EITF75 Systems and Signals

Lecture 3
The z-transform

Fredrik Rusek

Labs : online

sign up online opens next week

see web (soon) for when

Exercise : possibly: 1 session class room

1 sess. online

starting sept 15 Tuesday

see web (soon)

hand-in end of sept.

EITF75, z-transform

$H(z) \longleftrightarrow h(n)$

Definition

The z-transform of $h(n)$ is defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

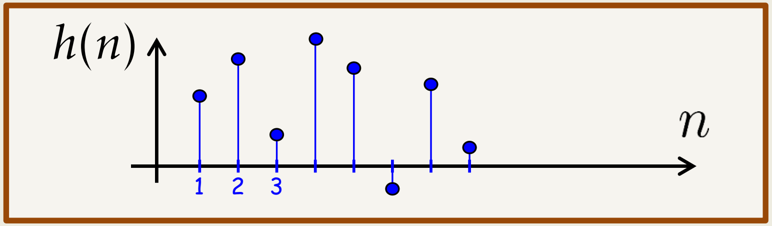
What is the z-transform?

- A map from sequences to complex valued functions

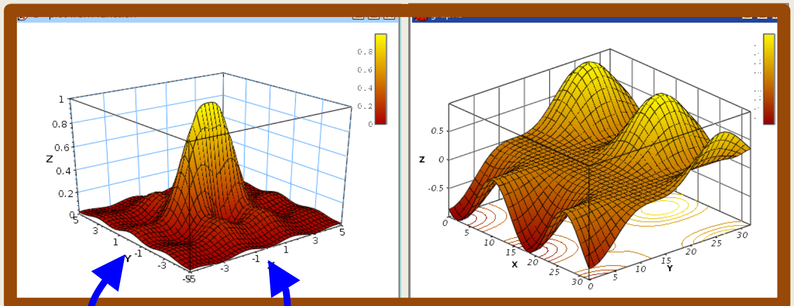
z is complex

What is H(z)?

- A complex function of a complex number



$\mathrm{Re}\{H(z)\}$ $\mathrm{Im}\{H(z)\}$



$\mathrm{Re}\{z\}$ $\mathrm{Im}\{z\}$

Illustration

Important: h(n) and H(z) contain the same information

EITF75, z-transform

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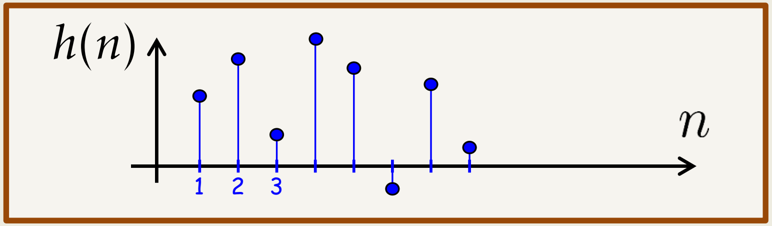
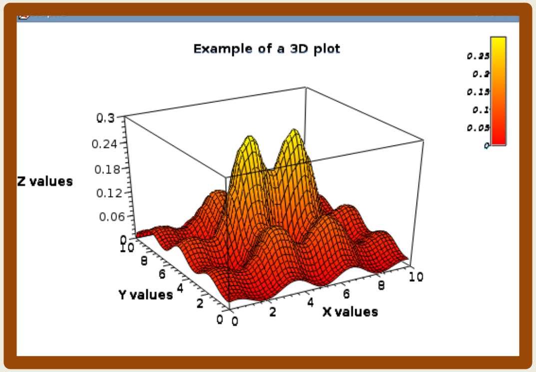


