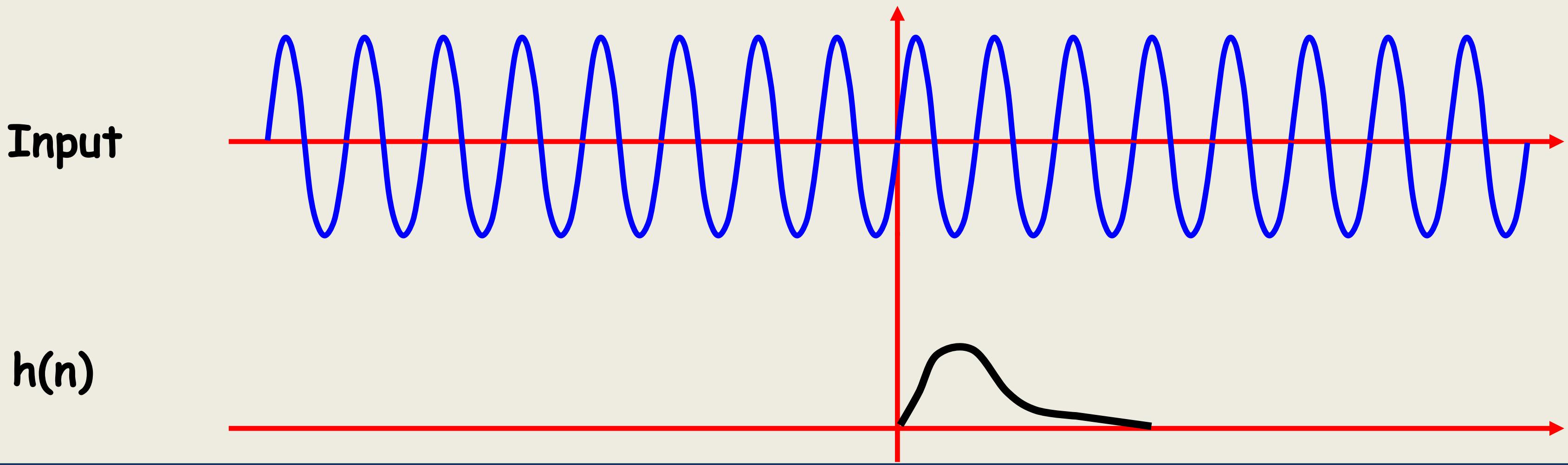

EITF75 Systems and Signals

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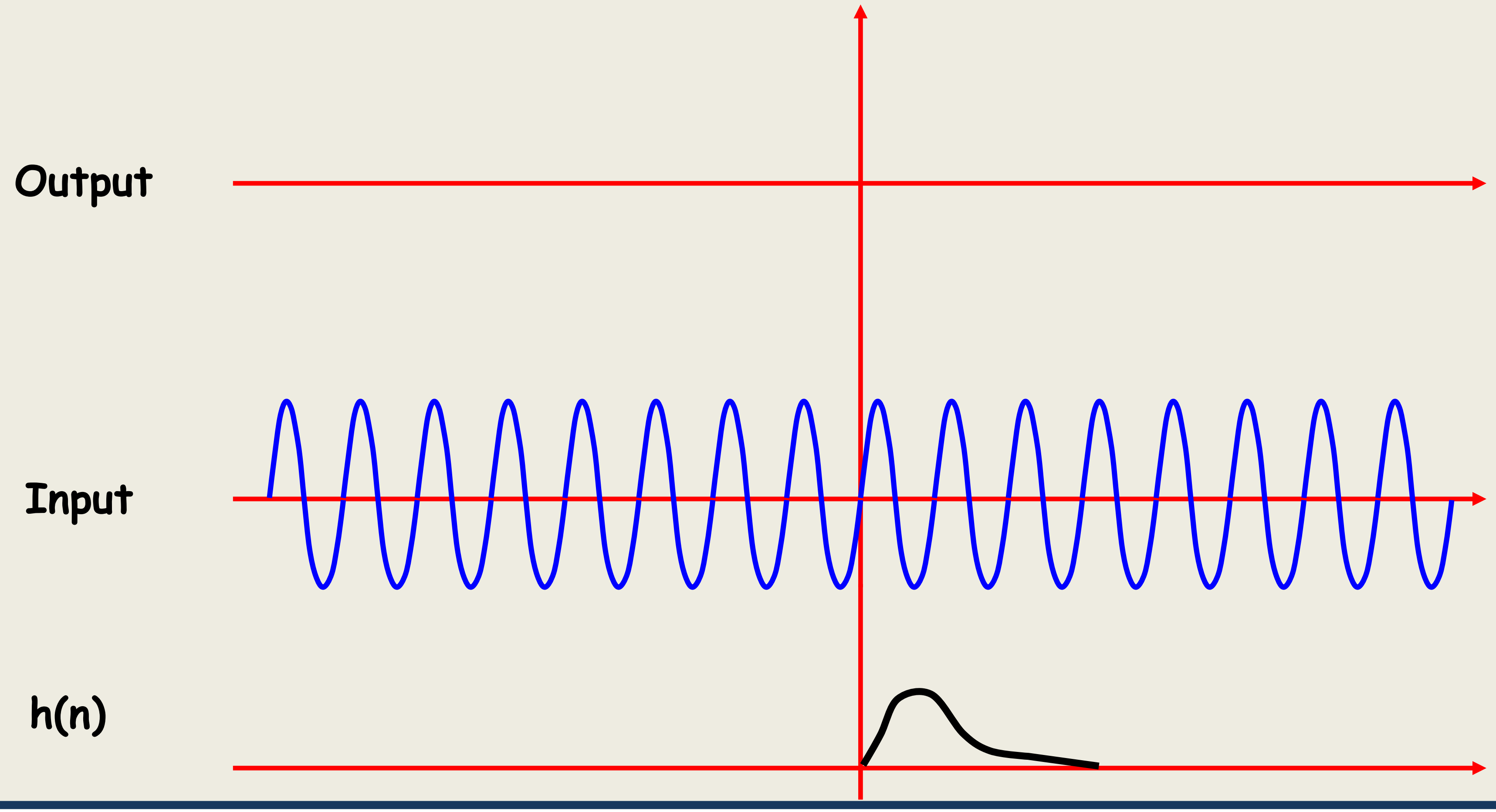

EITF75 Systems and Signals

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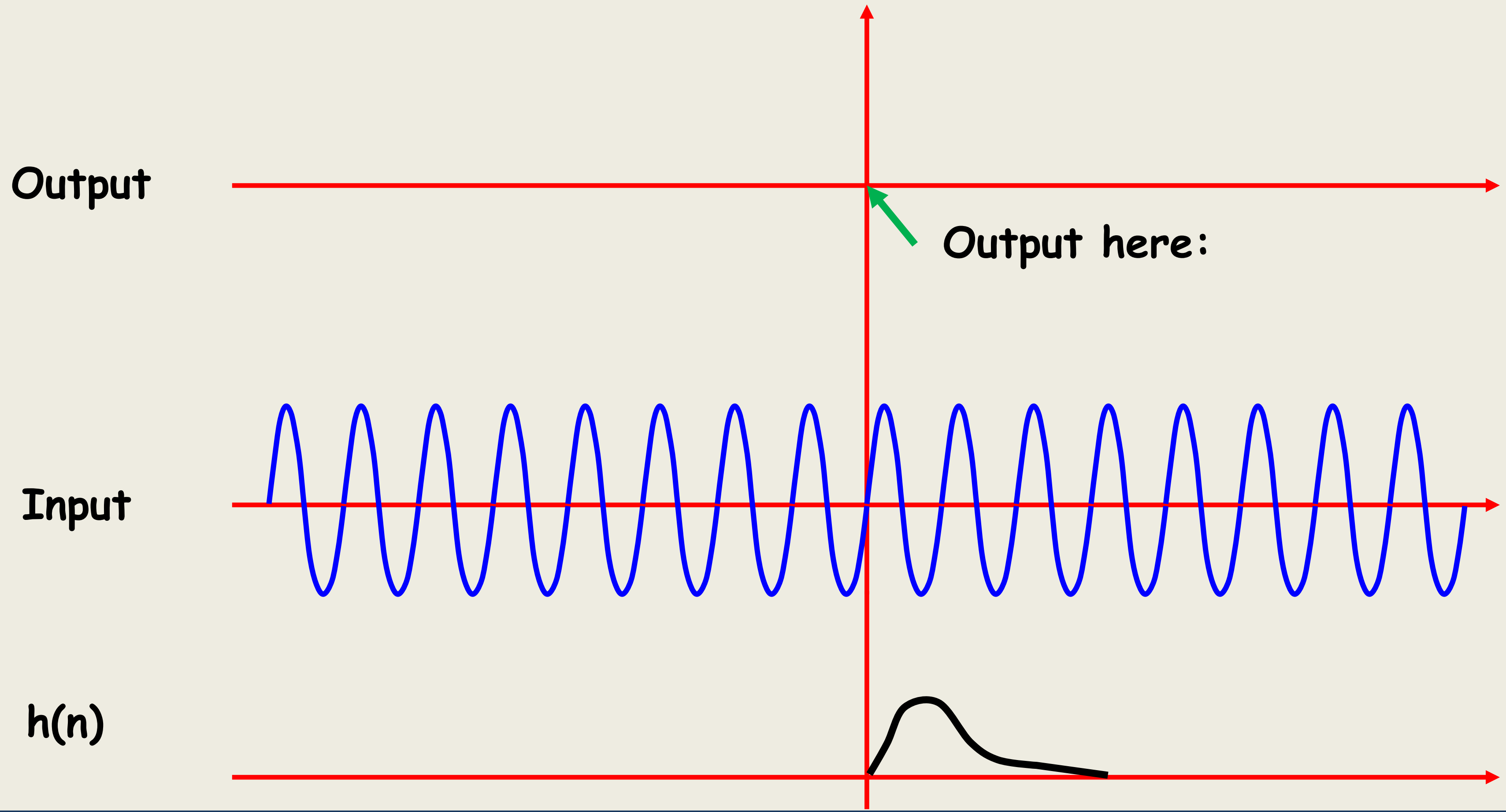

EITF75 Systems and Signals

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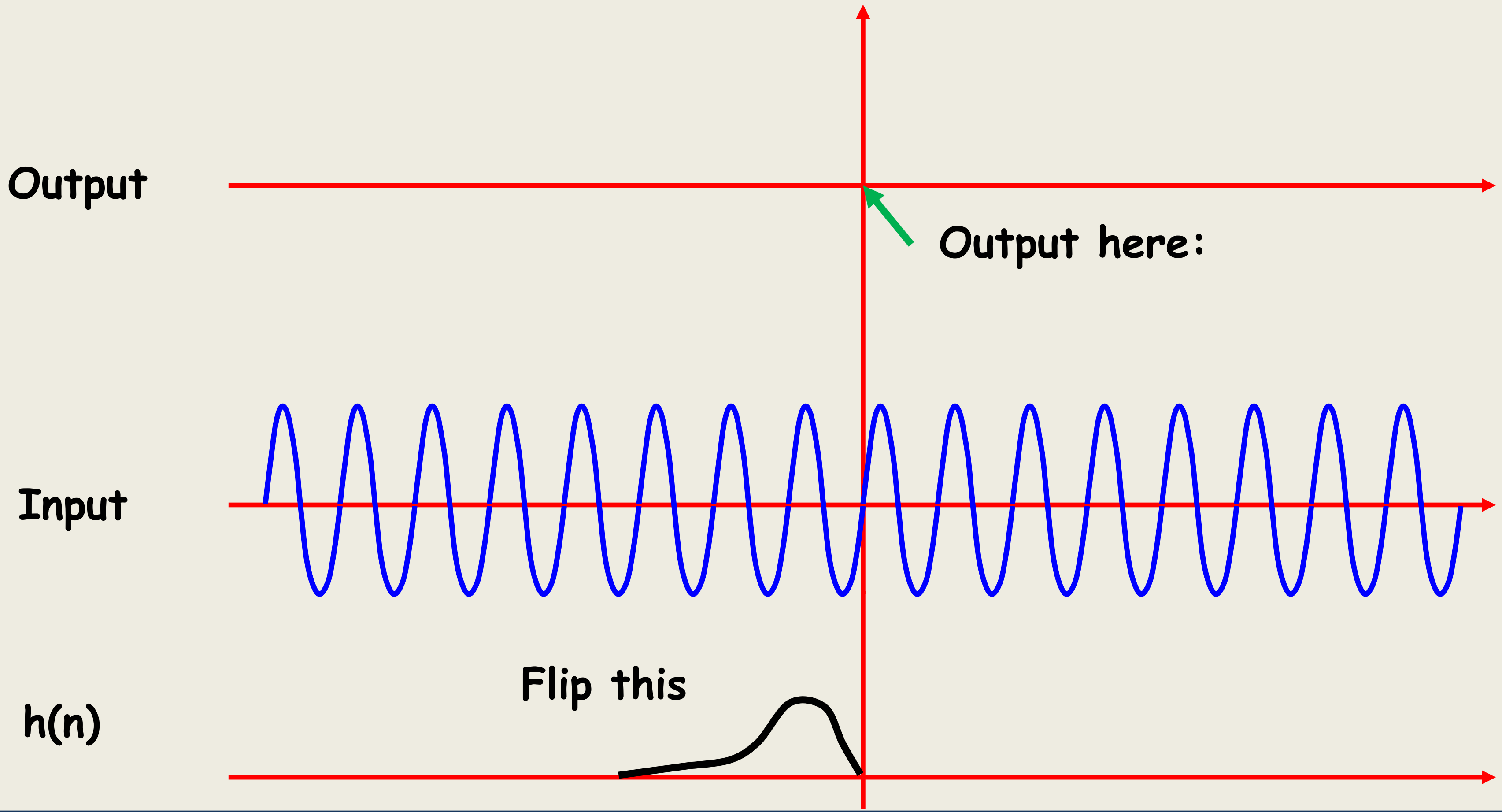

