EITF75 Systems and Signals

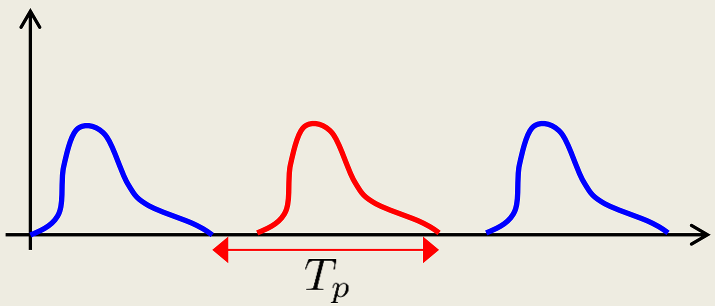
Lecture 5
The discrete-time Fourier transform

Fredrik Rusek

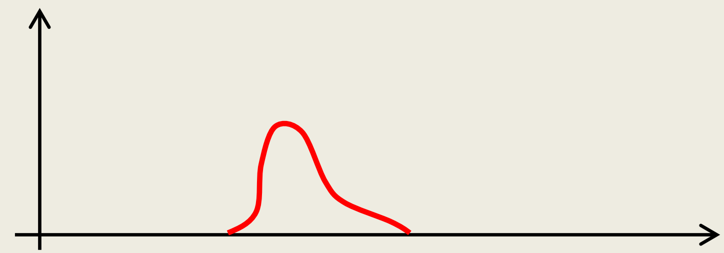
EITF75, Fourier transforms

4 different type of signals

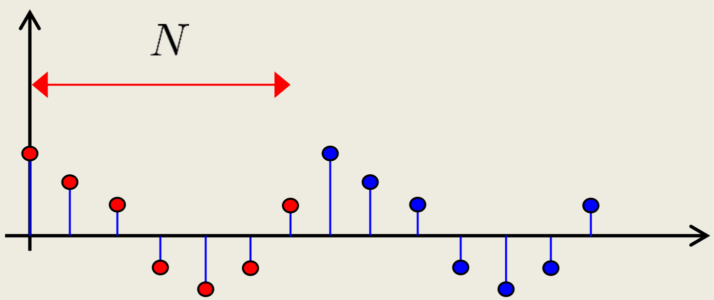
Continuous and periodic



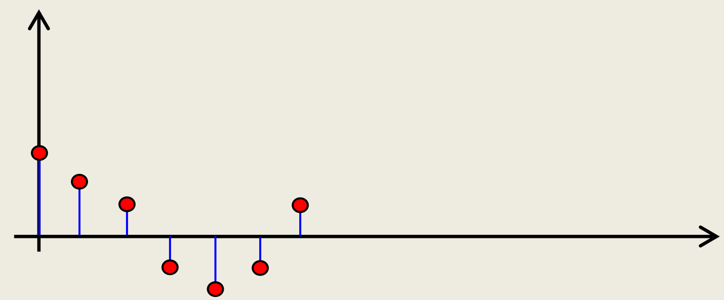
Continuous and aperiodic



Discrete and periodic



Discrete and aperiodic

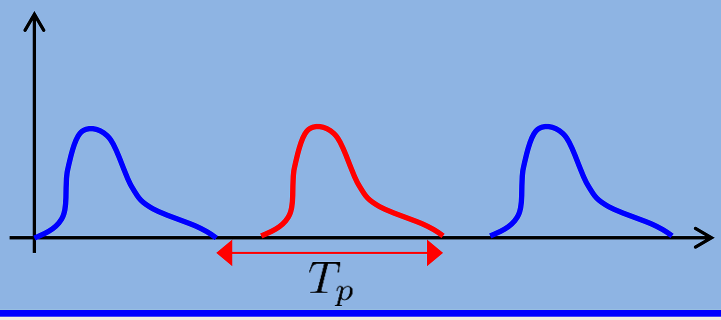


EITF75, Fourier transforms

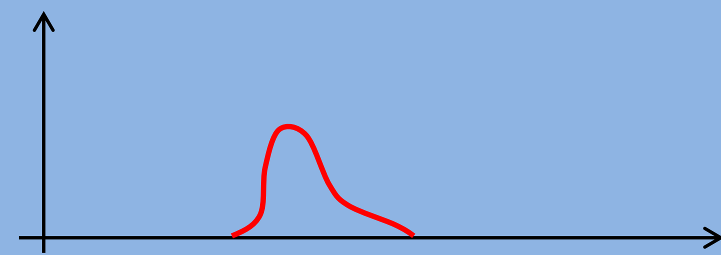
4 different type of signals

Studied before (funktionsteori, system o transformer)