$|H(z)|$



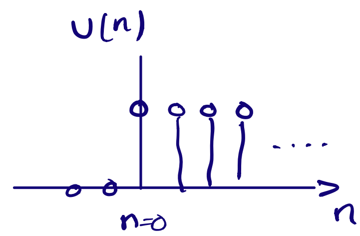
Magnitude of H(z) is typically shown

$$H(z) = \sum_n h(n) z^{-n}$$

seq	z-transform
$h(n)$	$H(z) = h(0) + h(1)z^{-1} + h(2)\cdot z^{-2} + \ldots$
	ass. $h(n)$ is causal
$\delta(n) = \{\underline{1}\ 0\ 0 ..\}$	$H(z) = 1 + 0 \ldots = \underline{\underline{1}}$
$\delta(n-k) = \{\underline{0} . 0 1 0 ..\}$ ↑ pos. $\underline{\underline{k}}$	$H(z) = z^{-k}$
$h(n-k) \Leftarrow ?$ $h(n) \Longleftrightarrow H(z)$	$y(n) = h(n-k)$ $Y(z) = \sum_n y(n) z^{-n} =$ $= \sum_n h(n-k) z^{-n}$ $= z^{-k} \sum_{n=-\infty}^{\infty} h(n-k) z^{-(n-k)}$ $= z^{-k} \sum_{m=-\infty}^{\infty} h(m) z^{-m}$ $= z^{-k} H(z)$
$h(n) = \{\underline{3}\ 2\ 1\}$	$3 + 2z^{-1} + 1\cdot z^{-2}$
$h(n) = \{\underline{0}\ 3\ 2\ 1\}$	$z^{-1}\cdot(\ \ ) = 3z^{-1} + 2z^{-2} + z^{-3}$

Important

$$h(n) = u(n) \qquad H(z) = ?$$



$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty} z^{-n} \qquad \text{(geom. series)}$$

$$= \frac{1 - (z^{-1})^{\infty}}{1 - z^{-1}} \qquad \underline{\underline{\text{if}}} \quad |z^{-1}| < 1$$

$$|z| > 1$$

$$= \frac{1}{1 - z^{-1}}$$

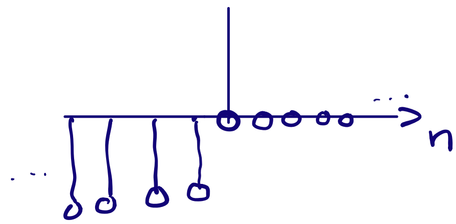
$$h(n) = u(n) \longleftrightarrow H(z) = \frac{1}{1 - z^{-1}} \qquad |z| > 1$$

Region of convergence

ROC

Important

$$\boxed{h(n) = -u(-n-1)}$$



"anti-causal step"

$$\boxed{H(z)} = \sum_{n=-\infty}^{\infty} -u(-n-1)z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (-1)z^{-n} = -\sum_{n=-\infty}^{-1} z^{-n} = -\sum_{n=1}^{\infty} z^{n}$$ "almost" geom. series

$$= -\sum_{n=0}^{\infty} z^{n} + 1 = -\frac{1}{1-z} + 1 = \frac{-z}{1-z} = \boxed{\frac{1}{1-z^{-1}}}$$

($|z|<1$ ROC)

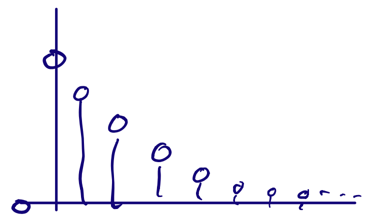
ROC $|z|<1$

Summary

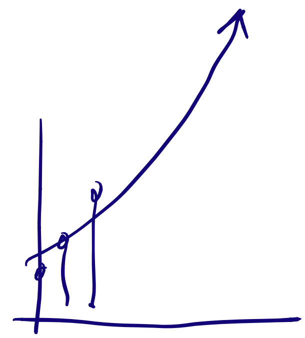
function		transform	ROC
$u(n)$	$\longleftrightarrow$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u(-n-1)$	$\longleftrightarrow$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$

(equal!)

$$h(n) = a^n \cdot u(n)$$



$|a|<1$



$|a|>1$

$H(z) = ?$

$$H(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= \frac{1}{1 - a z^{-1}}$$

if

$$|a z^{-1}| < 1$$

Result

$$H(z) = \frac{1}{1 - a z^{-1}}$$

ROC $|z| > |a|$

Homework

$$H(z) = \frac{1}{1 - a z^{-1}}$$

$|z| < |a|$ $h(n) = ?$

EITF75, z-transform

Some general rules about the ROC

$$X(z) = \ldots + x(-1001)z^{1001} + x(-1000)z^{1000} + \ldots + x(0) + \ldots + x(1000)z^{-1000} + x(1001)z^{-1001} + \ldots$$

Here is a formula for X(z) emphasizing x(n) far away from n=0

EITF75, z-transform

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Assume x(n) to be causal

EITF75, z-transform

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If X(z) exists (meaning that it is not infinity) then this cannot be too large

EITF75, z-transform

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If X(z) exists (meaning that it is not infinity) then this cannot be too large

Thus, z, cannot be too small.