EITF75 Systems and Signals

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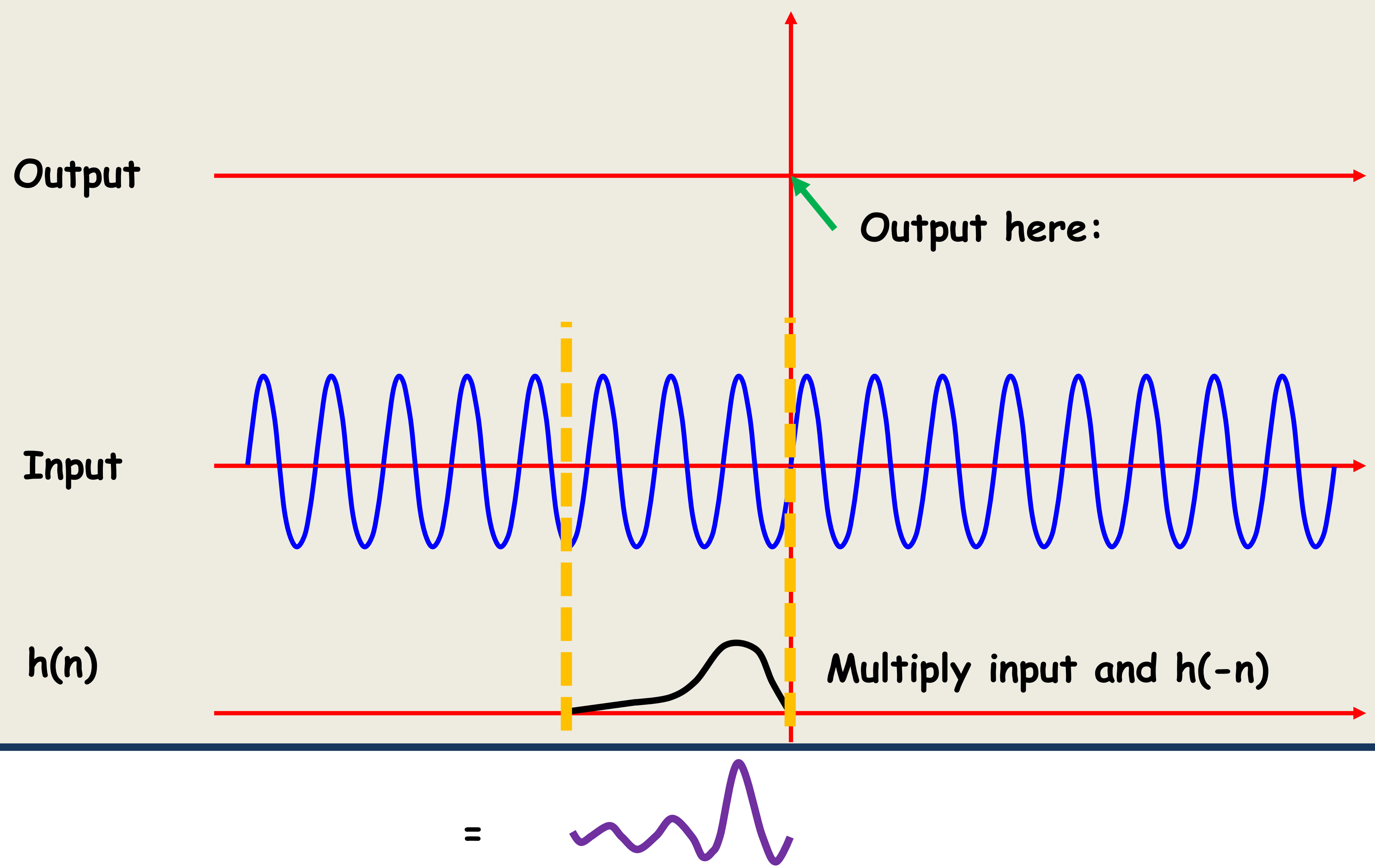

EITF75 Systems and Signals

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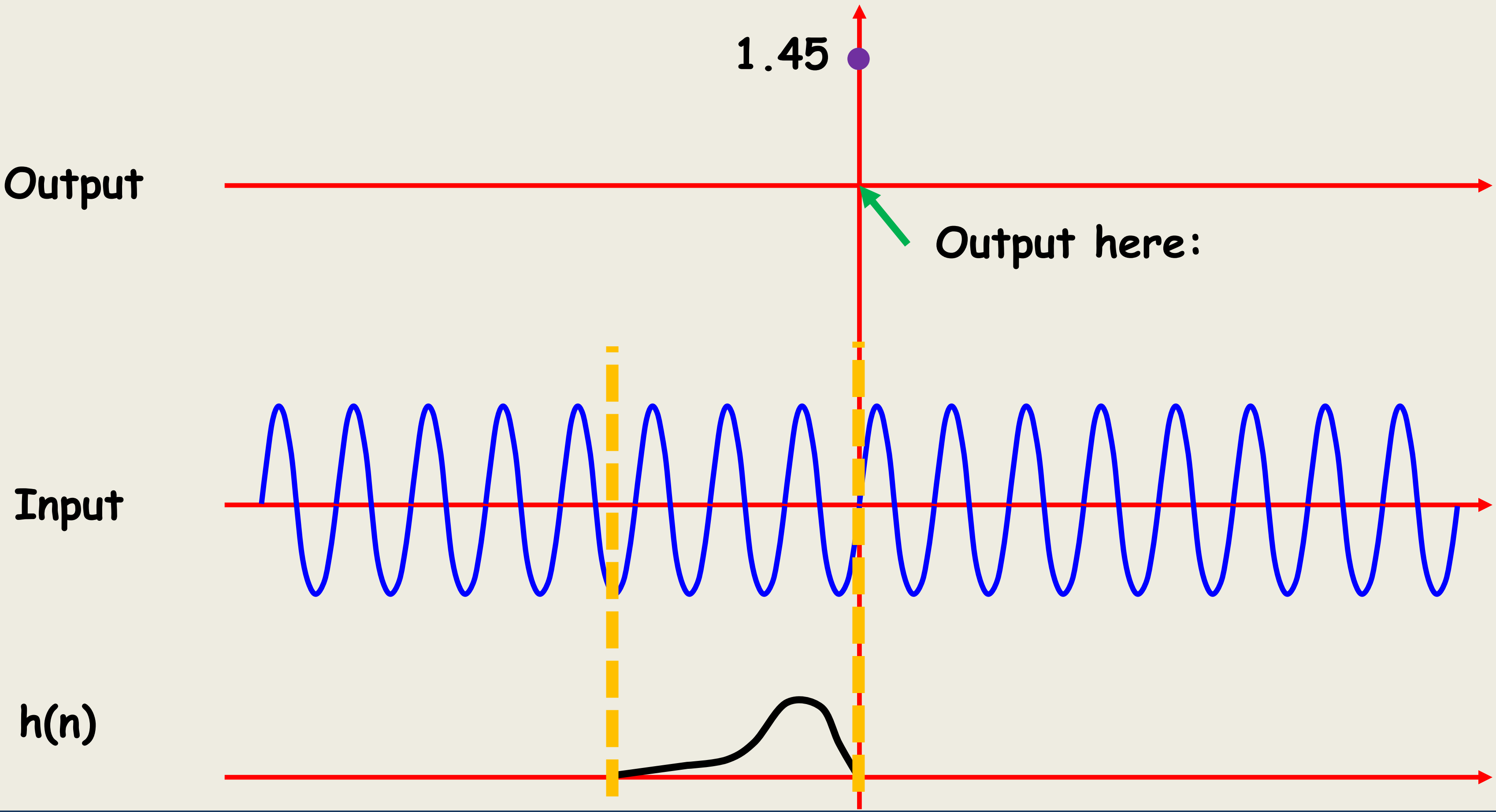
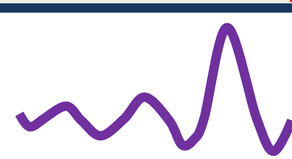


Integrate (sum for us)  = 1.45 (ex)

EITF75 Systems and Signals

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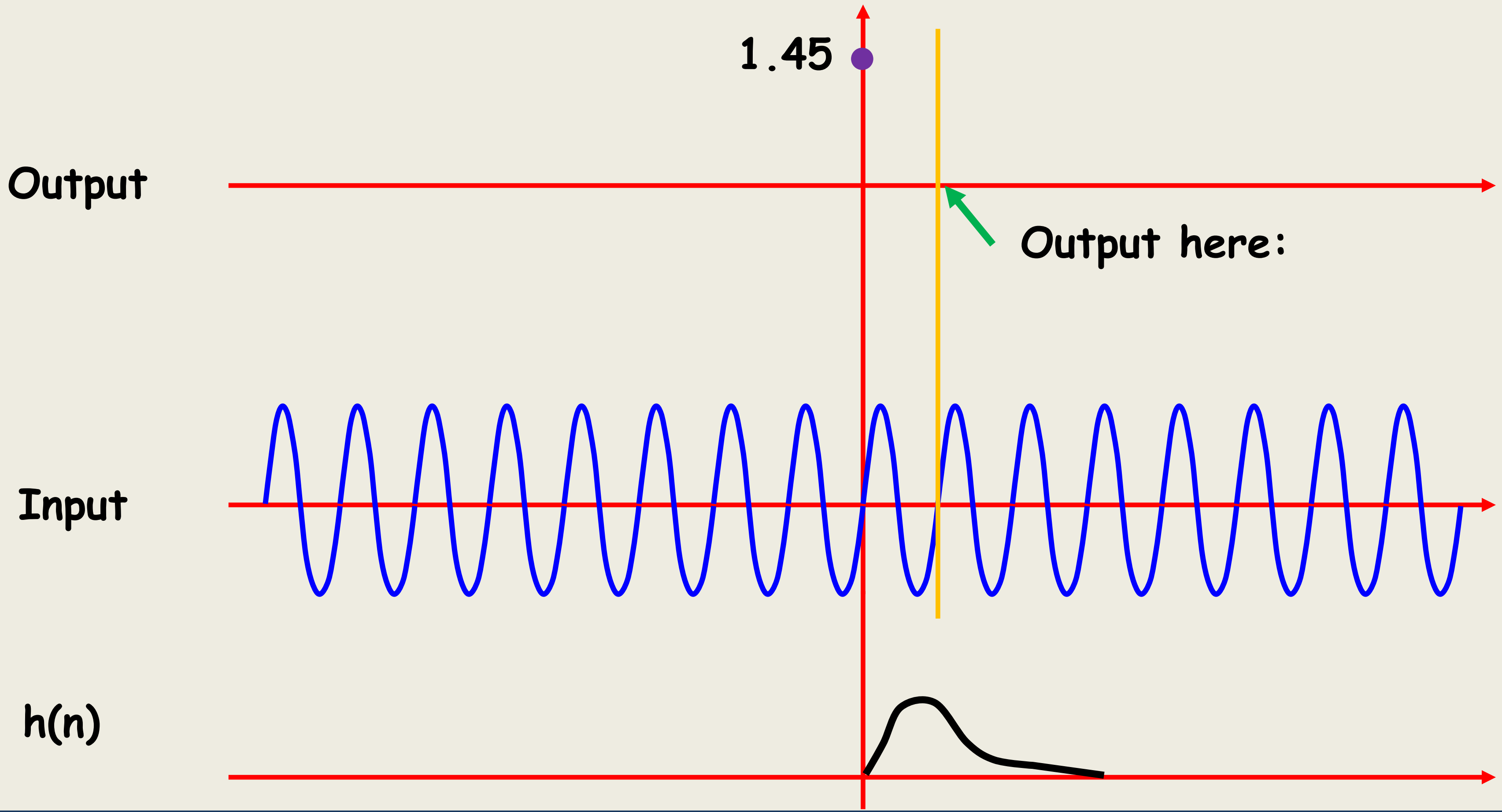