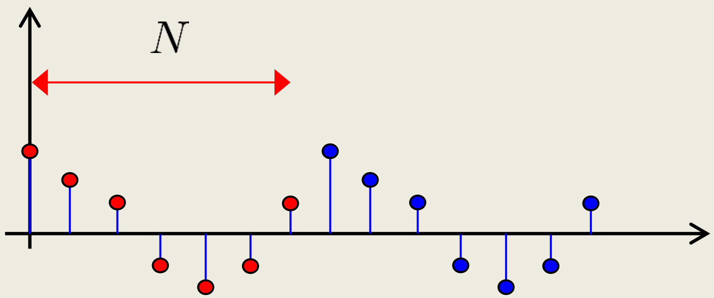
Continuous and periodic



Continuous and aperiodic

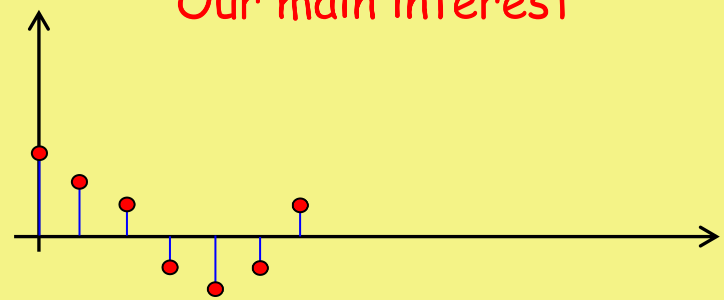


Discrete and periodic



Discrete and aperiodic

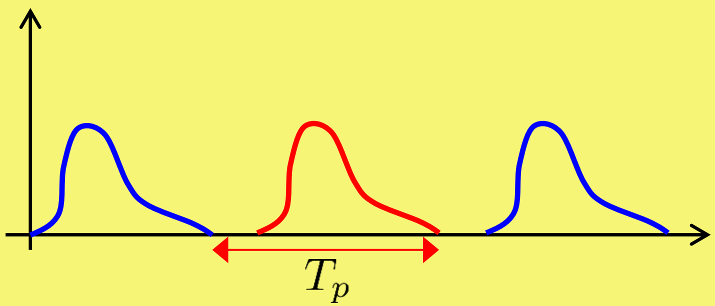
Our main interest



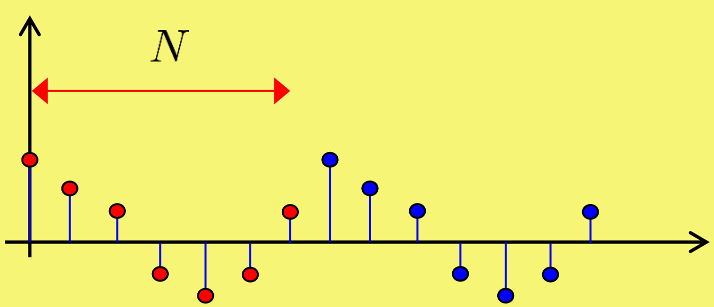
EITF75, Fourier transforms

4 different type of signals

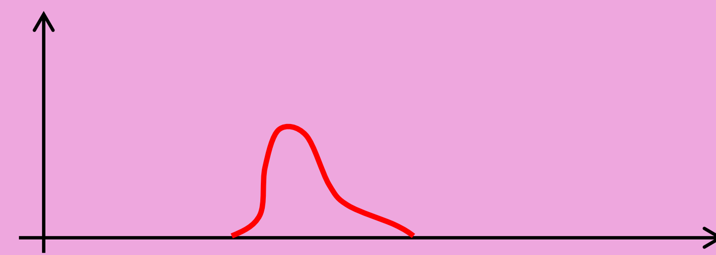
Continuous and periodic



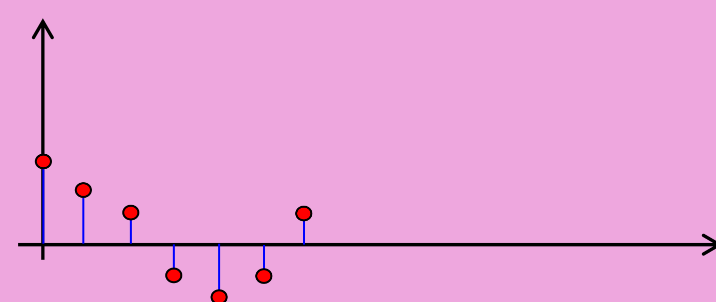
Discrete and periodic



Continuous and aperiodic



Discrete and aperiodic



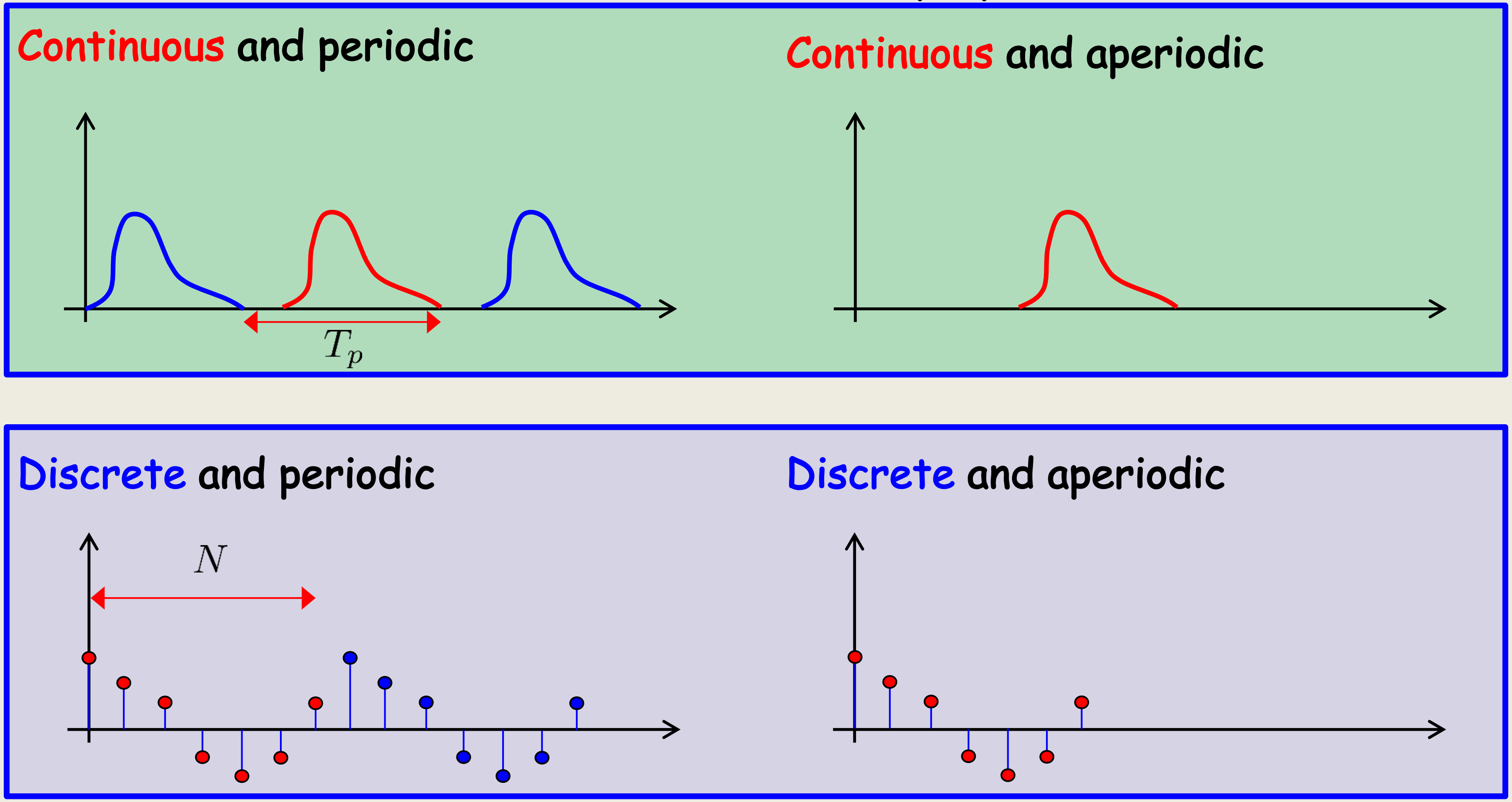
Some common properties

Some common properties

EITF75, Fourier transforms

4 different type of signals

Some common properties



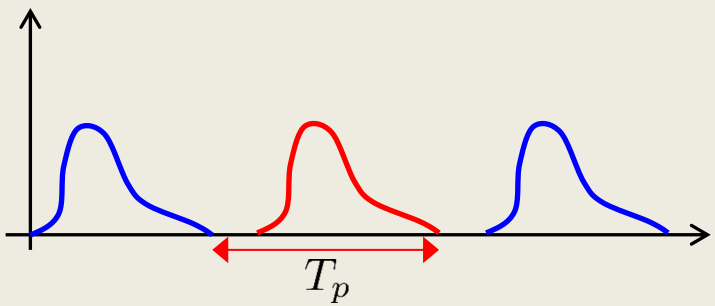
Some common properties

EITF75, Fourier transforms

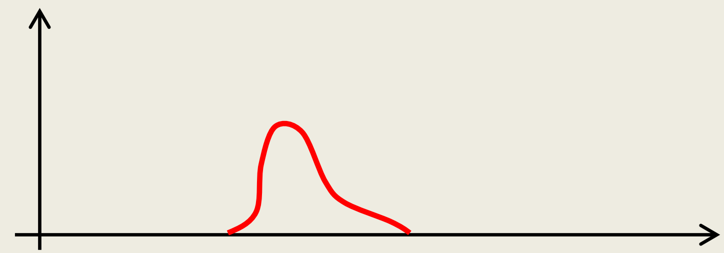
4 different type of signals

How to get aperiodic transforms:

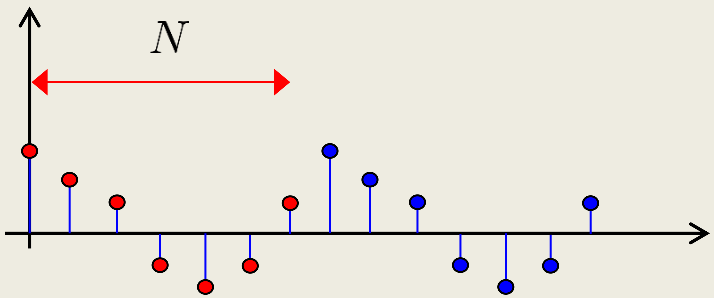
Continuous and periodic



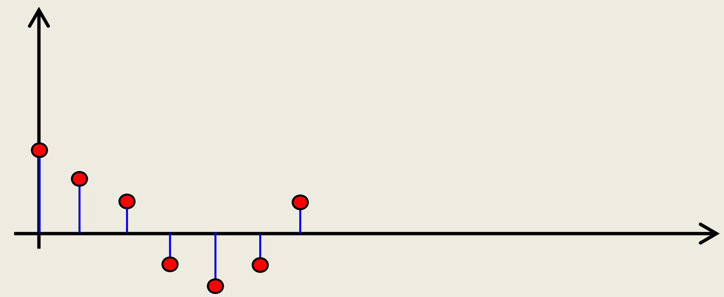
Continuous and aperiodic



Discrete and periodic



Discrete and aperiodic

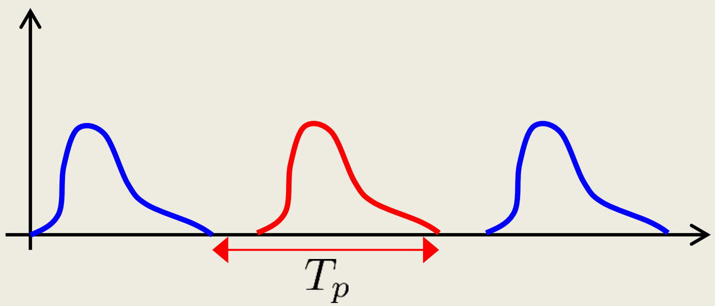


EITF75, Fourier transforms

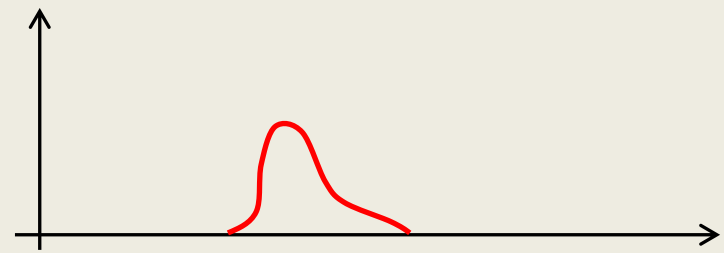
4 different type of signals

How to get aperiodic transforms:

Continuous and periodic

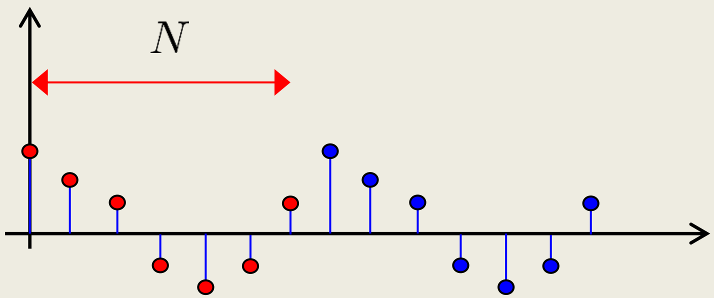


Continuous and aperiodic

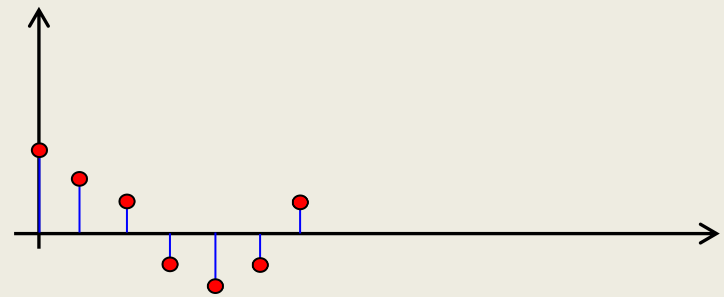


1. Obtain periodic transforms

Discrete and periodic



Discrete and aperiodic

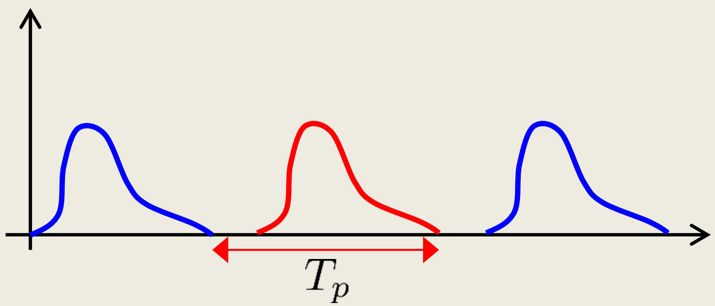


EITF75, Fourier transforms

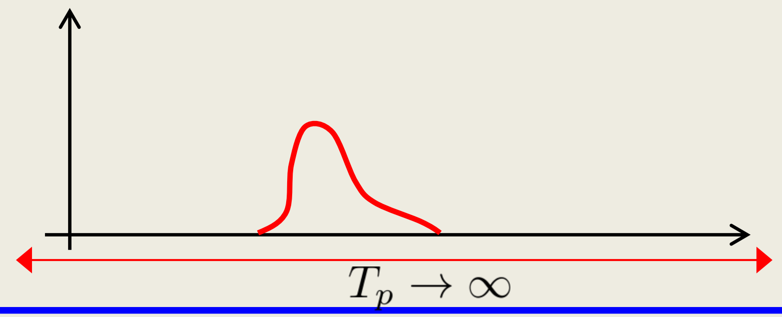
4 different type of signals

How to get aperiodic transforms:

Continuous and periodic



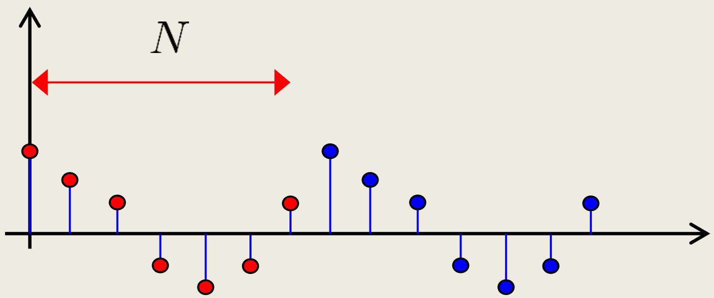
Continuous and aperiodic



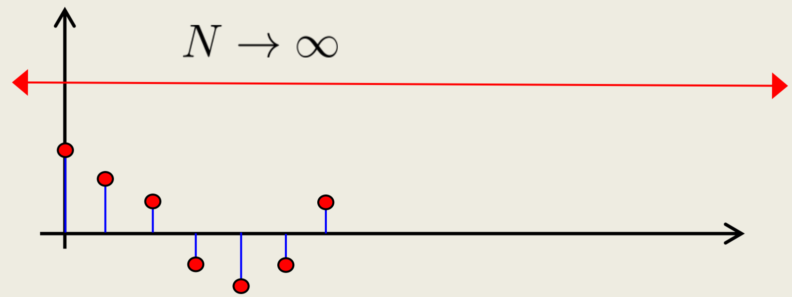
1. Obtain periodic transforms

2. Send period to infinity

Discrete and periodic



Discrete and aperiodic



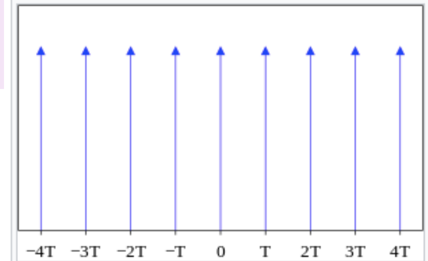
EITF75, Fourier transforms

WIKIPEDIA
The Free Encyclopedia

Dirac comb

$$\text{III}_T(t) \triangleq \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi n \frac{t}{T}}$$



A Dirac comb is an infinite series of Dirac delta functions spaced at intervals of *T*

$$\int_0^T x(\tau) \frac{1}{T} \sum_{k=-\infty}^{\infty} \mathrm{e}^{i2\pi k \frac{t-\tau}{T}} \mathrm{d}\tau = \int_0^T x(\tau) \sum_{k=-\infty}^{\infty} \delta(t - \tau - kT) \mathrm{d}\tau$$

EITF75, Fourier transforms

$$T_p \sum_{k=-\infty}^{\infty} |c_k|^2 = T_p \sum_{k=-\infty}^{\infty} \left| \frac{1}{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) \mathrm{d}t \right|^2$$

$$= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \int_0^{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) x^*(\tau) \exp(i2\pi k F_0 \tau) \mathrm{d}t \mathrm{d}\tau$$

$$= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \int_0^{T_p} \int_0^{T_p} x(t) x^*(\tau) \exp(i2\pi k F_0 (\tau - t)) \mathrm{d}t \mathrm{d}\tau$$

$$= \int_0^{T_p} \int_0^{T_p} x(t) x^*(\tau) Ш_{T_p}(\tau - t) \mathrm{d}t \mathrm{d}\tau = \int_0^{T_p} |x(t)|^2 \mathrm{d}t$$