How small? Depends on x(n)

EITF75, z-transform

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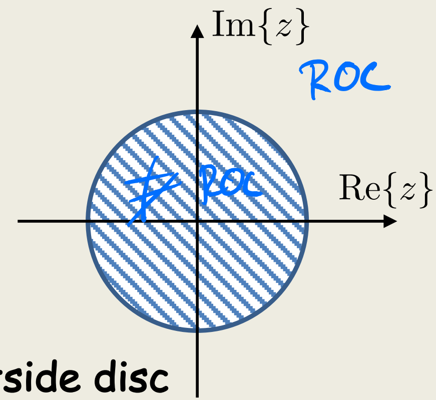

Hence, the ROC says that
"z should be larger than something"

ROC is outside disc

EITF75, z-transform

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$$X(z) = \ldots + x(-1001)z^{1001} + x(-1000)z^{1000} + \ldots + x(0)$$

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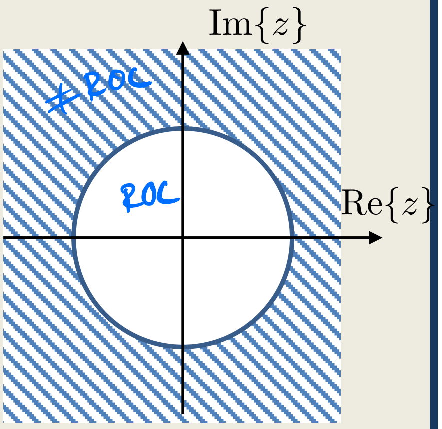
Assume x(n) to be anti-causal

If X(z) exists (meaning that it is not infinity) then this cannot be too large

Thus, z, cannot be too large.

Hence, the ROC says that
"z should be smaller than something"



ROC is inside disc

EITF75, z-transform

Some general rules about the ROC

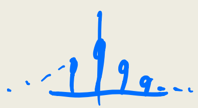
$$X(z) = \ldots + x(-1001)z^{1001} + x(-1000)z^{1000} + \ldots + x(0) + \ldots + x(1000)z^{-1000} + x(1001)z^{-1001} + \ldots$$

Here is a formula for X(z) emphasizing x(n) far away from n=0

Assume x(n) to be neither

If X(z) exists (meaning that it is not infinity) then these cannot be large

Thus, z, cannot be too large or too small



EITF75, z-transform

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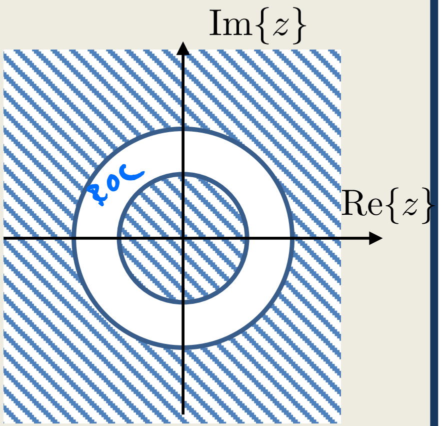
Assume x(n) to be neither