EITF75 Systems and Signals

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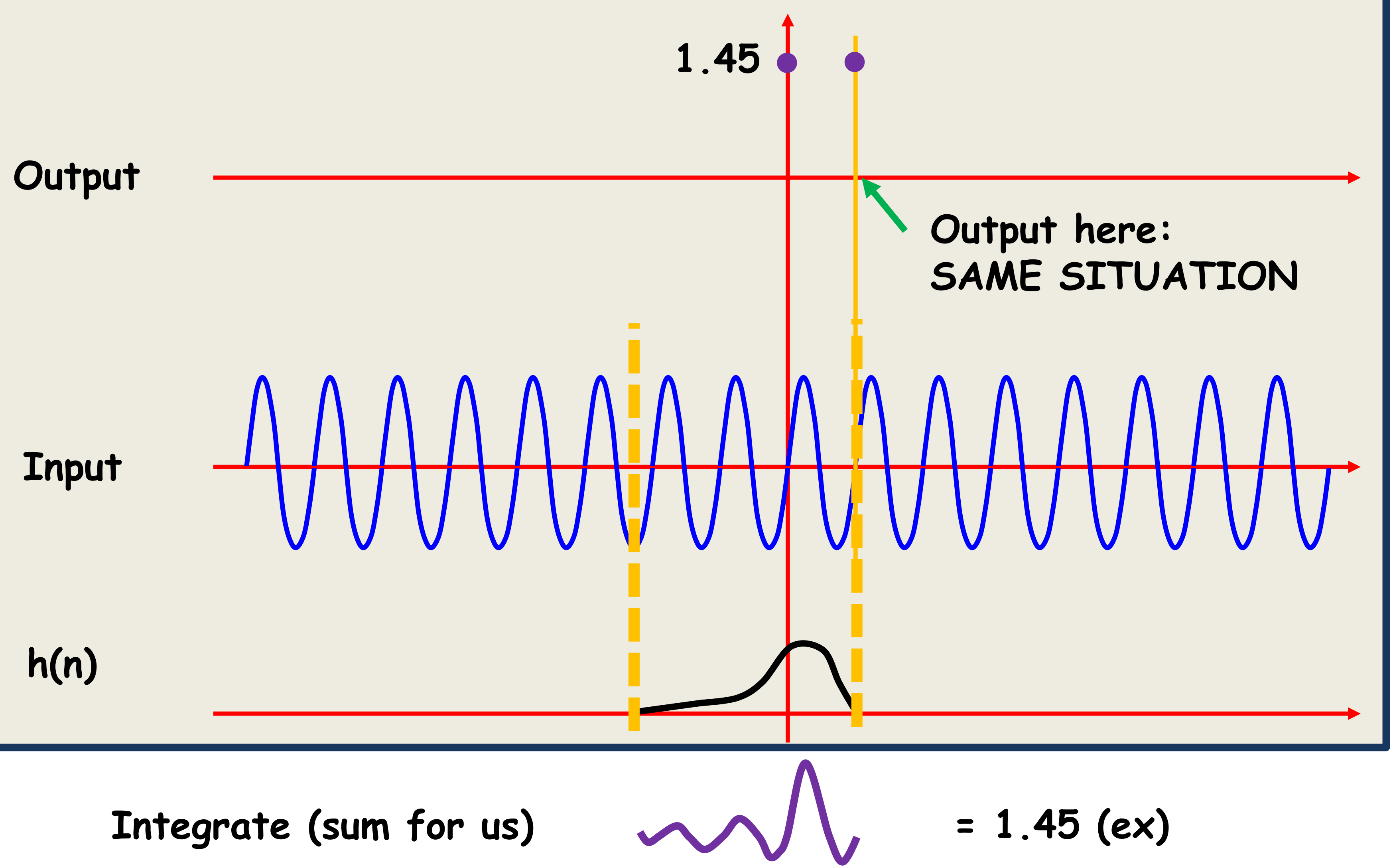

EITF75 Systems and Signals

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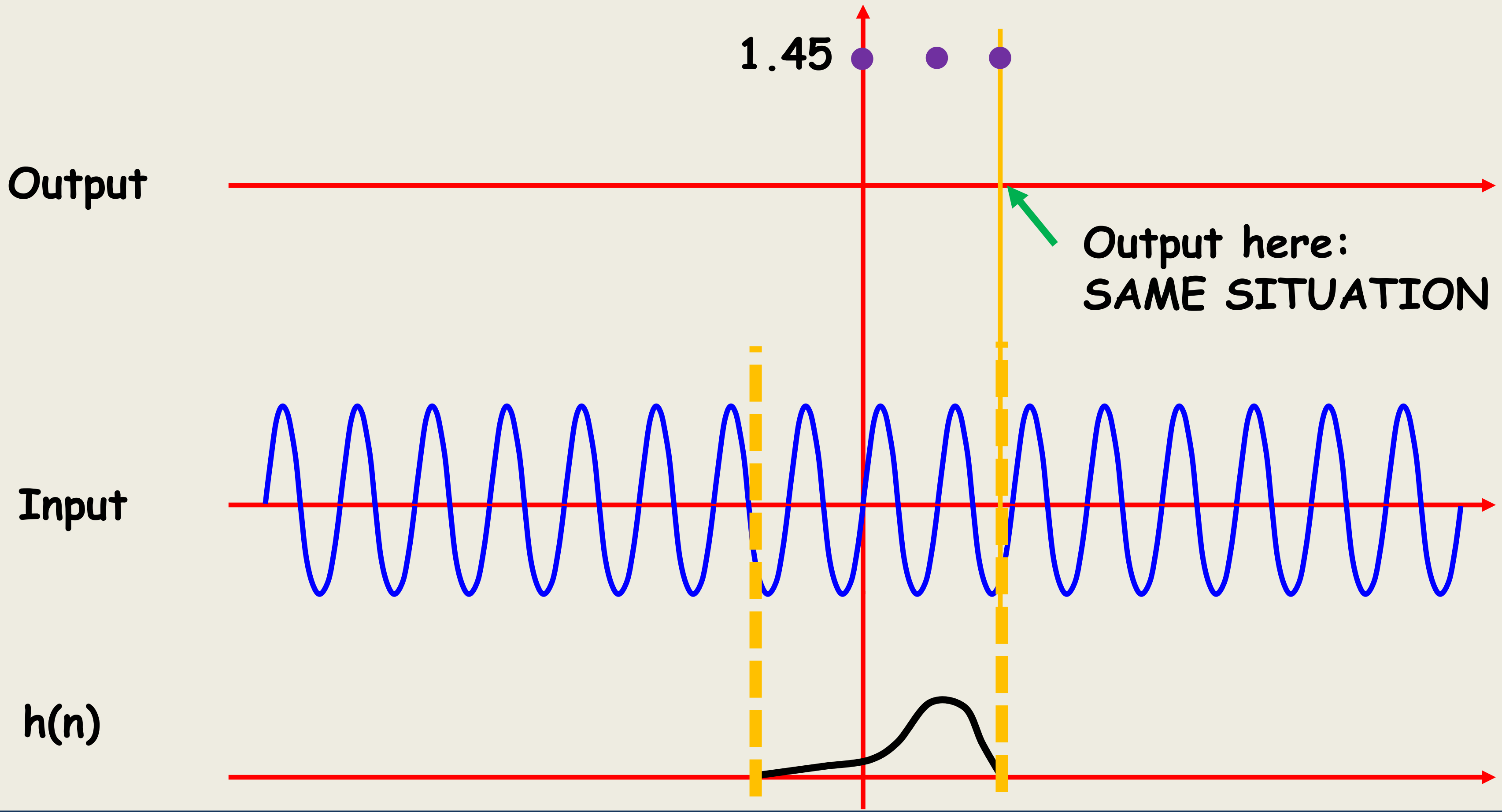

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EITF75 Systems and Signals

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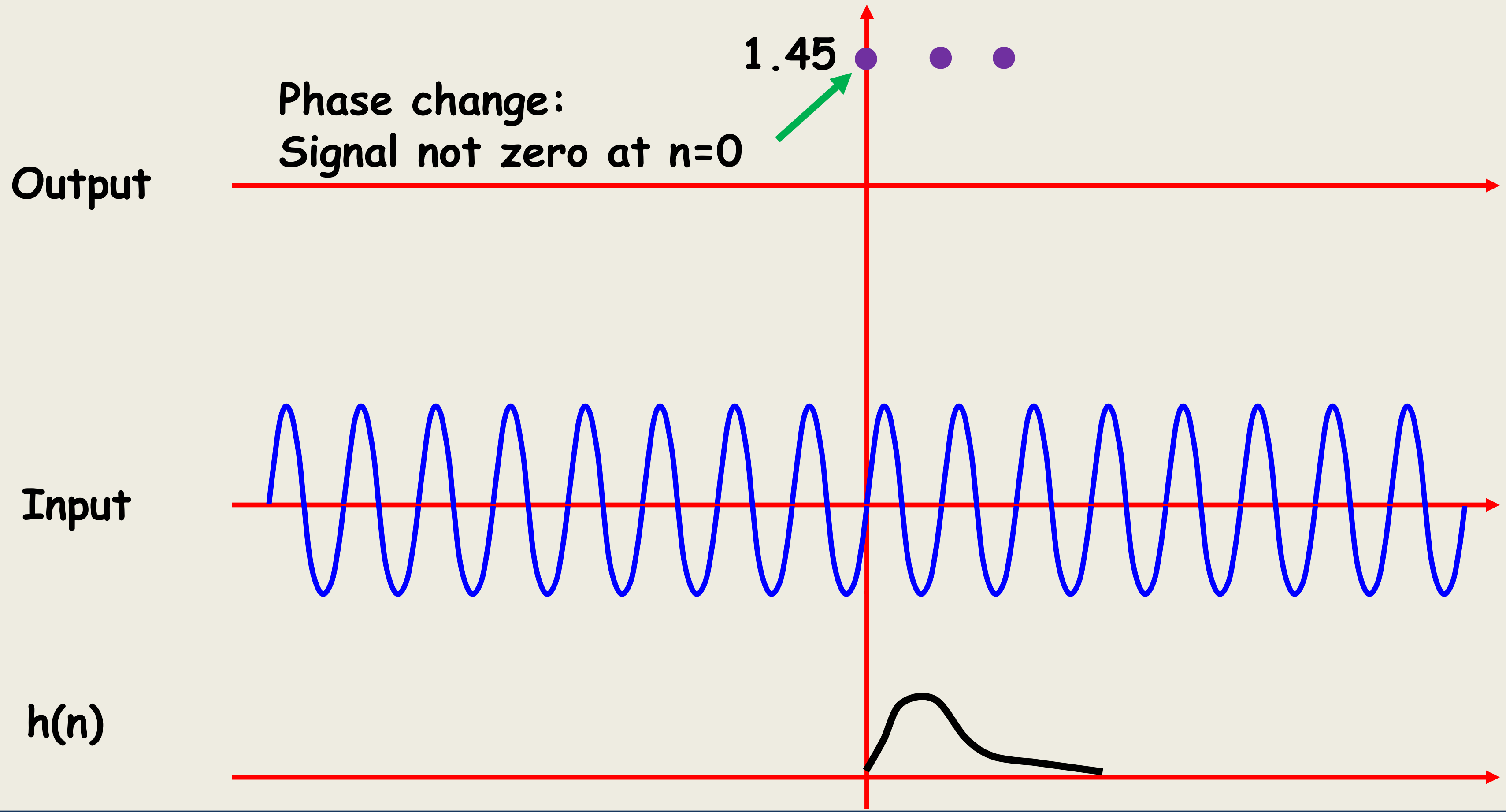

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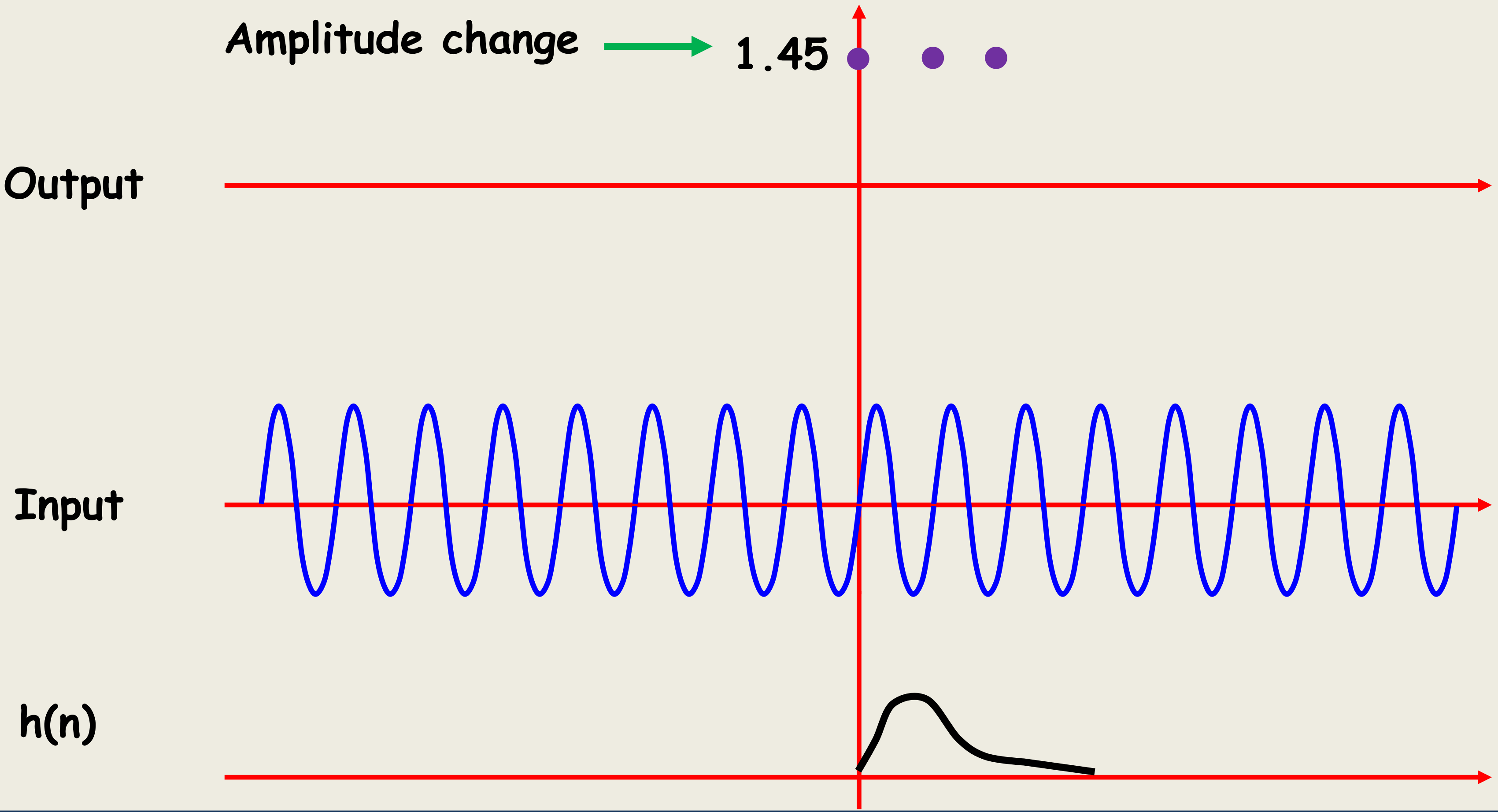