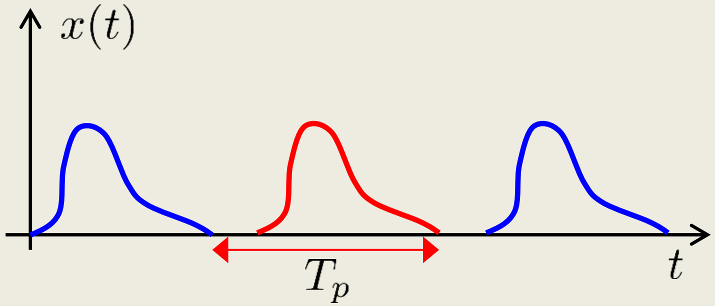
Parseval's formula

EITF75, Fourier transforms

Summary

Continuous and periodic

$$F_0 = \frac{1}{T_p}$$



Fourier series representation

Analysis equation

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t$$

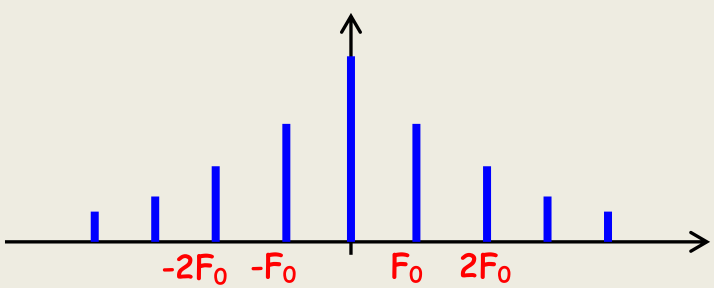
Synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

Parseval's identity

$$T_p \sum_{k=-\infty}^{\infty} |c_k|^2 = \int_0^{T_p} |x(t)|^2 \mathrm{d}t$$

Power spectrum



EITF75, Fourier transforms

4 different type of signals

Fourier transform

$$X(F) = \int_{-\infty}^{\infty} x(t) \exp(-i2\pi F t)\mathrm{d}t$$

Parseval's identity

$$\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |x(F)|^2 \mathrm{d}F$$

Energy in time, must be present in frequency as well

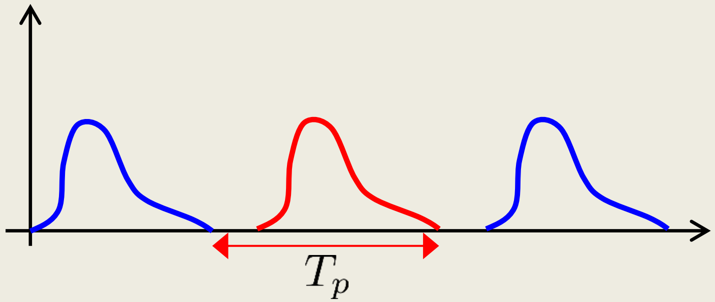
Inverse Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(F) \exp(i2\pi F t)\mathrm{d}F$$

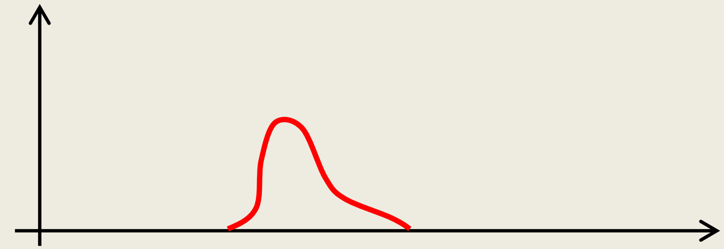
EITF75, Fourier transforms

So far, we did these two

Continuous and periodic



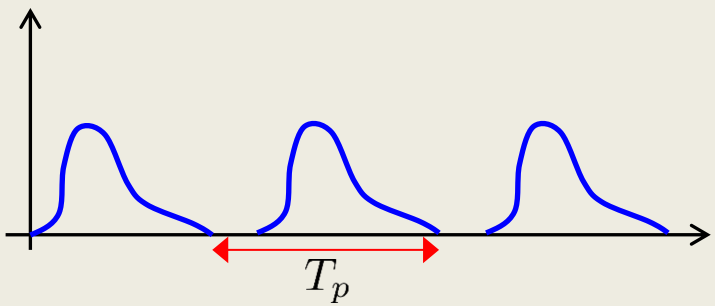
Continuous and aperiodic



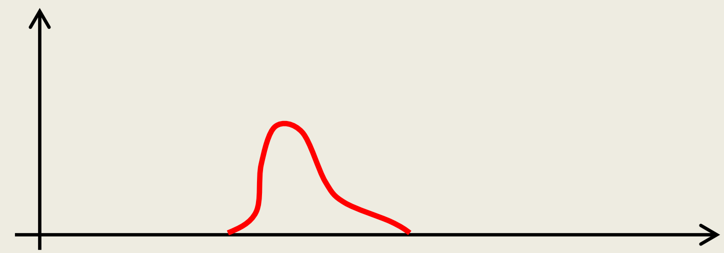
EITF75, Fourier transforms

So far, we did these two

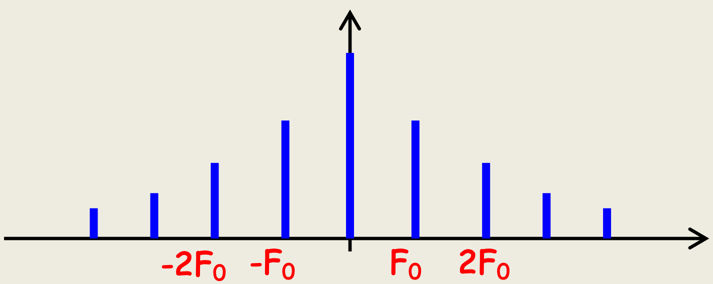
Continuous and periodic



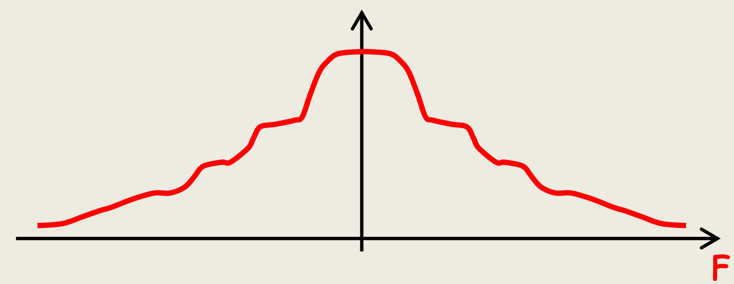
Continuous and aperiodic



Power spectrum

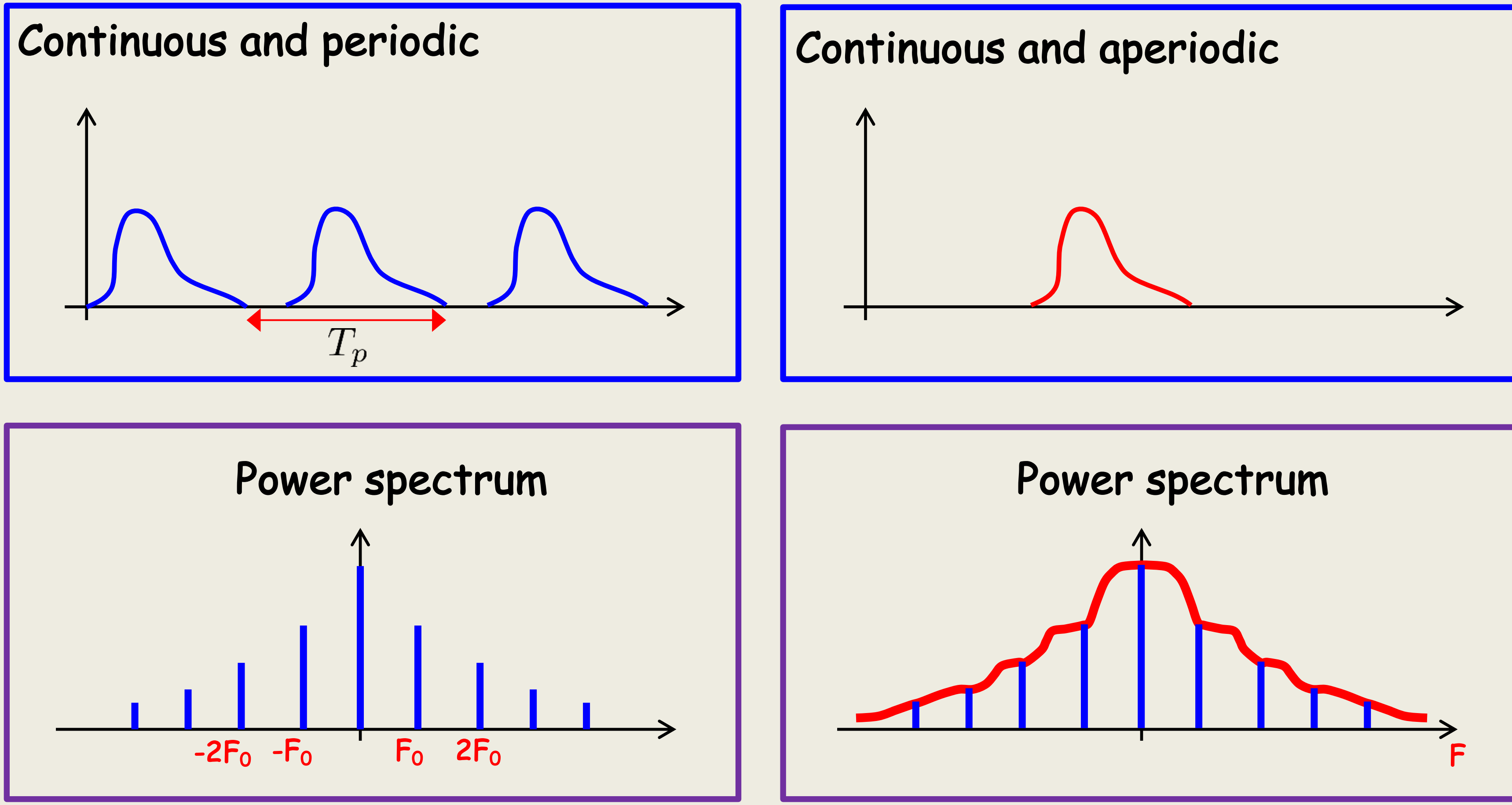


Power spectrum



EITF75, Fourier transforms

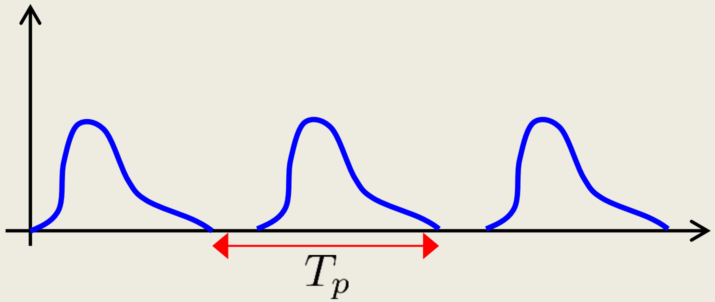
Question: Is this true? (The left spectrum is samples from the right)