If X(z) exists (meaning that it is not infinity) then these cannot be large

Thus, z, cannot be too large or too small

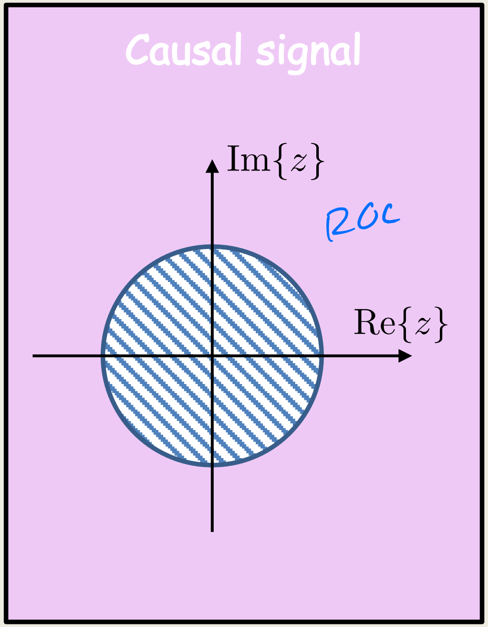
Hence, the ROC says that
"z should be smaller than something,
but larger than something else"

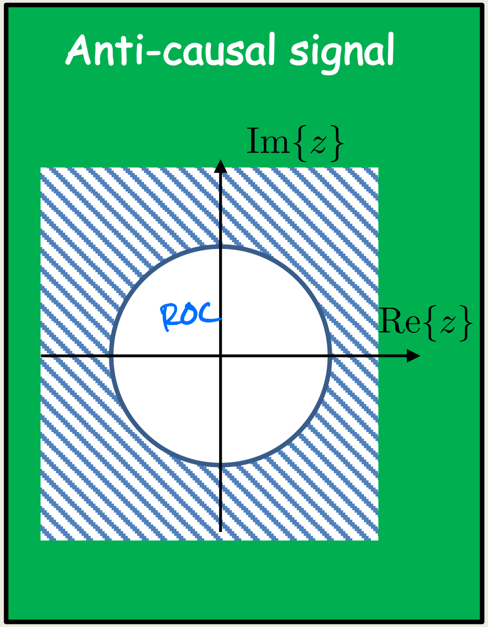


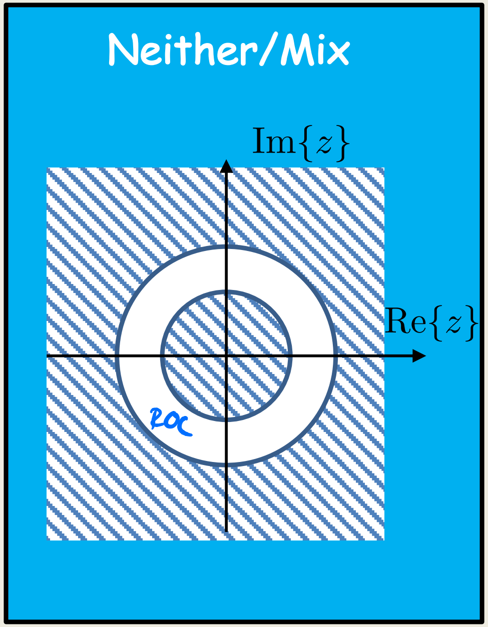
ROC is the white area

EITF75, z-transform

Summary: What is the general shape of the ROC?

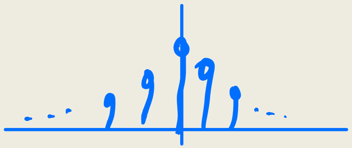






EITF75, z-transform

Homework



Given:

$$x(n) = \left(\frac{1}{2}\right)^{|n|} \qquad \text{for all } n$$

Find: The z-transform $X(z)$ of $x(n)$.

Including ROC

EITF75, z-transform

Convention

If we are given an X(z), and assume that the signal x(n) is causal, then we can be a bit sloppy with the ROC

This is what we do in this (most) of this course

In other words. There could be many x(n) for the same X(z), and the ROC specifies the particular one. However, there is only one that is causal.