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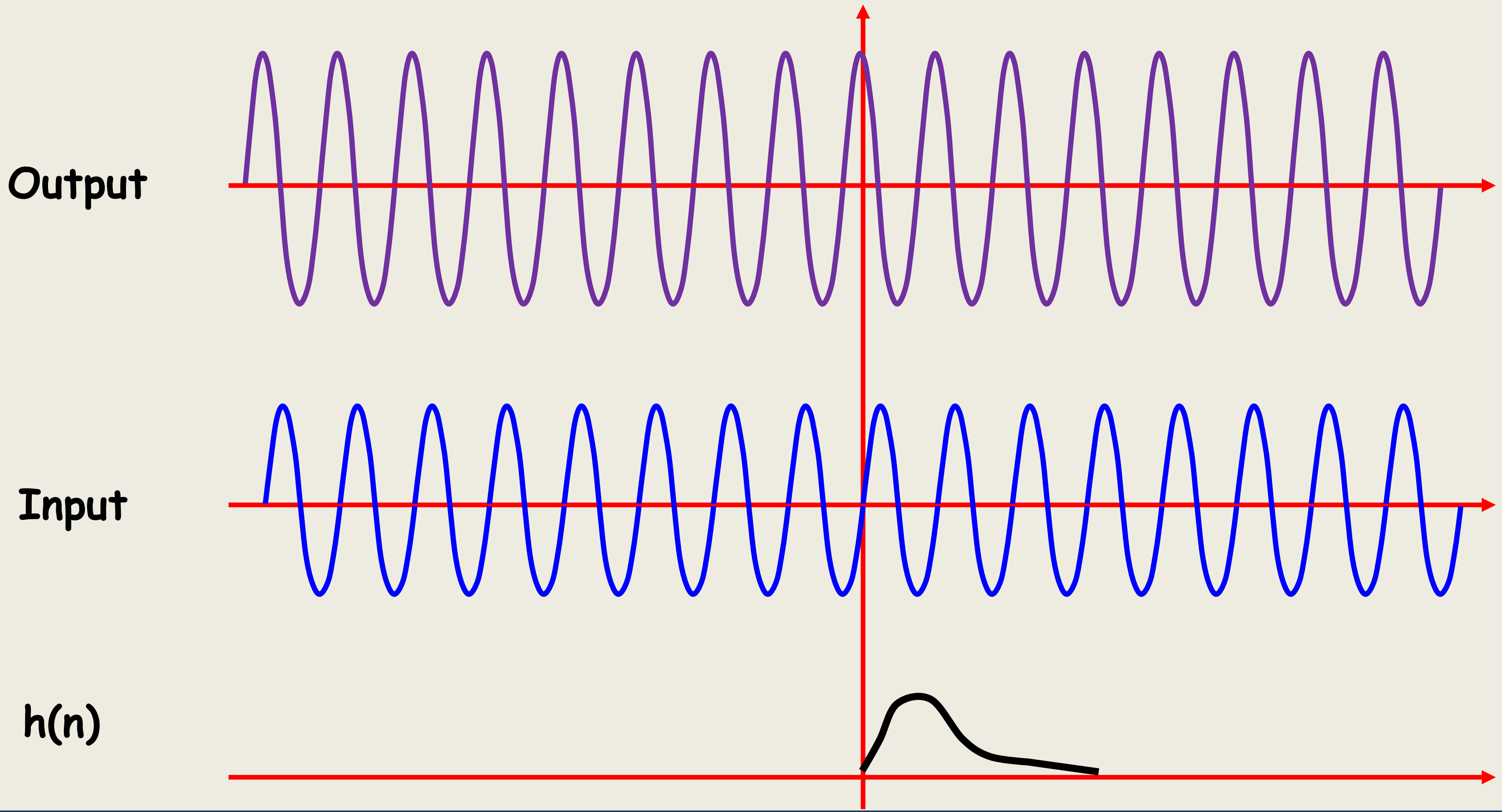
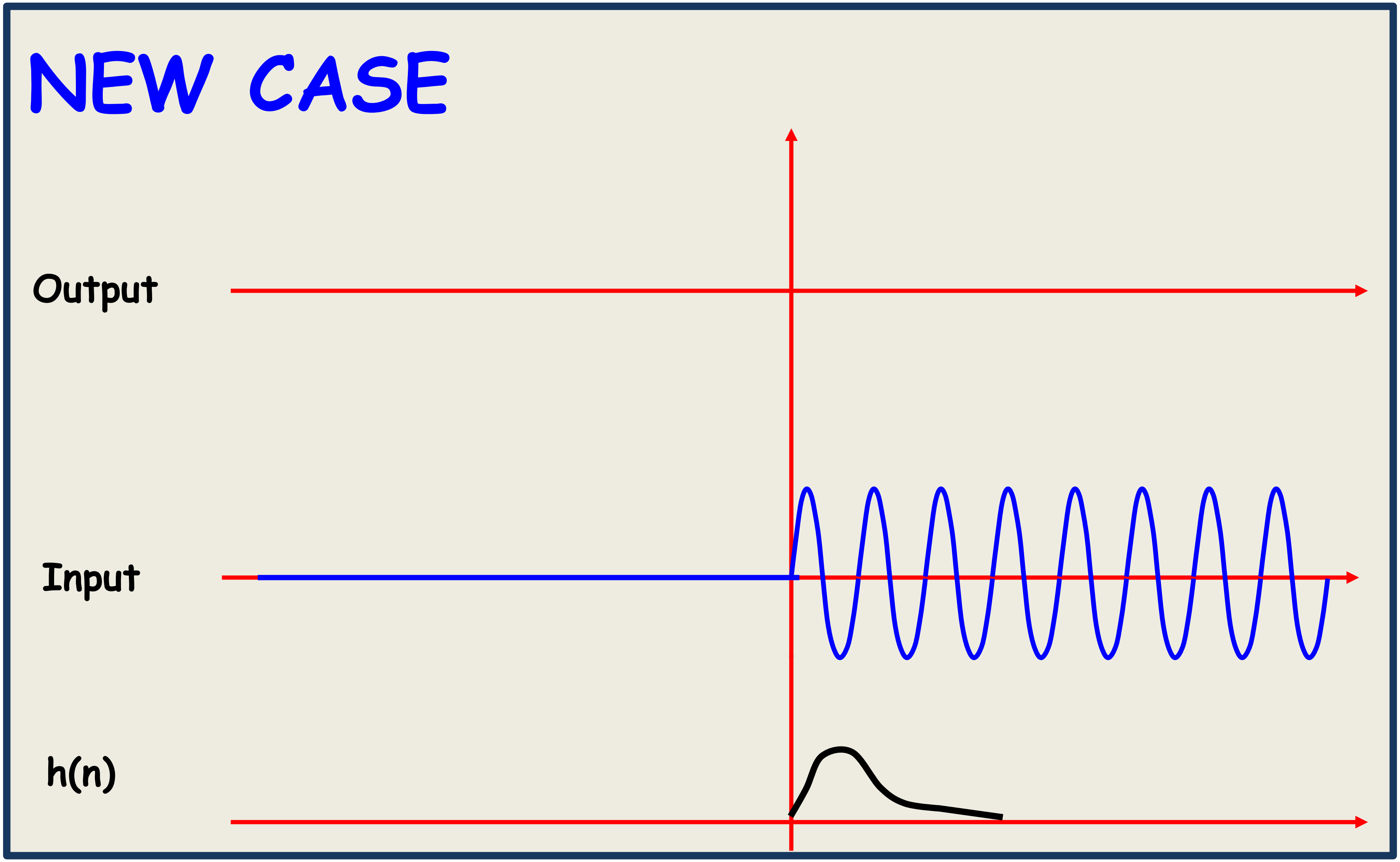
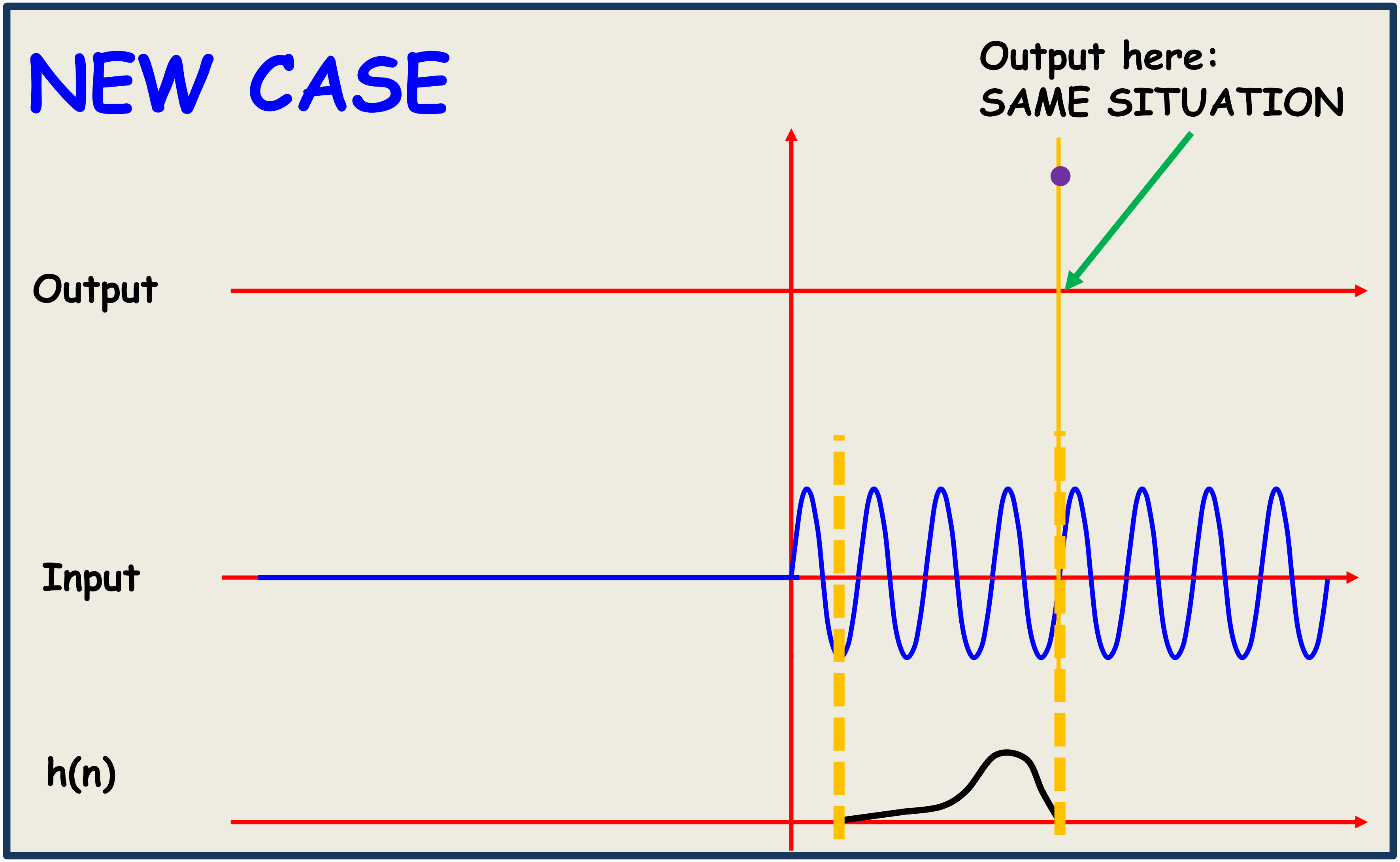