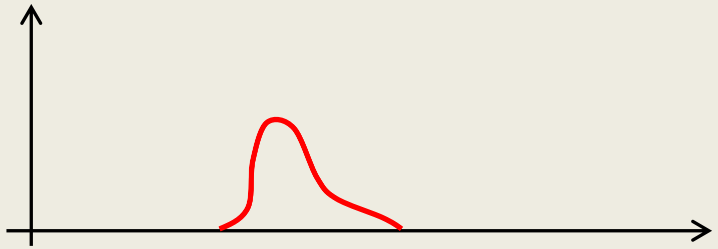
EITF75, Fourier transforms

Yes and no. Recall $\frac{1}{T_p} X(kF_0) = c_k$

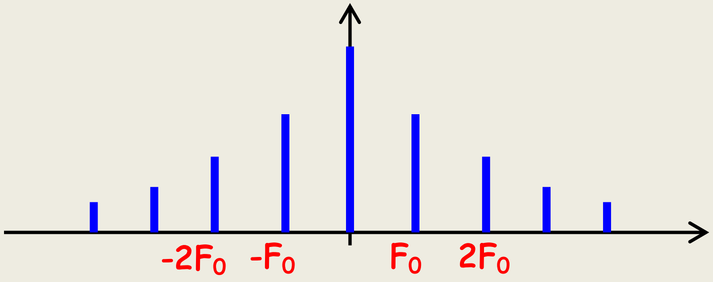
Continuous and periodic



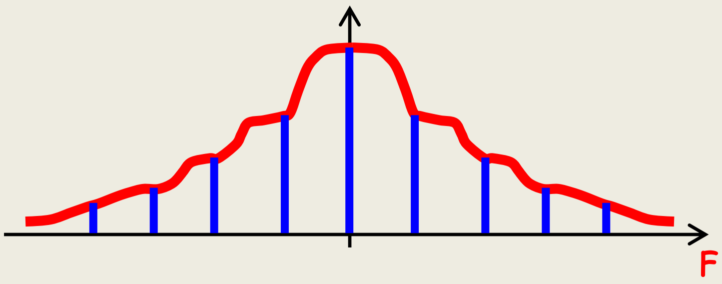
Continuous and aperiodic



Power spectrum



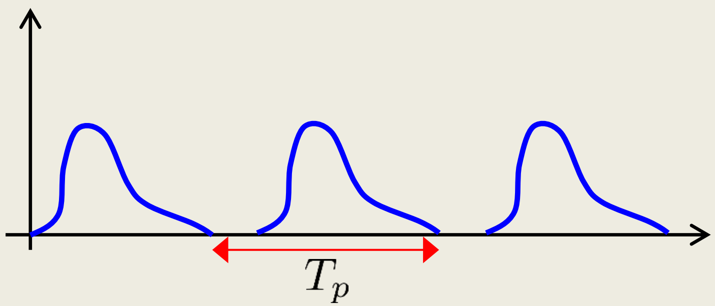
Power spectrum



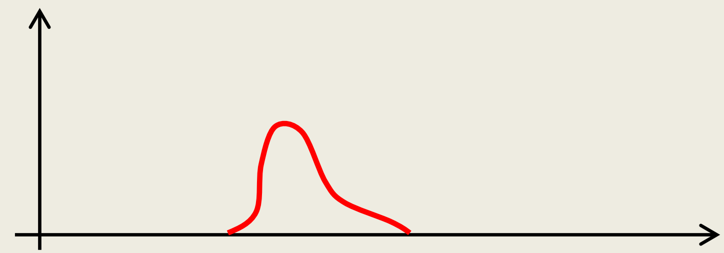
EITF75, Fourier transforms

Yes and no. Recall $\frac{1}{T_p} X(kF_0) = c_k$

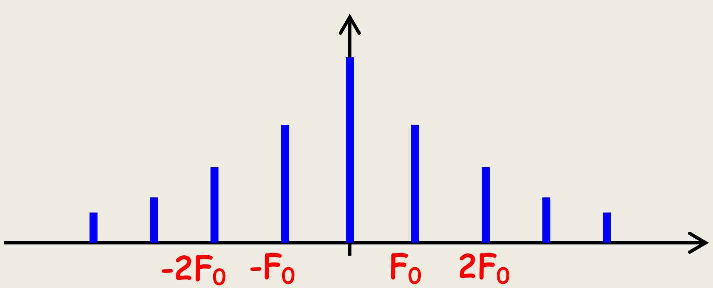
Continuous and periodic



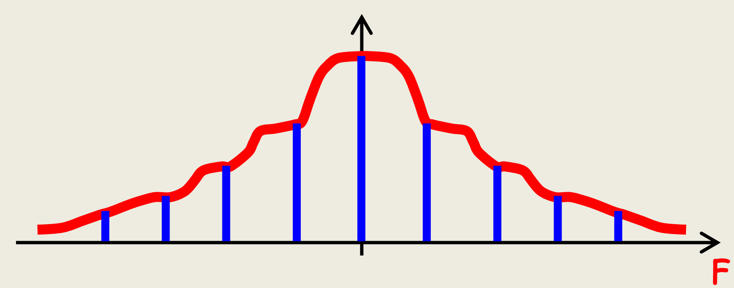
Continuous and aperiodic



Power spectrum



Power spectrum

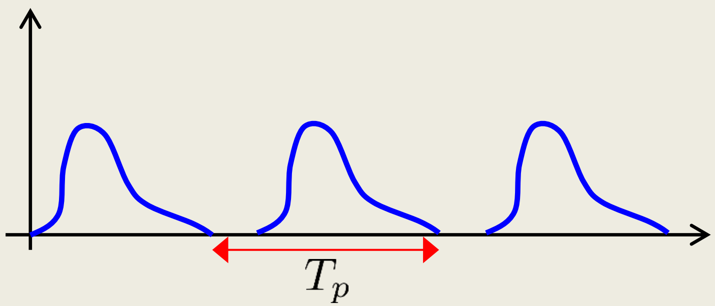


True if T_p=1

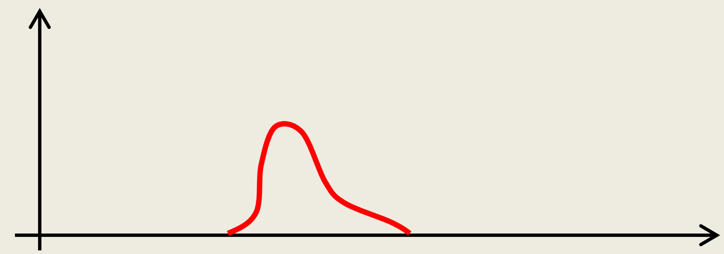
EITF75, Fourier transforms

Yes and no. Recall $\frac{1}{T_p} X(kF_0) = c_k$

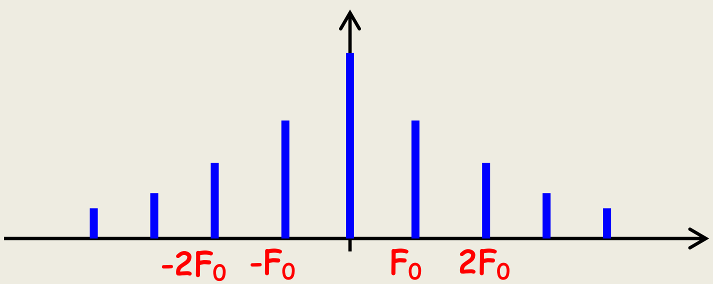
Continuous and periodic



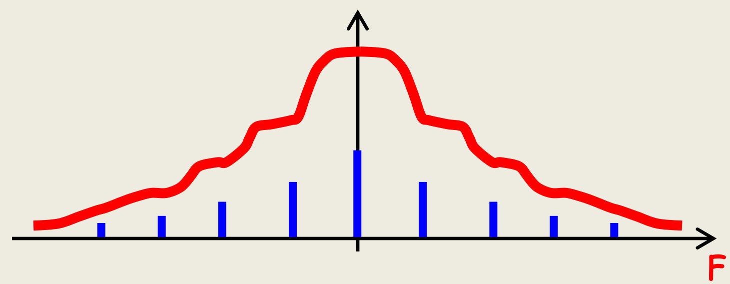
Continuous and aperiodic



Power spectrum



Power spectrum

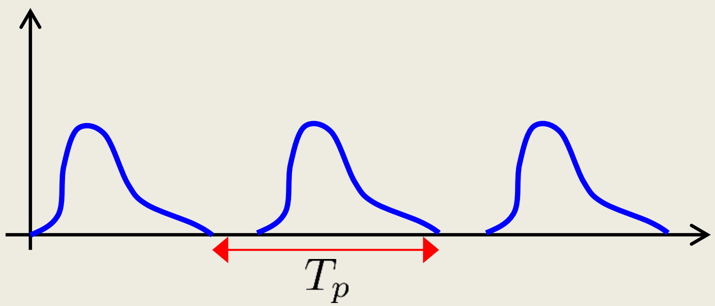


True if T_p=2 ???