EITF75, z-transform

Illustration

Sequence	Transform	ROC
$x_1(n)$		
$x_2(n)$		
$x_3(n)$		
$x_4(n)$		
$x_5(n)$		
$x_6(n)$		
$x_7(n)$		

Assume a bunch of *different* sequences

$u(n)$

$-u(-n-1)$ → $\frac{1}{1-z^{-1}}$

EITF75, z-transform

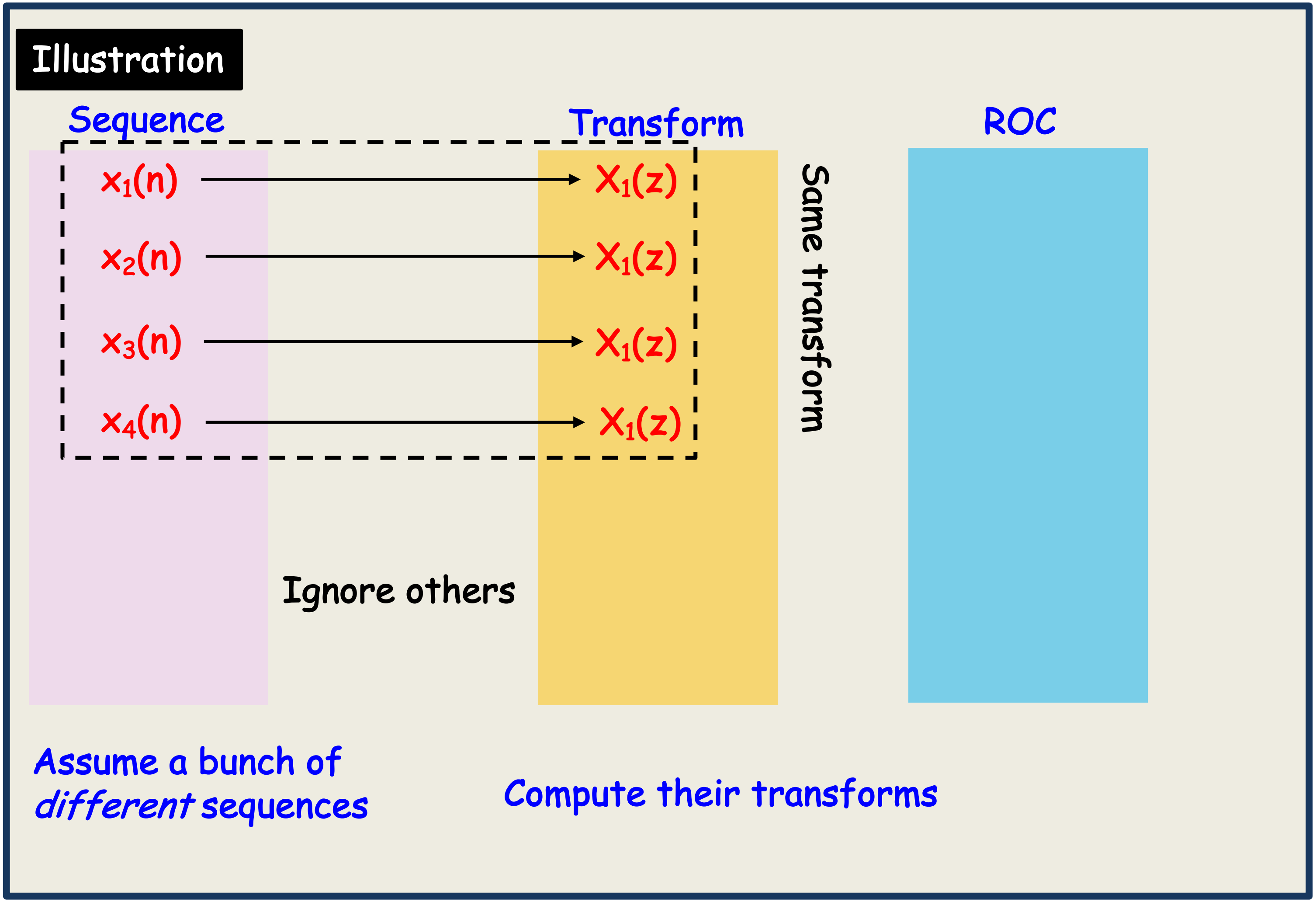
Illustration

Sequence		Transform	ROC
$x_1(n)$	→	$X_1(z)$	
$x_2(n)$	→	$X_1(z)$	
$x_3(n)$	→	$X_1(z)$	
$x_4(n)$	→	$X_1(z)$	
$x_5(n)$	→	$X_2(z)$	
$x_6(n)$	→	$X_2(z)$	
$x_7(n)$	→	$X_2(z)$	