EITF75 Systems and Signals

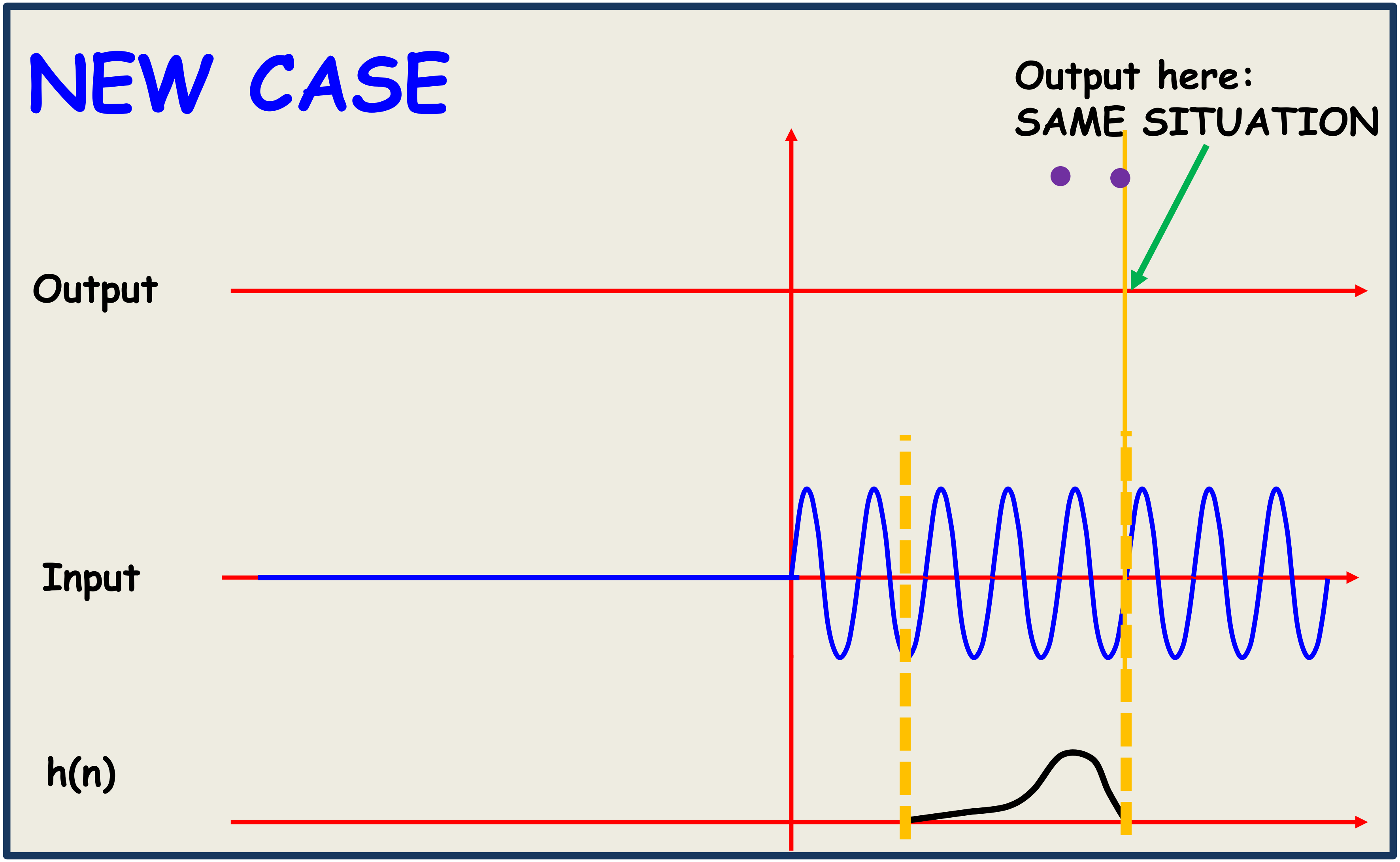


EITF75 Systems and Signals



Integrate (sum for us)  = 1.45 (ex)

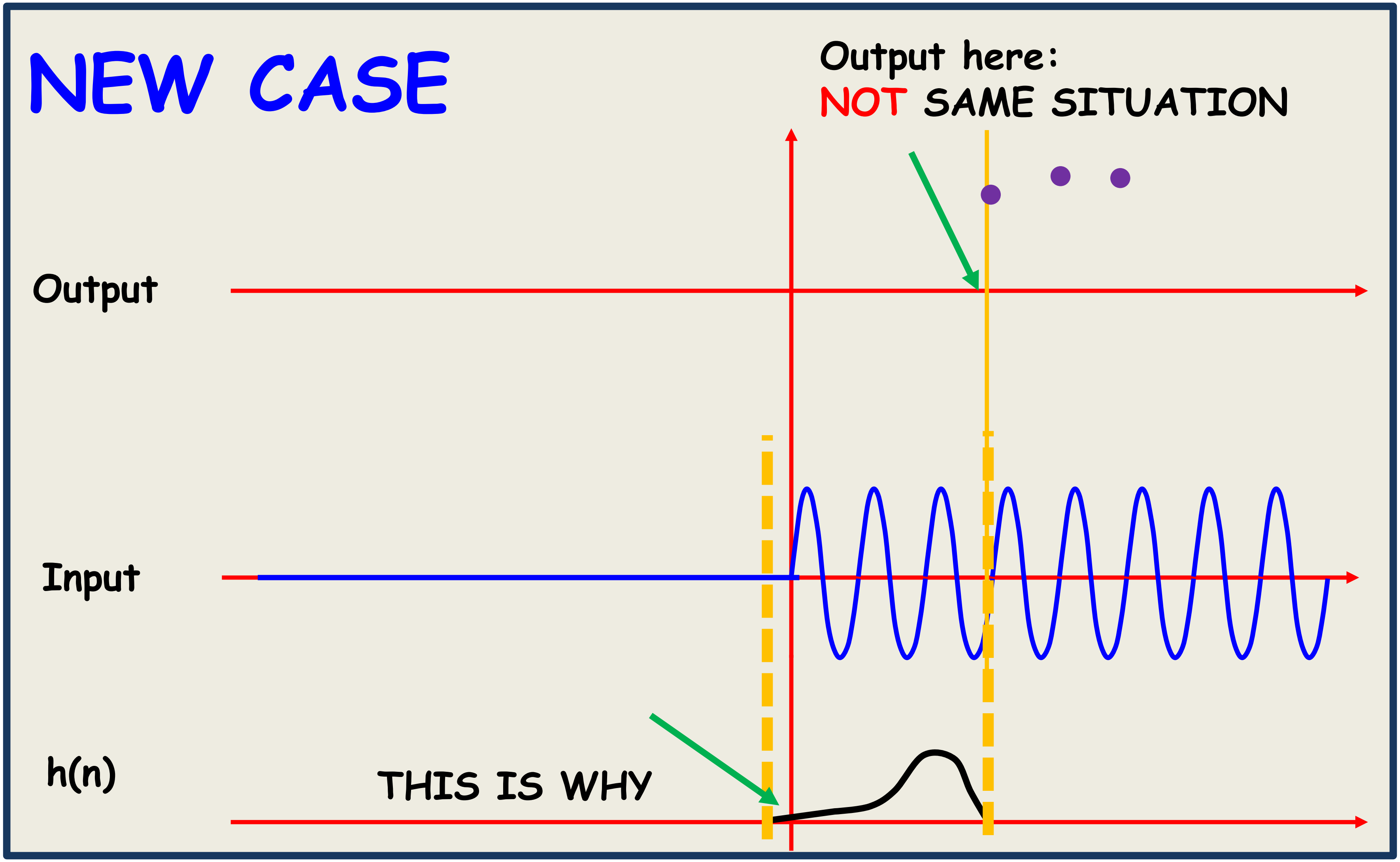
EITF75 Systems and Signals



Integrate (sum for us)

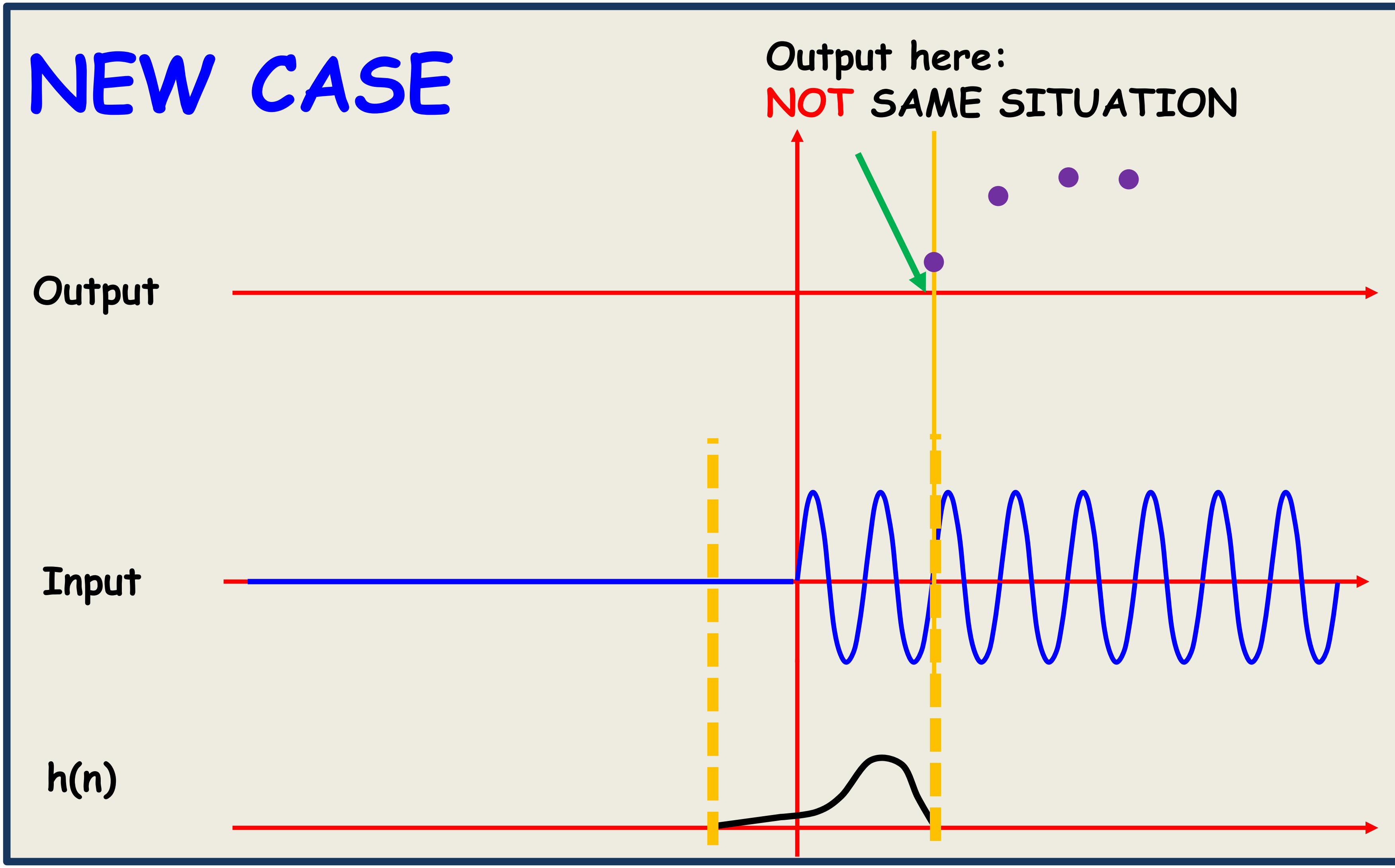
= 1.45 (ex)

EITF75 Systems and Signals

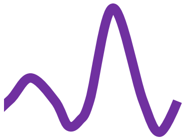


Integrate (sum for us)  = 1.3 (ex)

EITF75 Systems and Signals

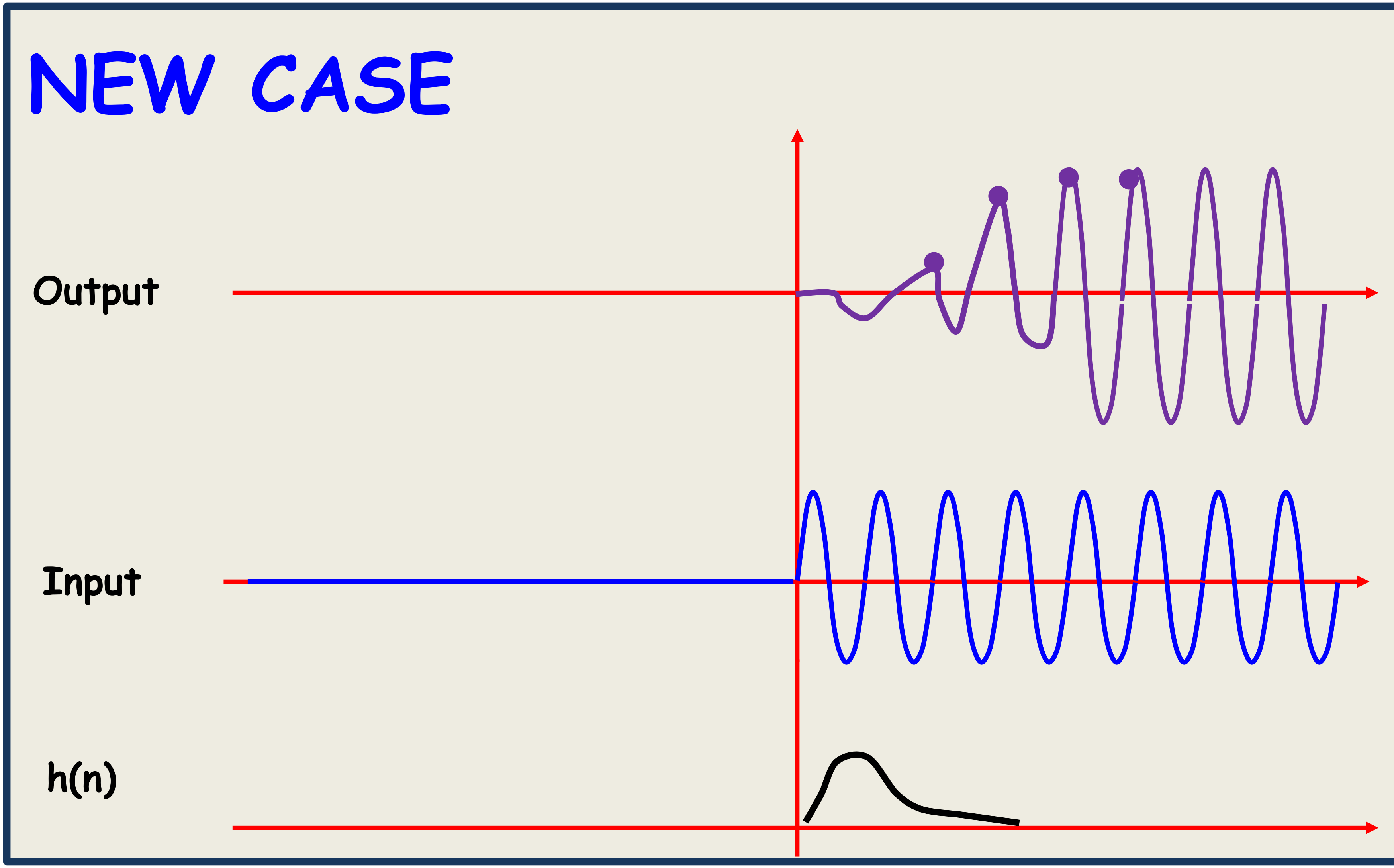


Integrate (sum for us)