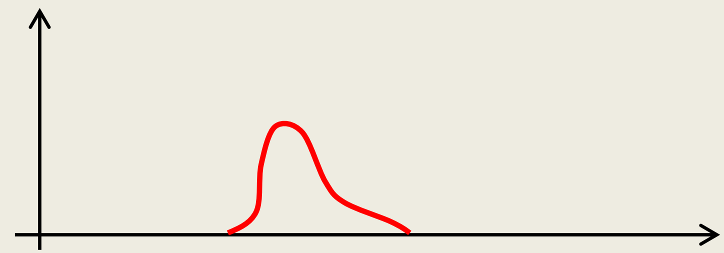
EITF75, Fourier transforms

Yes and no. Recall $\frac{1}{T_p} X(kF_0) = c_k$

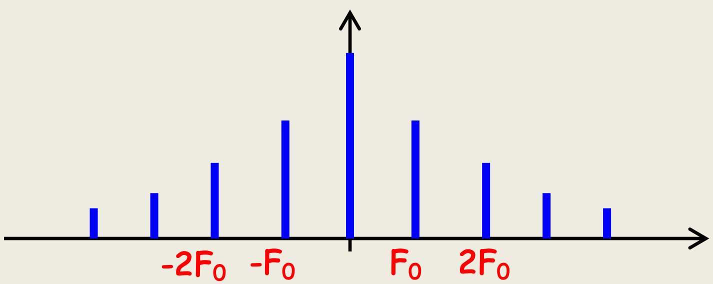
Continuous and periodic



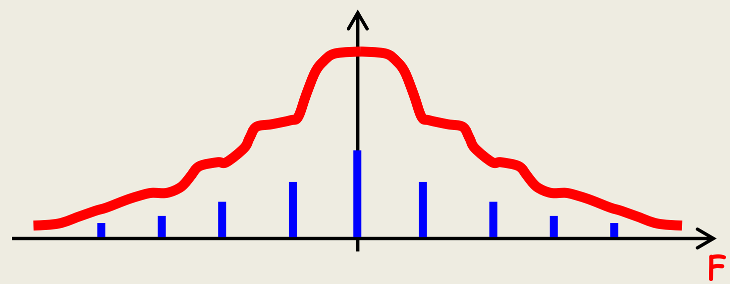
Continuous and aperiodic



Power spectrum



Power spectrum

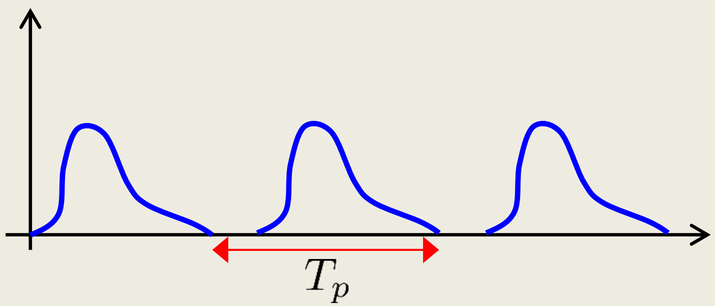


True if T_p=2 ??? NO

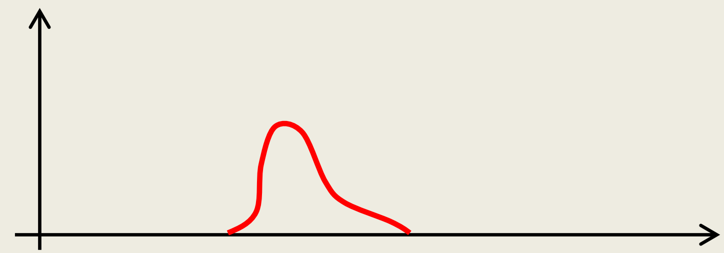
EITF75, Fourier transforms

Yes and no. Recall $\frac{1}{T_p} X(kF_0) = c_k$

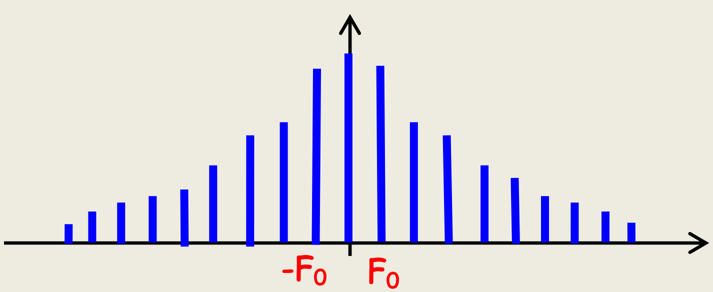
Continuous and periodic



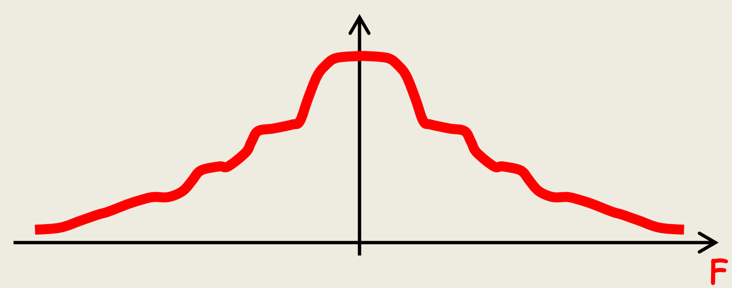
Continuous and aperiodic



Power spectrum



Power spectrum

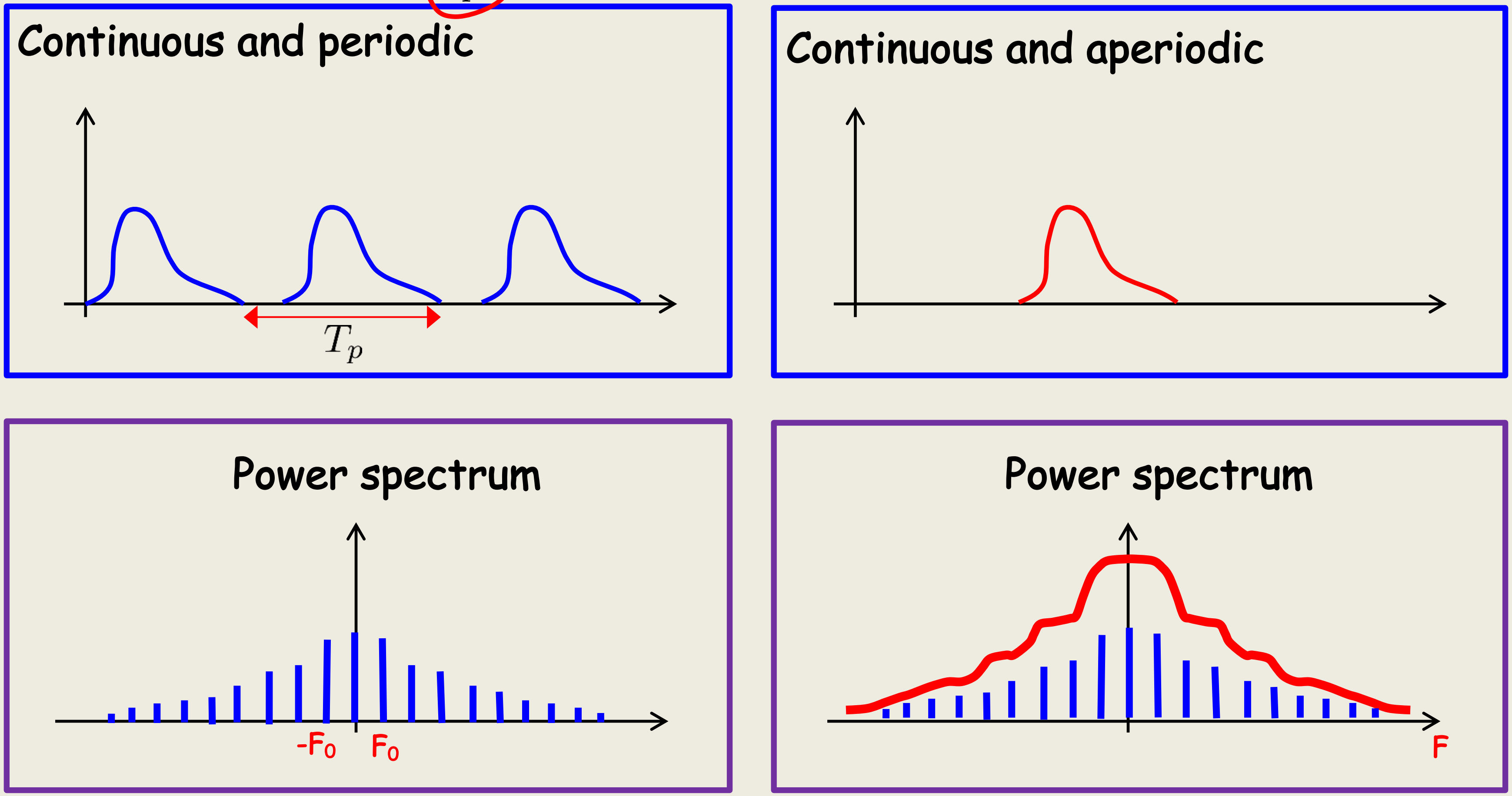


Effect 1: Denser sampling

$T_p=2$

EITF75, Fourier transforms

Yes and no. Recall $\frac{1}{T_p} X(kF_0) = c_k$



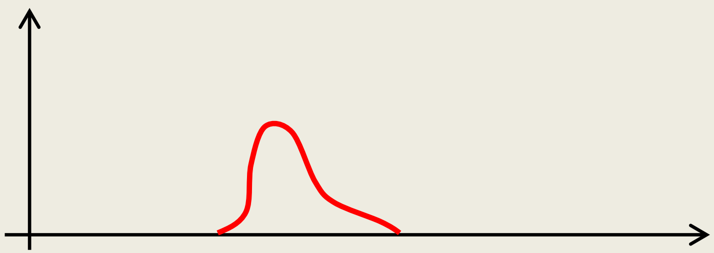
Effect 2: Scaled amplitude

T_p=2

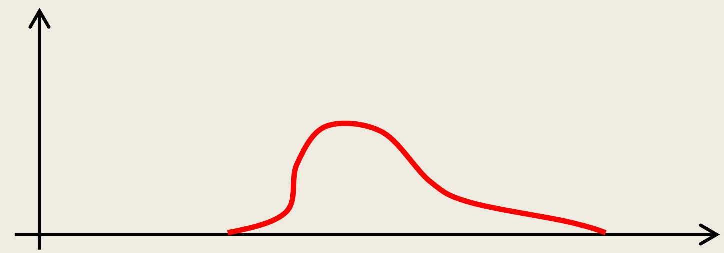
EITF75, Fourier transforms

Fundamental engineering knowledge

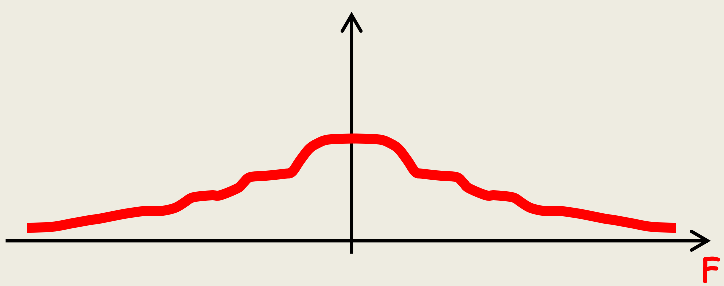
Continuous and aperiodic



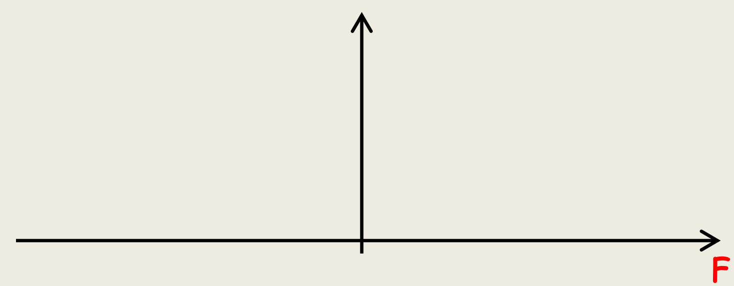
Same shape, twice the length



Power spectrum



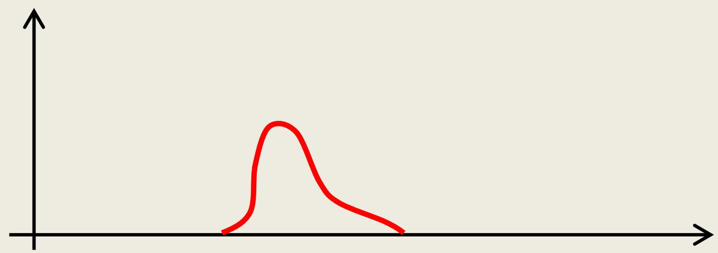
Power spectrum ?



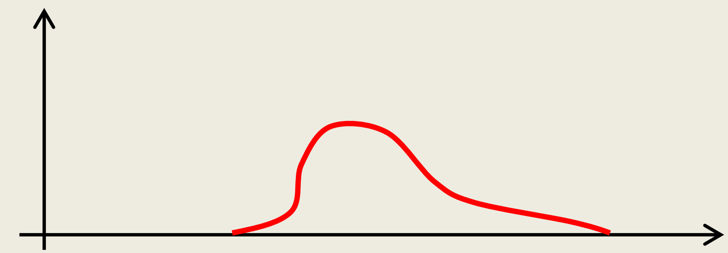
EITF75, Fourier transforms

Fundamental engineering knowledge

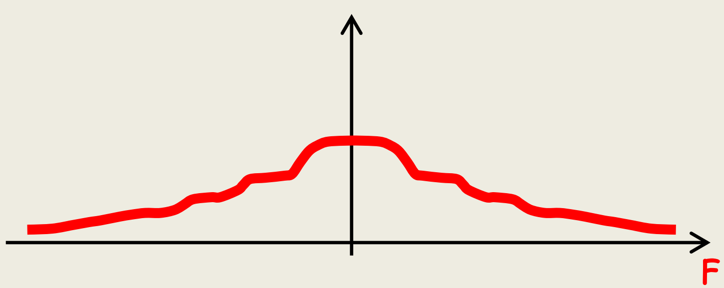
Continuous and aperiodic



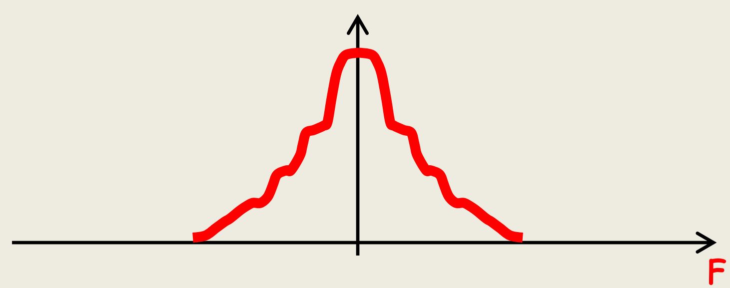
Same shape, twice the length



Power spectrum



Power spectrum ?

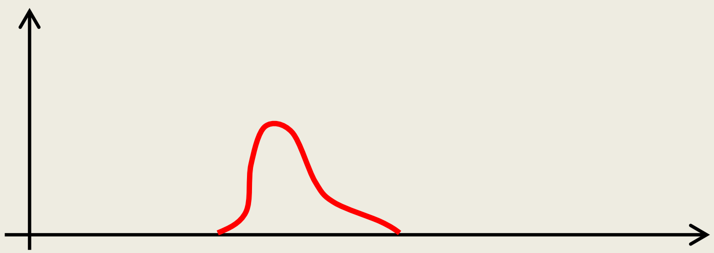


Strect in time is compression in frequency (and vica versa)