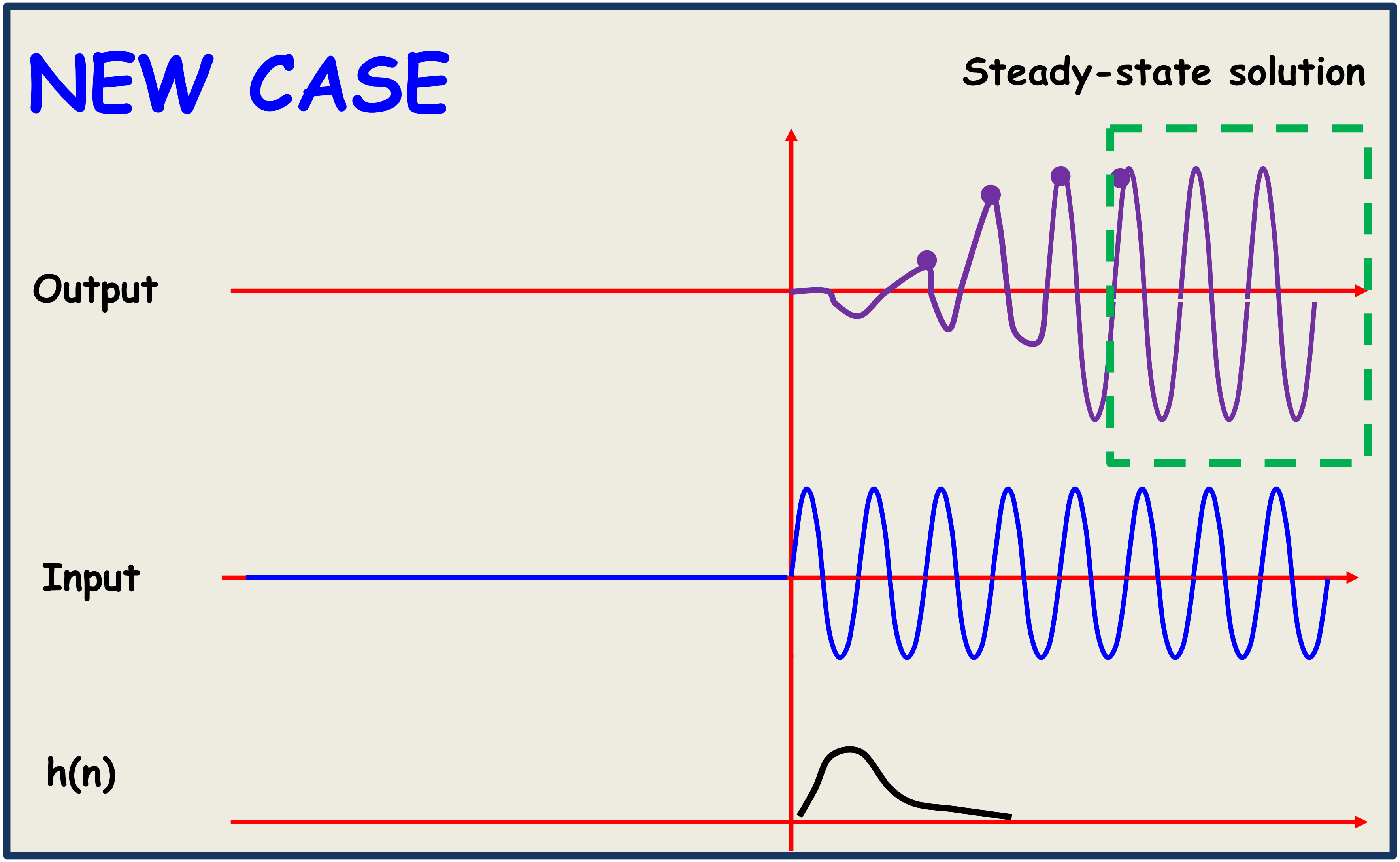


= 0.4 (ex)

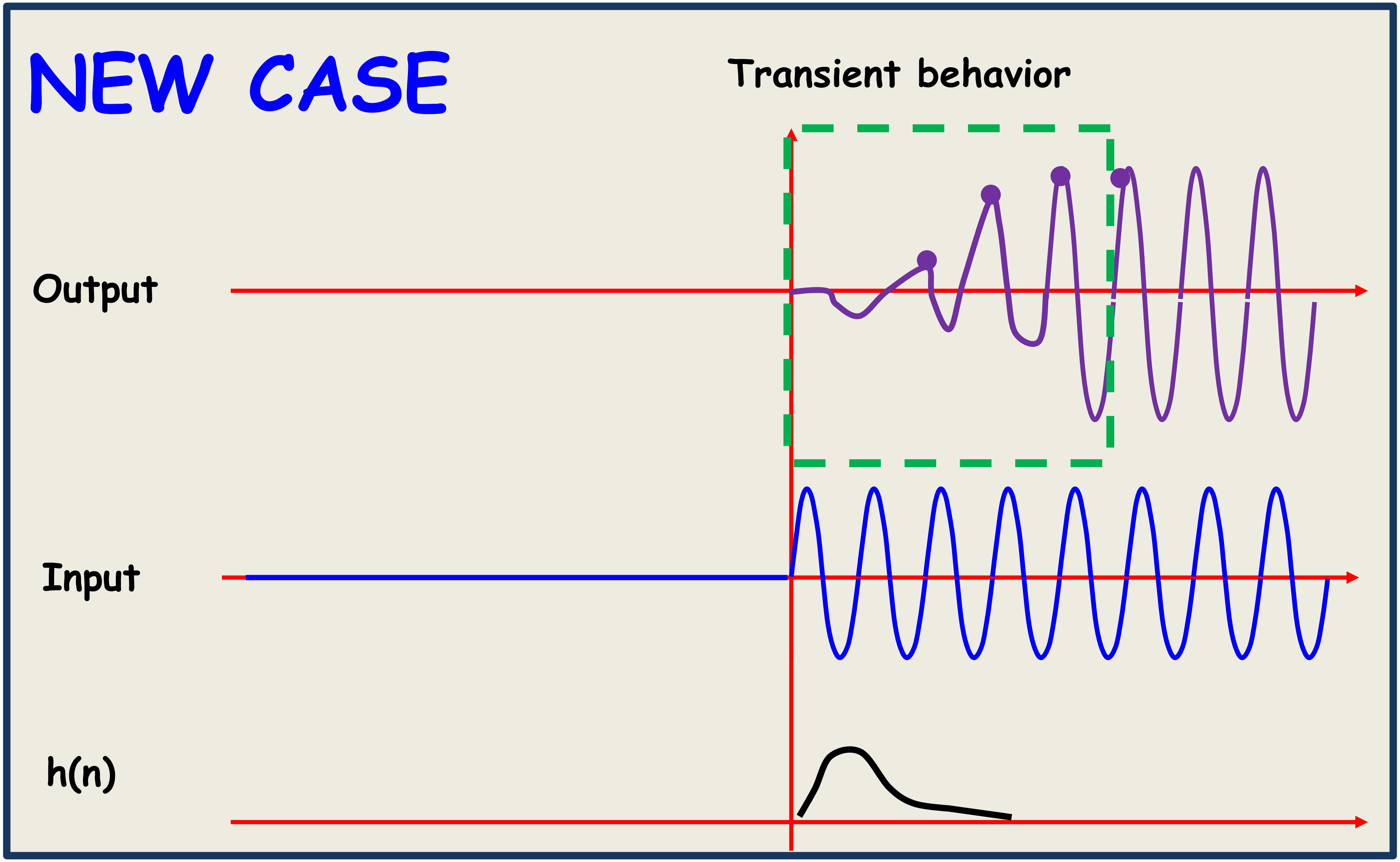
EITF75 Systems and Signals



EITF75 Systems and Signals



EITF75 Systems and Signals



EITF75 Systems and Signals

We have encountered a transient behavior once before. When?

EITF75 Systems and Signals

We have encountered a transient behavior once before. When?

For systems not at rest (has initial conditions)

EITF75 Systems and Signals

$$x(n) = \cos(\omega_0 n) \cdot u(n) \quad \omega_0 = \frac{2\pi}{16}$$

$$H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}}$$

Output

EITF75 Systems and Signals

$$x(n) = \cos(\omega_0 n) \cdot u(n) \quad \omega_0 = \frac{2\pi}{16}$$

$$H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}}$$

Output

Step 1: Copy from book $X(z) = \dfrac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$

A pair of complex conjugated poles at the unit circle

EITF75 Systems and Signals

$$x(n) = \cos(\omega_0 n) \cdot u(n) \quad \omega_0 = \frac{2\pi}{16}$$

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Step 1: Copy from book $X(z) = \dfrac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$

A pair of complex conjugated poles at the unit circle

Because "1" is multiplying z^{-2}

EITF75 Systems and Signals

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Step 1: Copy from book $X(z) = \dfrac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$

A pair of complex conjugated poles at the unit circle

Because $2\cos(w_0)$ < "two times square root of whatever multiplies z^{-2} "

EITF75 Systems and Signals

$$x(n) = \cos(\omega_0 n) \cdot u(n) \quad \omega_0 = \frac{2\pi}{16}$$

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Output

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A pair of complex conjugated poles at the unit circle

Requires "1" as constant term

EITF75 Systems and Signals

$$x(n) = \cos(\omega_0 n) \cdot u(n) \quad \omega_0 = \frac{2\pi}{16}$$

$$H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}}$$

Output

Step 1: Copy from book $X(z) = \dfrac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$

"1" as constant term: Satisfied

1.27 < "two times square root of 0.81" : satisfied

EITF75 Systems and Signals

$$x(n) = \cos(\omega_0 n) \cdot u(n) \quad \omega_0 = \frac{2\pi}{16}$$

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Implies pair of complex conjugated poles

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Output

Step 1: Copy from book $X(z) = \dfrac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$

"1" as constant term: Satisfied

1.27 < "two times square root of 0.81" : satisfied

Implies pair of complex conjugated poles

"1" is multiplying z^{-2} : No, so poles not at unit circle

EITF75 Systems and Signals

$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

$$H(z) = \frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}}$$

Output

Step 2: Form z-transform of output

$$Y(z) = H(z)X(z) = \frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}} \cdot \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

EITF75 Systems and Signals

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Step 3: Perform PFE

$$= -0.35 \cdot \frac{1-4.177z^{-1}}{1-1.27z^{-1}+0.81z^{-2}} + 0.35 \cdot \frac{1-1.896z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

Verify at home

EITF75 Systems and Signals

$$X(z)=\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0=\frac{2\pi}{16}$$

$$H(z)=\frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}}$$

Output

Step 2: Form z-transform of output

$$Y(z)=H(z)X(z)=\frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}}\cdot\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

$$=-0.35\cdot\frac{1-4.177z^{-1}}{1-1.27z^{-1}+0.81z^{-2}}+0.35\cdot\frac{1-1.896z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

Step 4: Take a break and study the above.

What properties can we identify?

EITF75 Systems and Signals

$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

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Step 2: Form z-transform of output

$$Y(z) = H(z)X(z) = \frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}} \cdot \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

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Poles at the unit circle

EITF75 Systems and Signals

$$X(z) = \frac{1 - \cos(\omega_0) z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

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Output

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$$Y(z) = H(z)X(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27 z^{-1} + 0.81 z^{-2}} \cdot \frac{1 - \cos(\omega_0) z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}}$$

$$= -0.35 \cdot \frac{1 - 4.177 z^{-1}}{1 - 1.27 z^{-1} + 0.81 z^{-2}} + 0.35 \cdot \frac{1 - 1.896 z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}}$$

Poles at the unit circle

Will be inversely transformed into a causal oscillating signal

EITF75 Systems and Signals

$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

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Poles at the unit circle

Will be inversely transformed into a causal oscillating signal

Steady state solution !!

EITF75 Systems and Signals

$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

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Step 2: Form z-transform of output

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Will it be a pure cosine?

(input was)

EITF75 Systems and Signals

$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

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Output

Step 2: Form z-transform of output

$$Y(z) = H(z)X(z) = \frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}} \cdot \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

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Will it be a pure cosine?

(input was)

No, since $1.896 \neq \cos(w_0)$

EITF75 Systems and Signals

$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

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Output

Step 2: Form z-transform of output

$$Y(z) = H(z)X(z) = \frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}} \cdot \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

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Will it be a pure cosine?