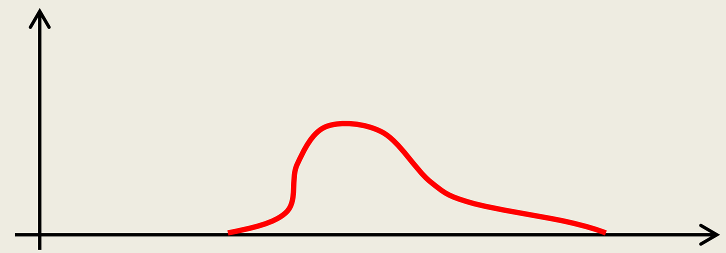
EITF75, Fourier transforms

Fundamental engineering knowledge

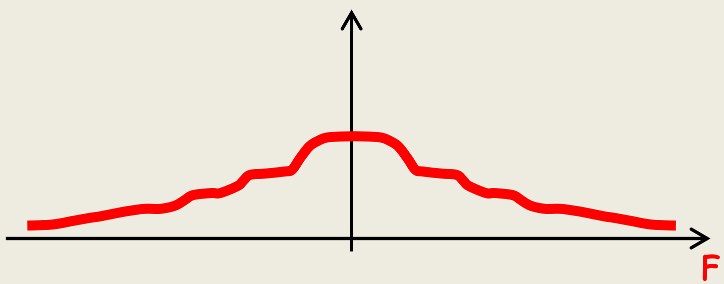
Continuous and aperiodic



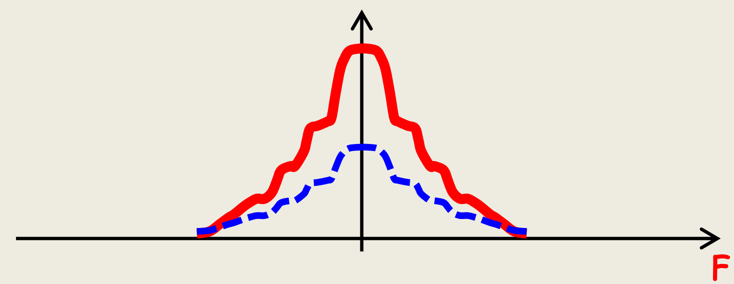
Same shape, twice the length



Power spectrum

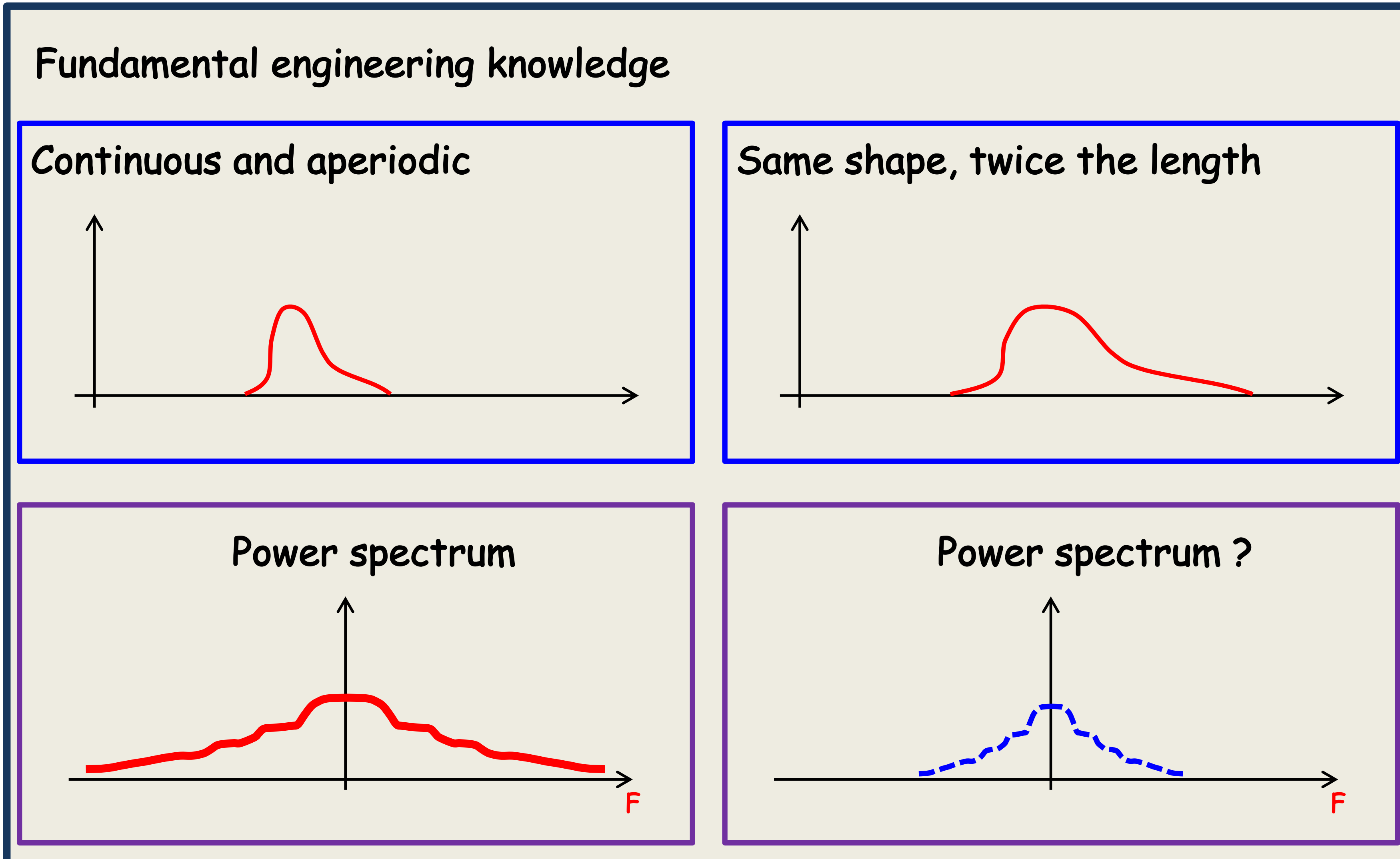


Power spectrum ?



Simplest way to understand the amplitude change?

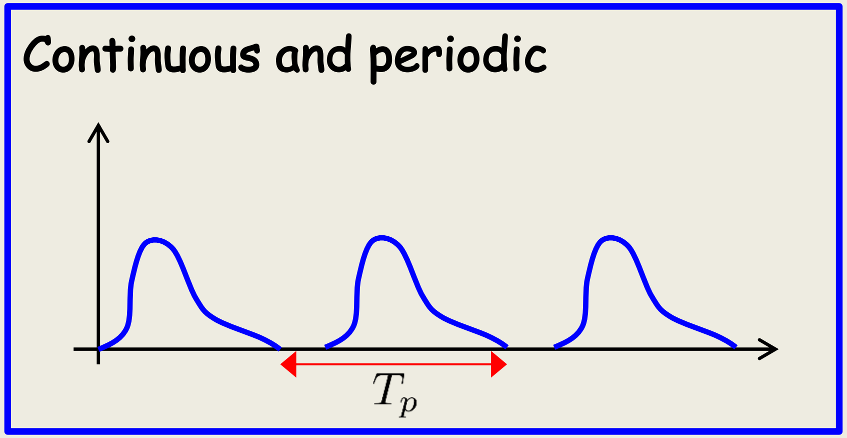
EITF75, Fourier transforms

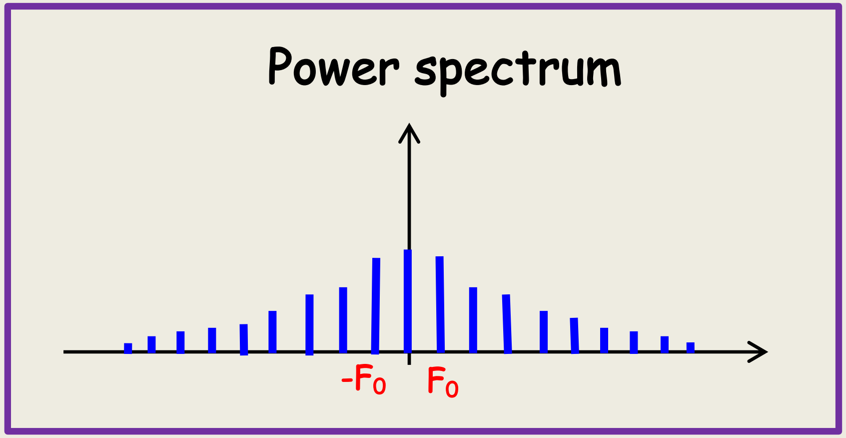


Simplest way to understand the amplitude change?

Parseval's identity: No way the integral-of-the-right-plot-squared equals the integral-of-the-left-plot-squared

EITF75, Fourier transforms





Why is the spectrum discrete?

We can write the signal as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

However, $\exp(i2\pi k F_0 t)$

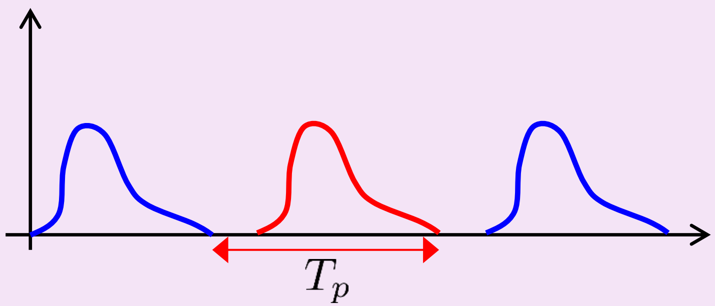
is not periodic with period Tp unless k is an integer.

Thus, there can be no non-integer components in the spectrum

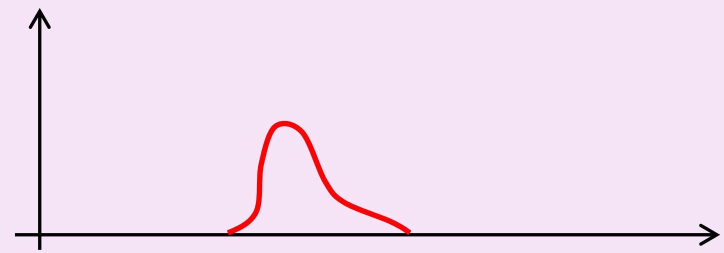
EITF75, Fourier transforms

4 different type of signals

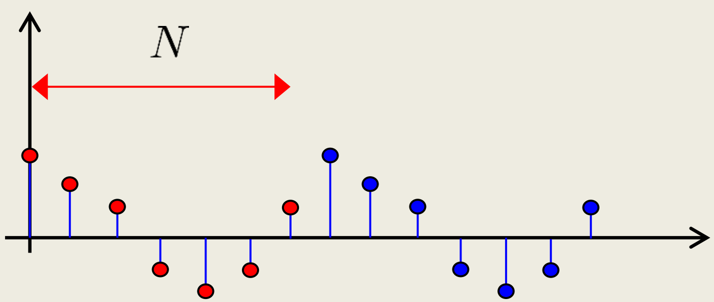
Continuous and periodic — Done



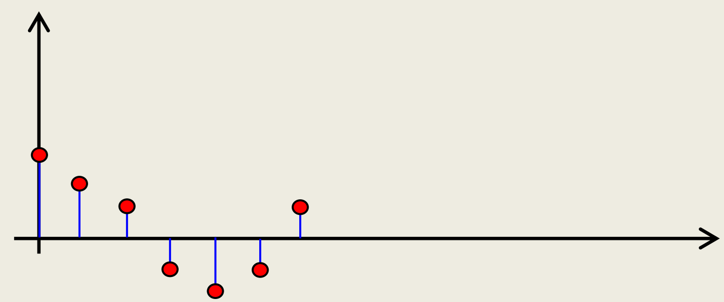
Continuous and aperiodic — Done



Discrete and periodic — Next case



Discrete and aperiodic



EITF75, Fourier transforms

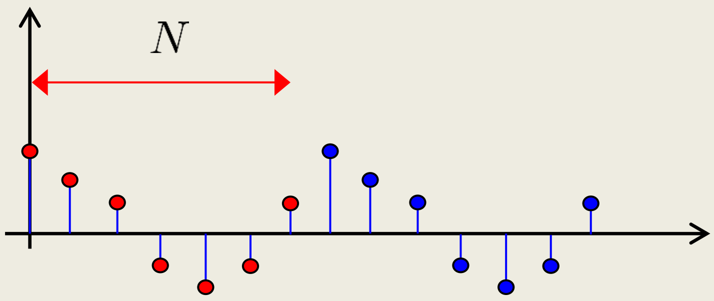
Recall what our goal is right now:

Given a periodic x(n), find either (we dont know which one yet)

- A set of coefficients c_k representing x(n)
- A continuous function X(F)

Discrete and periodic

Next case



EITF75, Fourier transforms

Recall what our goal is right now:

Given a periodic x(n), find either (we dont know which one yet)

- A set of N coefficients c_k representing x(n)

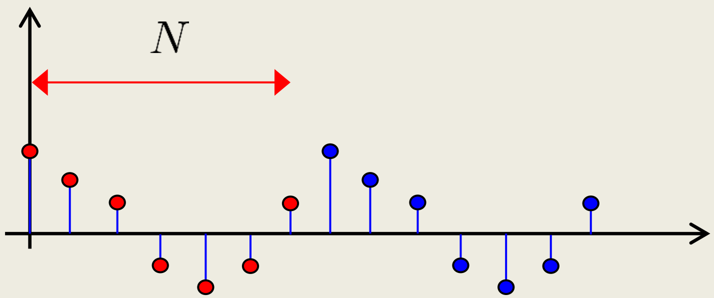
  Likely this should be the result

Don't forget (Lecture 1): For discrete signals, there is no difference between normalized frequency f=0.4 and f=…-0.6,1.4, 2.4,…

Therefore: Likely that the coefficients c_k are periodically extended.

Discrete and periodic

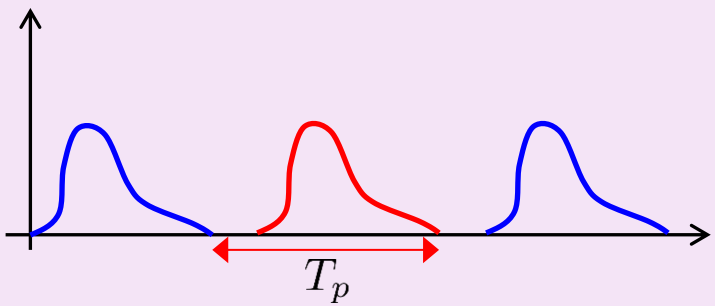
Next case



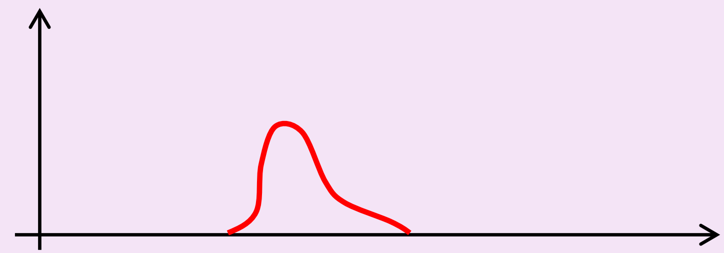
EITF75, Fourier transforms

4 different type of signals

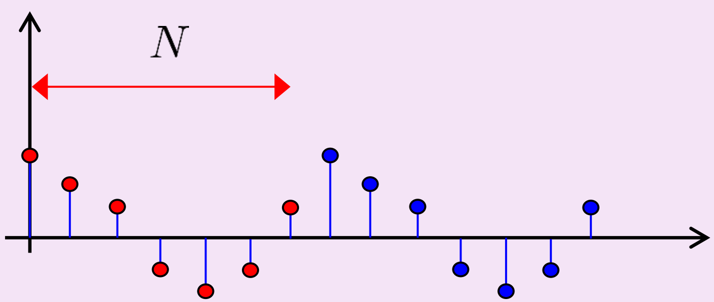
Continuous and periodic — Done



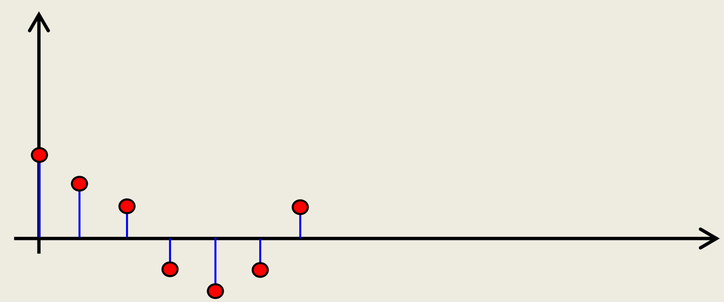
Continuous and aperiodic — Done



Discrete and periodic — Done



Discrete and aperiodic — Next case



EITF75, Fourier transforms

Convergence

Consider

$$X(f) = \sum_{n=0}^{N-1} x(n) \exp(-i2\pi n f)$$

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \exp(-i2\pi nk/N)$$

$$\hat{X}(f) = \sum_{n=0}^{N-1} c_n \exp(i2\pi n f)$$

What can we say about this one?

$$x(n) = \frac{1}{n+1} u(n)$$

Uniform convergence

$$\hat{X}(f) = X(f)$$

if, absolutely summable

$$\sum |x(n)| < \infty$$

Mean square sense convergence

$$\int_{-0.5}^{0.5} |\hat{X}(f) - X(f)|^2 \mathrm{d}f = 0$$

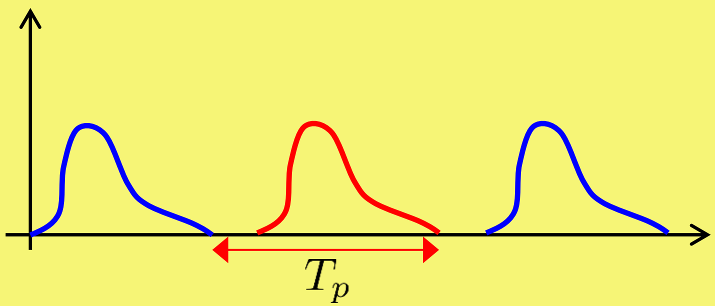
if, square summable

$$\sum |x(n)|^2 < \infty$$

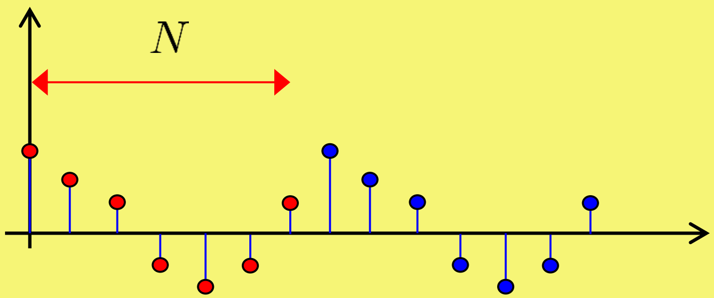
EITF75, Fourier transforms

4 different type of signals

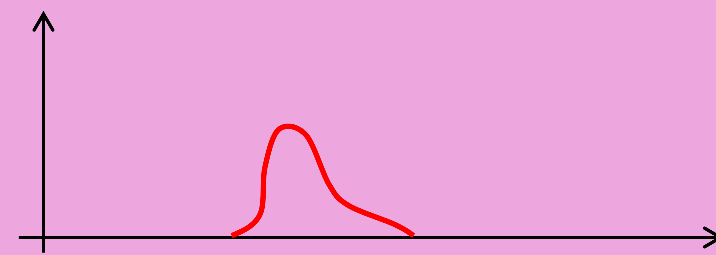
Continuous and periodic



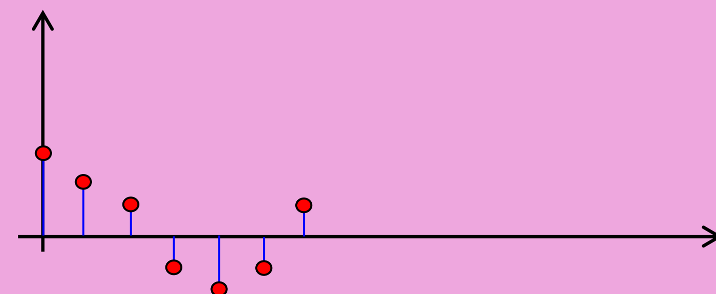
Discrete and periodic



Continuous and aperiodic



Discrete and aperiodic



Discrete spectra

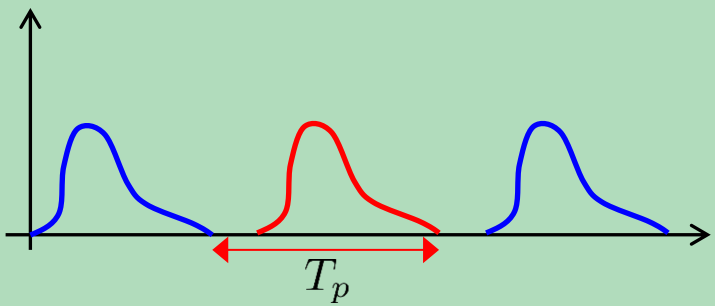
Continuous spectra

EITF75, Fourier transforms

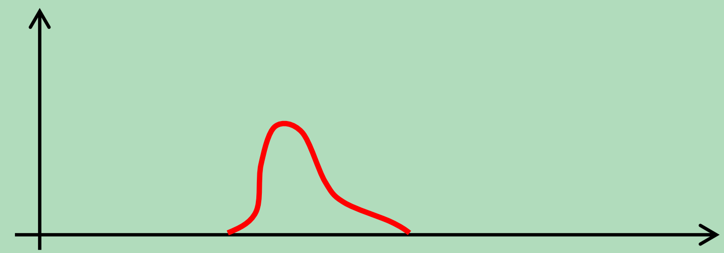
4 different type of signals

Aperiodic spectra

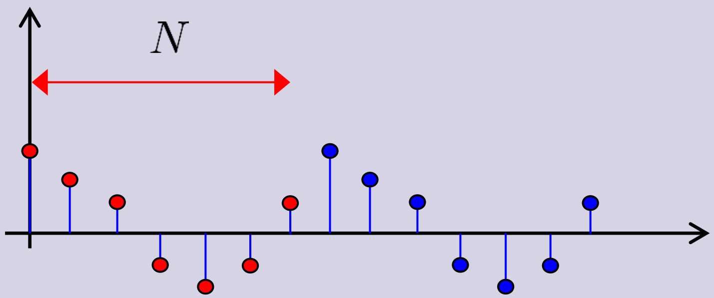
Continuous and periodic



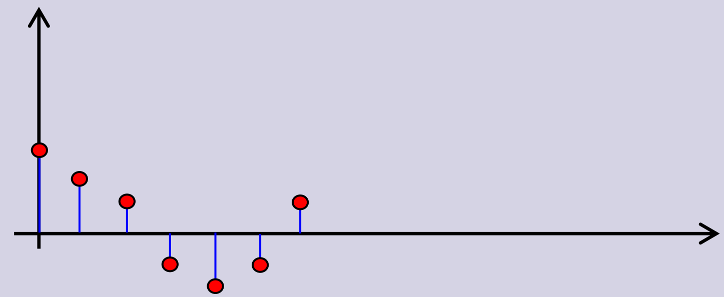
Continuous and aperiodic



Discrete and periodic



Discrete and aperiodic



Periodic spectra

EITF75, Fourier transforms

Example Find the Fourier transform of a square pulse

$$x(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Signal is continuous and aperiodic

Transform is continuous and aperiodic

EITF75, Fourier transforms

Example Find the Fourier transform of a square pulse

$$x(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Signal is continuous and aperiodic
Transform is continuous and aperiodic

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t}\, dt \quad = \int_{0}^{T} 1 \cdot e^{-j2\pi F t}\, dt$$

By definition

EITF75, Fourier transforms

Example Find the Fourier transform of a square pulse

$$x(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Signal is continuous and aperiodic
Transform is continuous and aperiodic

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} \, dt = \int_0^T 1 \cdot e^{-j2\pi F t} \, dt$$

$$= \frac{e^{-j2\pi F T} - 1}{-j2\pi F}$$

Elementary integral

EITF75, Fourier transforms

Example Find the Fourier transform of a square pulse

$$x(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Signal is continuous and aperiodic
Transform is continuous and aperiodic

$$X(F) = \int_{-\infty}^{\infty} x(t) \mathrm{e}^{-\mathrm{j}2\pi F t} \, \mathrm{d}t = \int_{0}^{T} 1 \cdot \mathrm{e}^{-\mathrm{j}2\pi F t} \, \mathrm{d}t$$

$$= \frac{\mathrm{e}^{-\mathrm{j}2\pi F T} - 1}{-\mathrm{j}2\pi F} = \frac{\mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{T}{2} \cdot F} \left(\mathrm{e}^{\mathrm{j}2\pi \cdot \frac{T}{2} \cdot F} - \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{T}{2} \cdot F} \right)}{\mathrm{j}2\pi F}$$

Minor manipulation

EITF75, Fourier transforms

Example Find the Fourier transform of a square pulse

$$x(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Signal is continuous and aperiodic
Transform is continuous and aperiodic

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} \, dt = \int_0^T 1 \cdot e^{-j2\pi Ft} \, dt$$

$$= \frac{e^{-j2\pi FT} - 1}{-j2\pi F} = \frac{e^{-j2\pi \cdot \frac{T}{2} \cdot F} \left(e^{j2\pi \cdot \frac{T}{2} \cdot F} - e^{-j2\pi \cdot \frac{T}{2} \cdot F} \right)}{j2\pi F}$$

$$= T \cdot \frac{\sin\left(2\pi \cdot \frac{T}{2} \cdot F\right)}{2\pi \cdot \frac{T}{2} \cdot F} \cdot e^{-j2\pi \cdot \frac{T}{2} \cdot F}$$