(input was)

No, since 1.896 ≠ $\cos(w_0)$

Therefore phase shift

EITF75 Systems and Signals

$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

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Output

Step 2: Form z-transform of output

$$Y(z) = H(z)X(z) = \frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}} \cdot \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

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Will it be a pure cosine?

(input was)

No, since $1.896 \neq \cos(w_0)$

Therefore phase shift

$\theta(w_0) \neq 0$

EITF75 Systems and Signals

$$X(z)=\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0=\frac{2\pi}{16}$$

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Output

Step 2: Form z-transform of output

$$Y(z)=H(z)X(z)=\frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}}\cdot\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

$$=-0.35\,\frac{1-4.177z^{-1}}{1-1.27z^{-1}+0.81z^{-2}}+0.35\cdot\frac{1-1.896z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

Poles inside the unit circle (r=0.9)

EITF75 Systems and Signals

$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

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Output

Step 2: Form z-transform of output

$$Y(z) = H(z)X(z) = \frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}} \cdot \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

$$= -0.35 \boxed{\frac{1-4.177z^{-1}}{1-1.27z^{-1}+0.81z^{-2}}} + 0.35 \cdot \frac{1-1.896z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

Poles inside the unit circle (r=0.9)

Will decay to 0 as n grows

Transient

EITF75 Systems and Signals

$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

$$H(z) = \frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}}$$

Output

Step 2: Form z-transform of output

$$Y(z) = H(z)X(z) = \frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}} \cdot \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

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Step 5: Perform inverse transform (messy, needs practice)

EITF75 Systems and Signals

$$X(z)=\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0=\frac{2\pi}{16}$$

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Step 2: Form z-transform of output

$$Y(z)=H(z)X(z)=\frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}}\cdot\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

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$$=-0.35\cdot\frac{1-1.27/2\;z^{-1}}{1-1.27z^{-1}+0.81z^{-2}}+0.35\cdot\frac{(4.177-1.27/2)\,z^{-1}}{1-1.27z^{-1}+0.81z^{-2}}$$

EITF75 Systems and Signals

$$X(z) = \frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

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$$= -0.35 \cdot \frac{1 - 4.177z^{-1}}{1 - 1.27z^{-1} + 0.81z^{-2}} + 0.35 \cdot \frac{1 - 1.896z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$$

$$= -0.35 \cdot \frac{1 - 1.27/2 \; z^{-1}}{1 - 1.27z^{-1} + 0.81z^{-2}} + 0.35 \cdot \frac{(4.177 - 1.27/2)\, z^{-1}}{1 - 1.27z^{-1} + 0.81z^{-2}}$$

$$-0.35 \cdot 0.9^n \cdot \cos\left(2\pi \cdot \frac{1}{8} \cdot n\right) \qquad 0.35 \cdot 5.562 \cdot 0.9^n \sin\left(2\pi \cdot \frac{1}{8} \cdot n\right)$$

EITF75 Systems and Signals

$$X(z)=\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}\quad \omega_0=\frac{2\pi}{16}$$

$$H(z)=\frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}}$$

Output

Step 2: Form z-transform of output

$$Y(z)=H(z)X(z)=\frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}}\cdot\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

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$$=-0.35\cdot\frac{1-\ 1.27/2\ z^{-1}}{1-1.27z^{-1}+0.81z^{-2}}+0.35\cdot\frac{(4.177-1.27/2)\,z^{-1}}{1-1.27z^{-1}+0.81z^{-2}}$$

$$-0.35\cdot 0.9^n\cdot\cos\left(2\pi\cdot\frac{1}{8}\cdot n\right)\qquad 0.35\cdot 5.562\cdot 0.9^n\sin\left(2\pi\cdot\frac{1}{8}\cdot n\right)$$ **Transients**

EITF75 Systems and Signals

$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

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$$+0.35 \cdot \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad +0.35 \cdot \frac{(\cos(\omega_0)-1.896)\,z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

EITF75 Systems and Signals

$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

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$$+0.35 \cdot \cos(\omega_0 \cdot n) - 0.35 \cdot 2.5392 \cdot \sin(\omega_0 \cdot n)$$

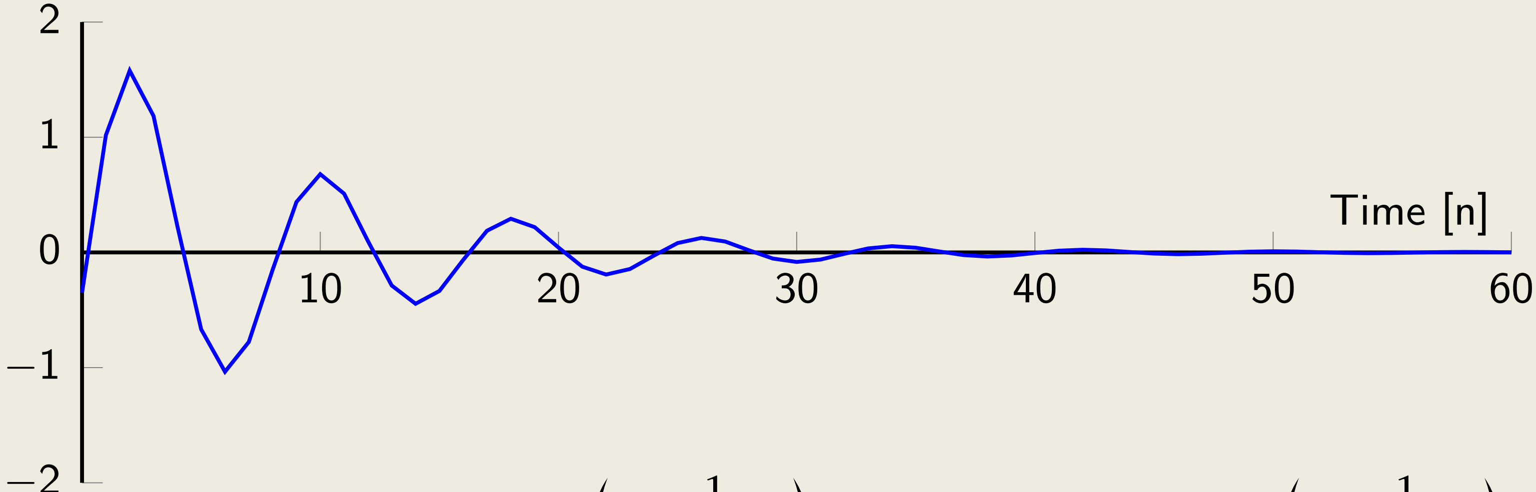
Steady state

EITF75 Systems and Signals

$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

$$H(z) = \frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}}$$

Transient



$$-0.35\cdot 0.9^n \cdot \cos\left(2\pi\cdot\frac{1}{8}\cdot n\right) + 0.35\cdot 5.562\cdot 0.9^n \sin\left(2\pi\cdot\frac{1}{8}\cdot n\right)$$

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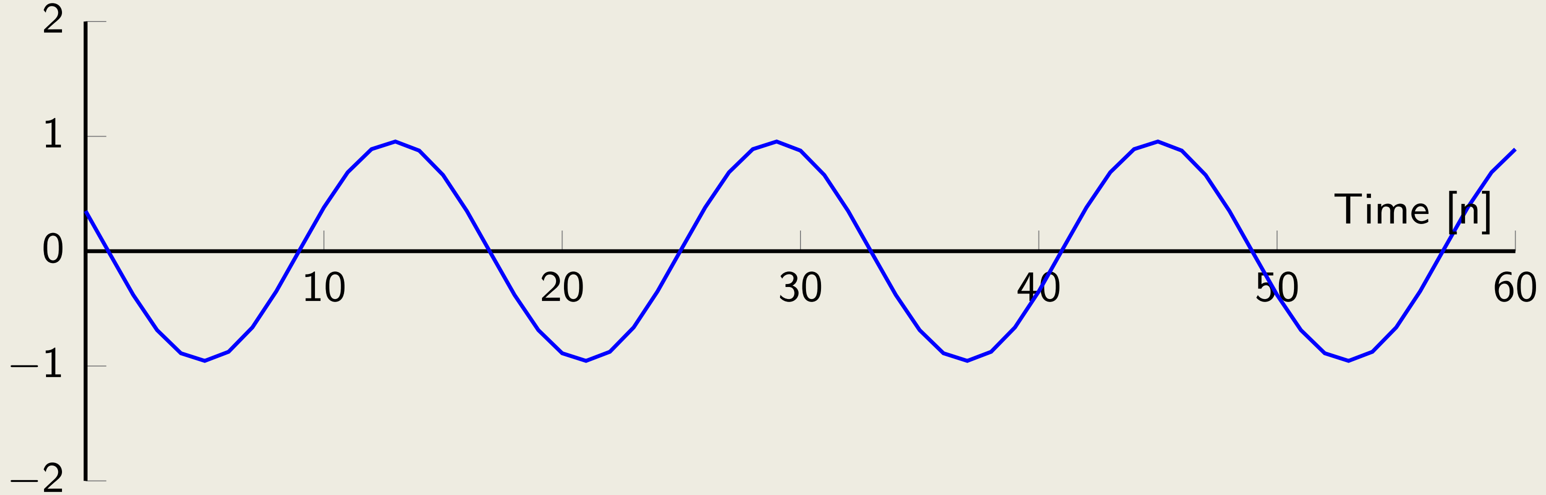
$$X(z) = \frac{1 - \cos(\omega_0) z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

$$H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27 z^{-1} + 0.81 z^{-2}}$$

Steady state



$$0.35 \cdot \cos(\omega_0 \cdot n) - 0.35 \cdot 2.5392 \cdot \sin(\omega_0 \cdot n)$$

EITF75 Systems and Signals

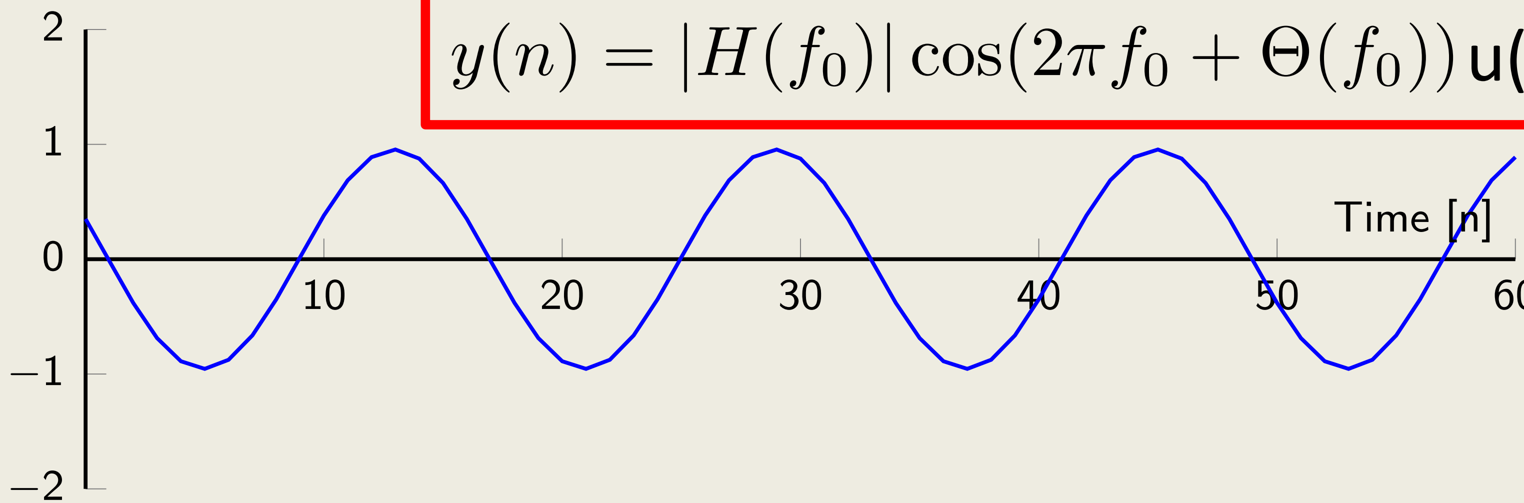
$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

$$H(z) = \frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}}$$

Steady state

Important:
The steady state solution can be computed via

$$y(n) = |H(f_0)|\cos(2\pi f_0 + \Theta(f_0))\,\mathrm{u(n)}$$



$$0.35\cdot\cos(\omega_0\cdot n) - 0.35\cdot 2.5392\cdot\sin(\omega_0\cdot n)$$

Group and phase delay

Remark: In videos/old lecture notes/book, the material on group delay is incomplete and becomes confusing

EITF75 Systems and Signals

Group and phase delay

Motivation:

We have seen that a pure tone becomes phase delayed after a filter

Any interesting signal comprices several tones

What if these would be phase delayed differently?

EIT F75 Systems and Signals

Group and phase delay

Motivation:

We have seen that a pure tone becomes phase delayed after a filter

Any interesting signal comprices several tones

What if these would be phase delayed differently?