Euler's formula

EITF75, Fourier transforms

Example Find the Fourier transform of a square pulse

$$x(t)=\begin{cases}1 & 0\le t<T\\ 0 & \text{otherwise}\end{cases}$$

Signal is continuous and aperiodic
Transform is continuous and aperiodic

$$X(F)=\int_{-\infty}^{\infty}x(t)\mathrm{e}^{-\mathrm{j}2\pi Ft}\,\mathrm{d}t \quad =\int_{0}^{T}1\cdot\mathrm{e}^{-\mathrm{j}2\pi Ft}\,\mathrm{d}t$$

$$=\frac{\mathrm{e}^{-\mathrm{j}2\pi FT}-1}{-\mathrm{j}2\pi F} \quad =\frac{\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{T}{2}\cdot F}\left(\mathrm{e}^{\mathrm{j}2\pi\cdot\frac{T}{2}\cdot F}-\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{T}{2}\cdot F}\right)}{\mathrm{j}2\pi F}$$

$$=T\cdot\frac{\sin\left(2\pi\cdot\frac{T}{2}\cdot F\right)}{2\pi\cdot\frac{T}{2}\cdot F}\cdot\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{T}{2}\cdot F}$$

$$=T\cdot\operatorname{sinc}\left(2\cdot\frac{T}{2}\cdot F\right)\cdot\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{T}{2}\cdot F}$$

$$\operatorname{sinc}(x)=\frac{\sin(\pi x)}{\pi x}$$

$$\lim_{x\to 0}\frac{\sin(\pi x)}{\pi x}=1$$

Definition of a sinc-pulse

In 1948, Shannon used this pulse to derive the ultimate limit, in bits/sec, of communication. Super-important pulse in EE

Emre Telatar:
(In brief: Superstar)

"What Shannon's 48 paper has done for communication engineering has no parallel in any engineering field"

EITF75, Fourier transforms

Example Find the Fourier transform of a square pulse

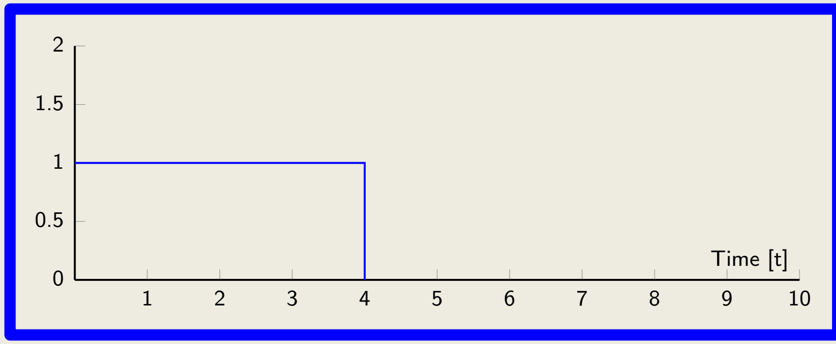
$$x(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Signal is continuous and aperiodic
Transform is continuous and aperiodic

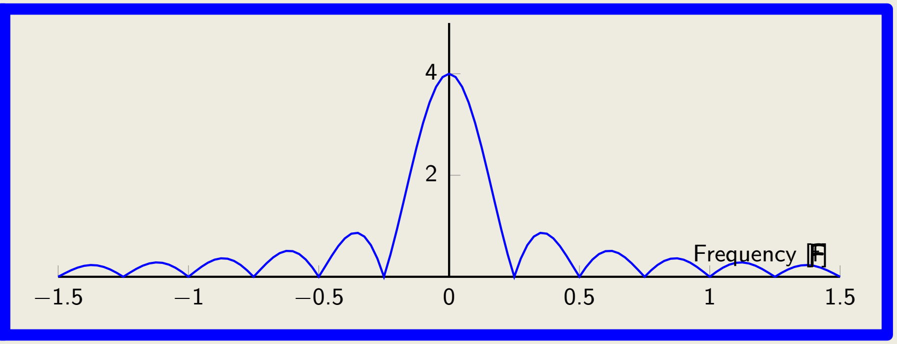
$$X(F) = T \cdot \text{sinc}\left(2 \cdot \frac{T}{2} \cdot F\right) \cdot e^{-j2\pi \cdot \frac{T}{2} \cdot F}$$

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$x(t)$ $T = 4$



$|X(F)|$



EITF75, Fourier transforms

Example Find the Fourier transform of a discrete square pulse

$$x(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Signal is discrete and aperiodic
Transform is continuous and periodic

EITF75, Fourier transforms

Example Find the Fourier transform of a discrete square pulse

$$x(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Signal is discrete and aperiodic
Transform is continuous and periodic

Note that the periodicity rules out the sinc-shape from the continuous case

EITF75, Fourier transforms

Example Find the Fourier transform of a discrete square pulse

$$x(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Signal is discrete and aperiodic
Transform is continuous and periodic

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=0}^{N-1} 1 \cdot e^{-j\omega n}$$

By definition

EITF75, Fourier transforms

Example Find the Fourier transform of a discrete square pulse

$$x(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Signal is discrete and aperiodic
Transform is continuous and periodic

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\mathrm{e}^{-\mathrm{j}\omega n} = \sum_{n=0}^{N-1} 1 \cdot \mathrm{e}^{-\mathrm{j}\omega n}$$

$$= \frac{1-\mathrm{e}^{-\mathrm{j}\omega N}}{1-\mathrm{e}^{-\mathrm{j}\omega}}$$

Geometric series

EITF75, Fourier transforms

Example Find the Fourier transform of a discrete square pulse

$$x(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Signal is discrete and aperiodic
Transform is continuous and periodic

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=0}^{N-1} 1 \cdot e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

$$= \frac{e^{j\omega N/2}\left(e^{j\omega \cdot \frac{N}{2}} - e^{-j\omega \cdot \frac{N}{2}}\right)}{e^{j\omega/2}\left(e^{j\omega \cdot \frac{1}{2}} - e^{-j\omega \cdot \frac{1}{2}}\right)}$$

Manipulation to reach Euler's

EITF75, Fourier transforms

Example Find the Fourier transform of a discrete square pulse

$$x(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Signal is discrete and aperiodic
Transform is continuous and periodic

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\mathrm{e}^{-\mathrm{j}\omega n} = \sum_{n=0}^{N-1} 1 \cdot \mathrm{e}^{-\mathrm{j}\omega n}$$

$$= \frac{1 - \mathrm{e}^{-\mathrm{j}\omega N}}{1 - \mathrm{e}^{-\mathrm{j}\omega}}$$

$$= \frac{\mathrm{e}^{\mathrm{j}\omega N/2}\left(\mathrm{e}^{\mathrm{j}\omega \cdot \frac{N}{2}} - \mathrm{e}^{-\mathrm{j}\omega \cdot \frac{N}{2}}\right)}{\mathrm{e}^{\mathrm{j}\omega/2}\left(\mathrm{e}^{\mathrm{j}\omega \cdot \frac{1}{2}} - \mathrm{e}^{-\mathrm{j}\omega \cdot \frac{1}{2}}\right)}$$

$$= N \cdot \frac{\sin\left(\omega \cdot \frac{N}{2}\right)}{N \sin\left(\omega \cdot \frac{1}{2}\right)} \cdot \mathrm{e}^{-\mathrm{j}\omega(N-1)/2}$$

Period = 1

EITF75, Fourier transforms

Example Find the Fourier transform of a discrete square pulse

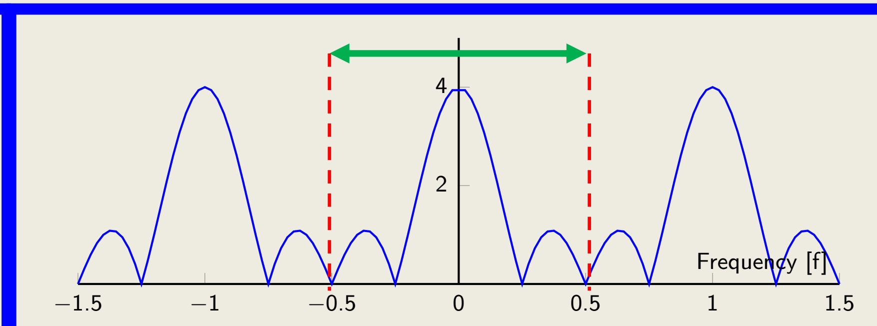
$$x(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Signal is discrete and aperiodic
Transform is continuous and periodic

$$X(\omega) = N \cdot \frac{\sin\left(\omega \cdot \frac{N}{2}\right)}{N \sin\left(\omega \cdot \frac{1}{2}\right)} \cdot e^{-j\omega(N-1)/2}$$

$x(n)$ $N = 4$

$|X(f)|$



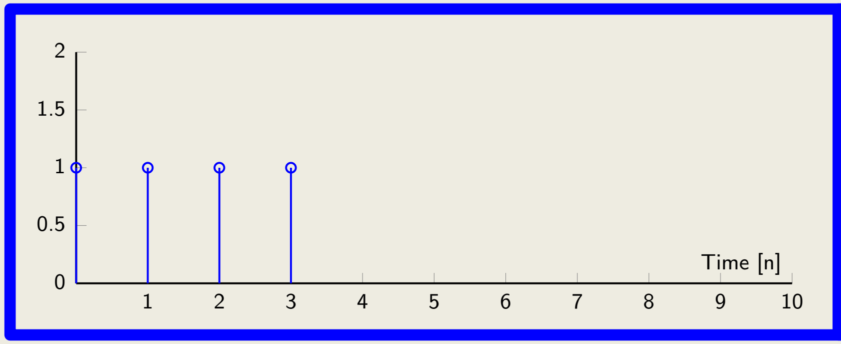
Period = 1

EITF75, Fourier transforms

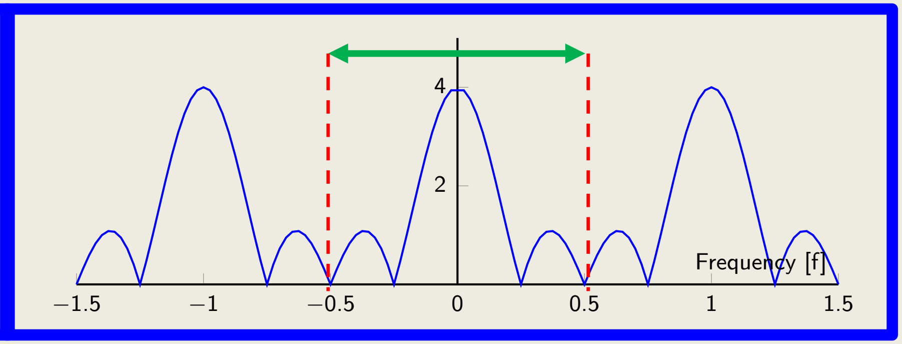
Remember?

$$\int_0^1 X(f)\exp(i2\pi nf)\mathrm{d}f = \int_{-0.5}^{0.5} X(f)\exp(i2\pi nf)\mathrm{d}f$$

$x(n)$ $N = 4$



$|X(f)|$



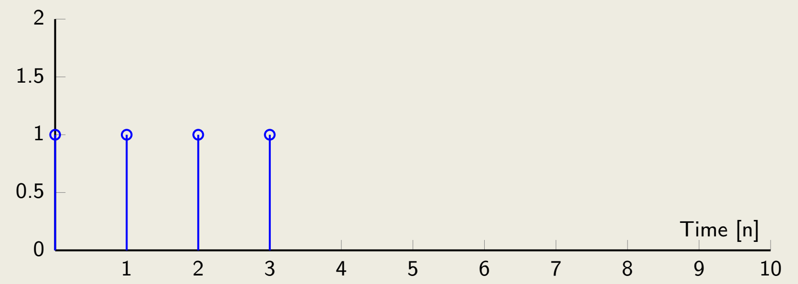
Period = 1

EITF75, Fourier transforms

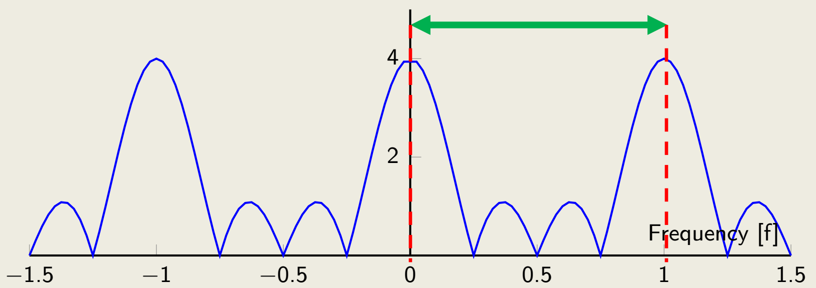
Remember?

$$\int_0^1 X(f)\exp(i2\pi nf)\mathrm{d}f = \int_{-0.5}^{0.5} X(f)\exp(i2\pi nf)\mathrm{d}f$$

$x(n)\ \ N=4$



$|X(f)|$



Same content, different order