For example: Human speech has frequencies 20-5000 Hz

If we filter a sound signal, we would like all frequencies to be phase delayed equally much

EITF75 Systems and Signals

Group and phase delay

$$x(n) \longrightarrow \boxed{H(\omega) = A(\omega)\mathrm{e}^{\mathrm{j}\Phi(\omega)}} \longrightarrow y(n)$$

$x(n) = \sin(\omega_0 n)$ **(ignore transients)**

EITF75 Systems and Signals

Group and phase delay

$$x(n) \longrightarrow \boxed{H(\omega) = A(\omega)\mathrm{e}^{\mathrm{j}\Phi(\omega)}} \longrightarrow y(n)$$

$$x(n) = \sin(\omega_0 n)$$

We know

$$y(n) = A(\omega_0)\sin(\omega_0 n + \Phi(\omega_0))$$

EITF75 Systems and Signals

Group and phase delay

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$$x(n) = \sin(\omega_0 n)$$

We know

$$y(n) = A(\omega_0)\sin(\omega_0 n + \Phi(\omega_0))$$

$$= A(\omega_0)\sin\left(\omega_0\left(n + \frac{\Phi(\omega_0)}{\omega_0}\right)\right)$$

Manipulation

Group and phase delay

$$x(n) \longrightarrow \boxed{H(\omega) = A(\omega)\mathrm{e}^{\mathrm{j}\Phi(\omega)}} \longrightarrow y(n)$$

$$x(n) = \sin(\omega_0 n)$$

We know

$$y(n) = A(\omega_0)\sin(\omega_0 n + \Phi(\omega_0))$$

$$= A(\omega_0)\sin\left(\omega_0\left(n + \frac{\Phi(\omega_0)}{\omega_0}\right)\right)$$

Time-Delay

Tolerable if equal for all w_0

EITF75 Systems and Signals

Group and phase delay

$$x(n) \longrightarrow \boxed{H(\omega) = A(\omega)e^{j\Phi(\omega)}} \longrightarrow y(n)$$

$$x(n) = \sin(\omega_0 n)$$

We know

$$y(n) = A(\omega_0)\sin(\omega_0 n + \Phi(\omega_0))$$
$$= A(\omega_0)\sin\left(\omega_0\left(n + \frac{\Phi(\omega_0)}{\omega_0}\right)\right)$$

$\frac{\Phi(\omega_0)}{\omega_0}$ equal for all w_0 if $\Phi(\omega_0)$ is a straight line

EITF75 Systems and Signals

Group and phase delay

$$x(n) \longrightarrow \boxed{H(\omega) = A(\omega)e^{j\Phi(\omega)}} \longrightarrow y(n)$$

$\Phi(\omega_0) = C\ \omega_0$ **Linear phase filter**

$$y(n) = A(\omega_0)\sin(\omega_0 n + \Phi(\omega_0))$$

$$= A(\omega_0)\sin\left(\omega_0\left(n + \frac{\Phi(\omega_0)}{\omega_0}\right)\right)$$

$\dfrac{\Phi(\omega_0)}{\omega_0}$ **equal for all w_0 if $\Phi(\omega_0)$ is a straight line**

EITF75 Systems and Signals

Group and phase delay

$$h(n) = \left\{ \begin{array}{ccc} 1 & 2 & 1 \end{array} \right\}$$

Linear phase filter ?

$\Phi(\omega_0) = C\ \omega_0$ **Linear phase filter**

EITF75 Systems and Signals

Group and phase delay

$$h(n) = \left\{ \begin{matrix} 1 & 2 & 1 \end{matrix} \right\}$$

Linear phase filter ?

$$H(\omega) = 1 + 2e^{-j\omega} + e^{-j2\omega}$$

$\Phi(\omega_0) = C\,\omega_0$ **Linear phase filter**

EITF75 Systems and Signals

Group and phase delay

$$h(n) = \{\ 1 \quad 2 \quad 1\ \}$$

Linear phase filter ?

$$H(\omega) = 1 + 2e^{-j\omega} + e^{-j2\omega}$$

$$= e^{-j\omega} \cdot \left(e^{j\omega} + 2 + e^{-j\omega}\right)$$

$\Phi(\omega_0) = C\ \omega_0$ Linear phase filter

EITF75 Systems and Signals

Group and phase delay

$$h(n) = \{\ 1 \quad 2 \quad 1\ \}$$

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$$= e^{-j\omega} \cdot \left(e^{j\omega} + 2 + e^{-j\omega}\right)$$

$$= e^{-j\omega} \cdot (2 + 2\cos(\omega))$$

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EITF75 Systems and Signals

Group and phase delay

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$$= e^{-j\omega} \cdot (2 + 2\cos(\omega))$$

$$= A(\omega) \cdot e^{j\Phi(\omega)}$$

$\Phi(\omega_0) = C\,\omega_0$ **Linear phase filter**

EITF75 Systems and Signals

Group and phase delay

$$h(n) = \{\ 1 \quad 2 \quad 1\ \}$$

Linear phase filter ?

$$H(\omega) = 1 + 2e^{-j\omega} + e^{-j2\omega}$$

$$= e^{-j\omega} \cdot \left(e^{j\omega} + 2 + e^{-j\omega}\right)$$

$$= e^{-j\omega} \cdot (2 + 2\cos(\omega))$$

Non-negative.
NOT ENOUGH IF REAL-VALUED

$$= A(\omega) \cdot e^{j\Phi(\omega)}$$

$\Phi(\omega_0) = C\ \omega_0$ Linear phase filter

EITF75 Systems and Signals

Group and phase delay

$$h(n) = \{\ 1 \quad 2 \quad 1\ \}$$

Linear phase filter ? YES.

$$H(\omega) = 1 + 2e^{-j\omega} + e^{-j2\omega}$$

$$= e^{-j\omega} \cdot \left(e^{j\omega} + 2 + e^{-j\omega}\right)$$

$$= e^{-j\omega} \cdot (2 + 2\cos(\omega))$$

$$= A(\omega) \cdot e^{j\Phi(\omega)}$$

$$\Phi(\omega) = -\omega$$

$\Phi(\omega_0) = C\ \omega_0$ Linear phase filter

EITF75 Systems and Signals

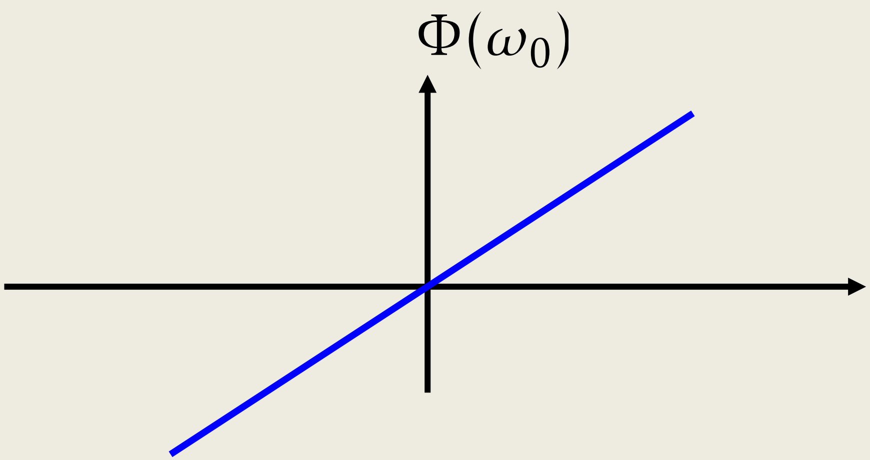
Group and phase delay

Definition $$\tau_g = -\frac{\mathrm{d}\Phi(\omega)}{\mathrm{d}\omega}$$

Not the same. Book is highly unclear about what this is

Group and phase delay

Definition $$\tau_g = -\frac{\mathrm{d}\Phi(\omega)}{\mathrm{d}\omega}$$



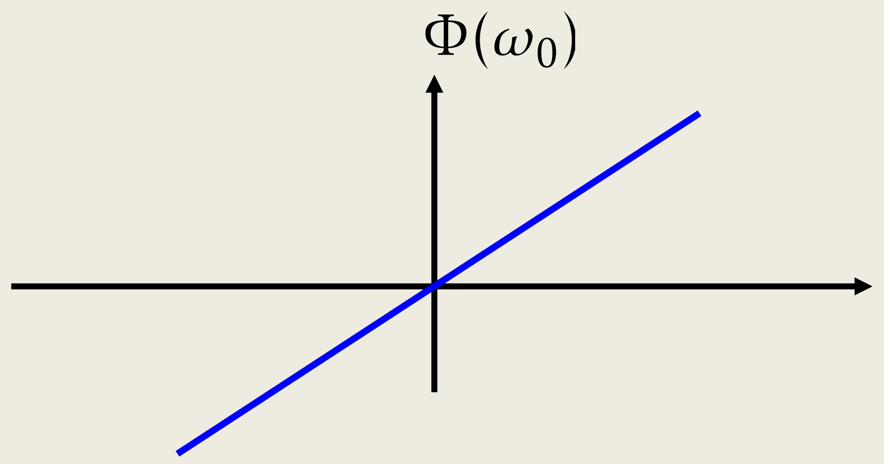
$\frac{\Phi(\omega_0)}{\omega_0}$ **Constant**

All w_0 have same delay

EITF75 Systems and Signals

Group and phase delay

Definition $$\tau_g = -\frac{d\Phi(\omega)}{d\omega}$$ = constant



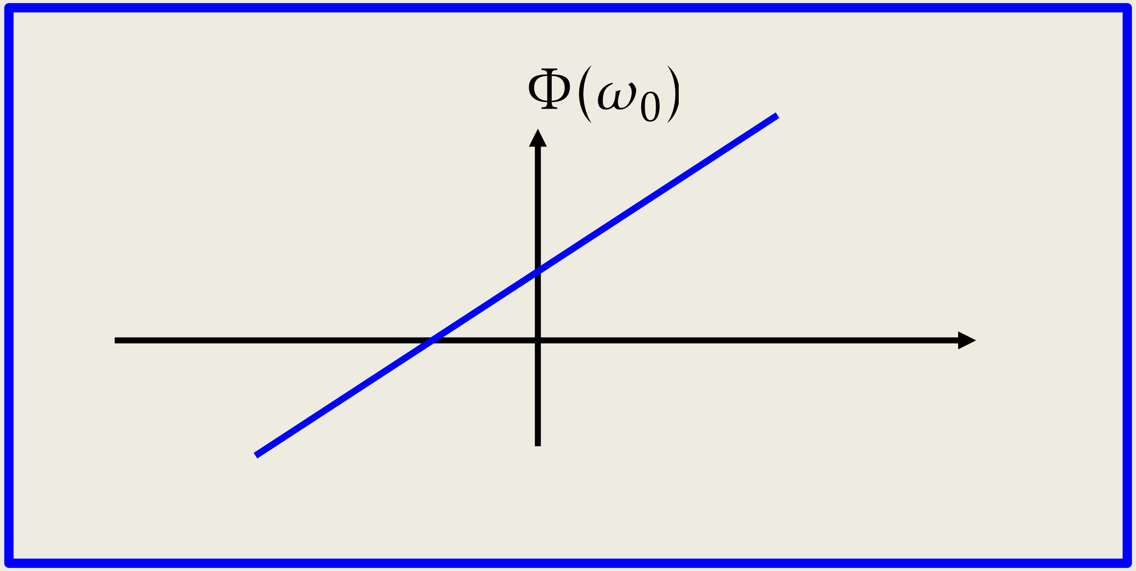
$$\frac{\Phi(\omega_0)}{\omega_0}$$ Constant

All w_0 have same delay

EITF75 Systems and Signals

Group and phase delay

Definition $$\tau_g = -\frac{\mathrm{d}\Phi(\omega)}{\mathrm{d}\omega}$$ = constant



$$\frac{\Phi(\omega_0)}{\omega_0}$$ Not Constant

All w_0 don't have same delay

EITF75 Systems and Signals

Group and phase delay

Definition $$\tau_g = -\frac{\mathrm{d}\Phi(\omega)}{\mathrm{d}\omega}$$

What is group delay? Assume x(n) = A(n) sin(wn)

EITF75 Systems and Signals

Group and phase delay

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What is group delay? Assume x(n) = A(n) sin(wn)

sin(wn) acts as a carrier frequency

A(t) is a data signal. Assumed that A(n) changes slowly compared with sin(wn)

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Group and phase delay

Definition $$\tau_g = -\frac{\mathrm{d}\Phi(\omega)}{\mathrm{d}\omega}$$

What is group delay? Assume x(n) = A(n) sin(wn)

sin(wn) acts as a carrier frequency

A(t) is a data signal. Assumed that A(n) changes slowly compared with sin(wn)

Output is $y(n) = |H(w)| A(n-\tau_g) \sin(w(n-\tau_p))$

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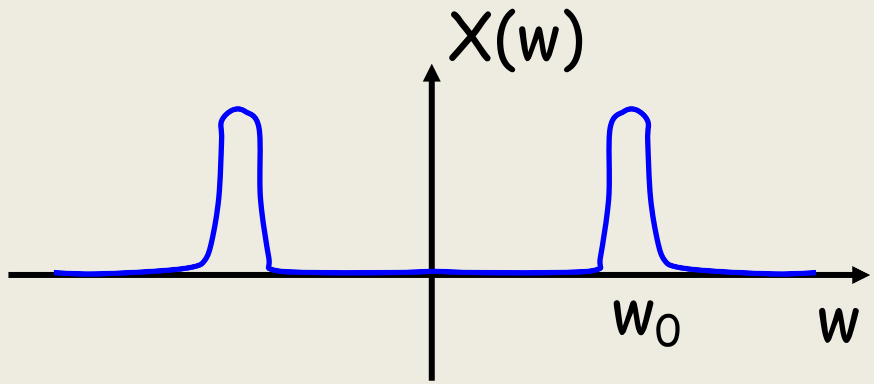
Group and phase delay

Applies to these kind of situations



Output is $y(n) = |H(w)| A(n-\tau_g) \sin(w(n-\tau_p))$

Group delay is NOT in the core of the course