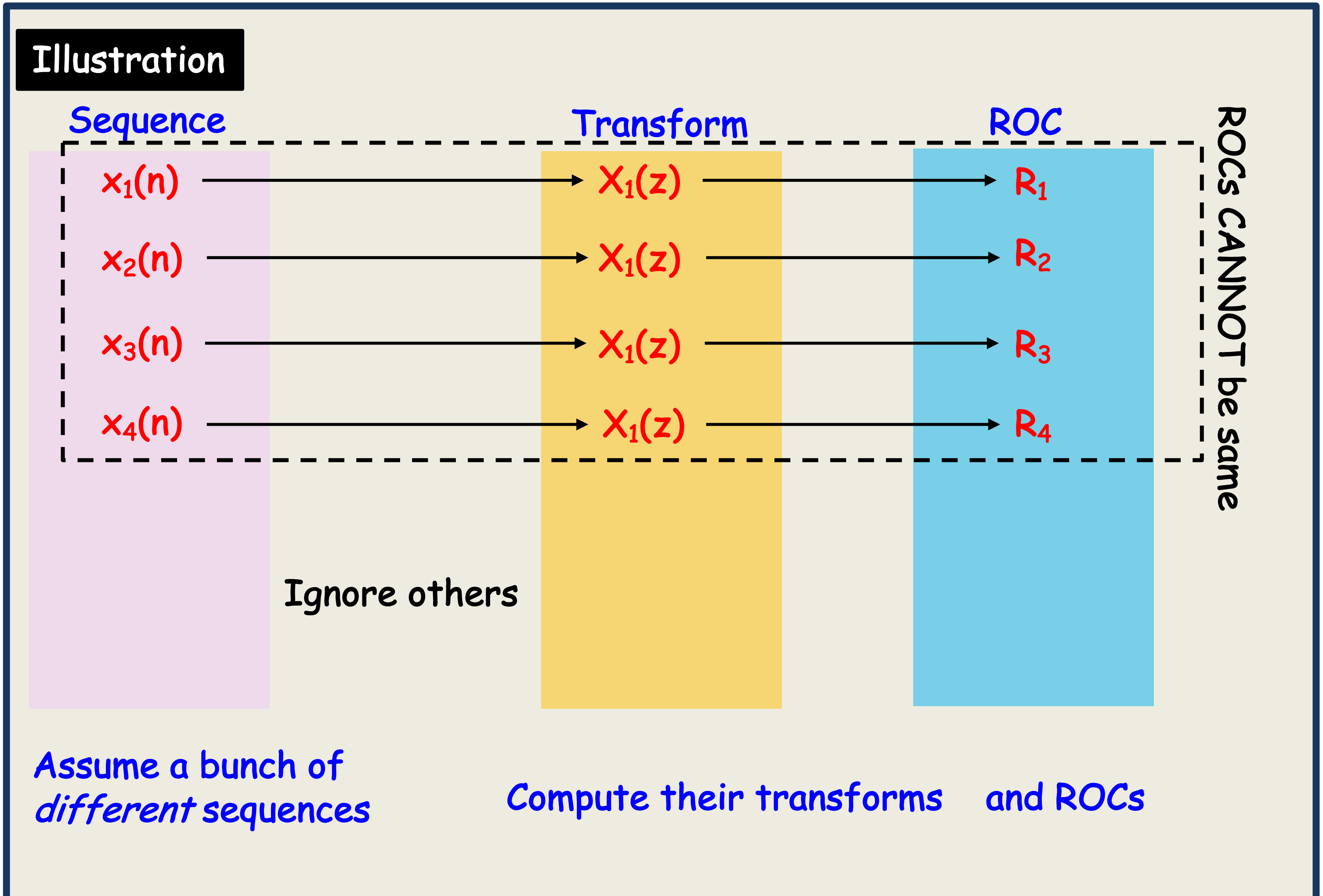
Assume a bunch of *different* sequences

Compute their transforms

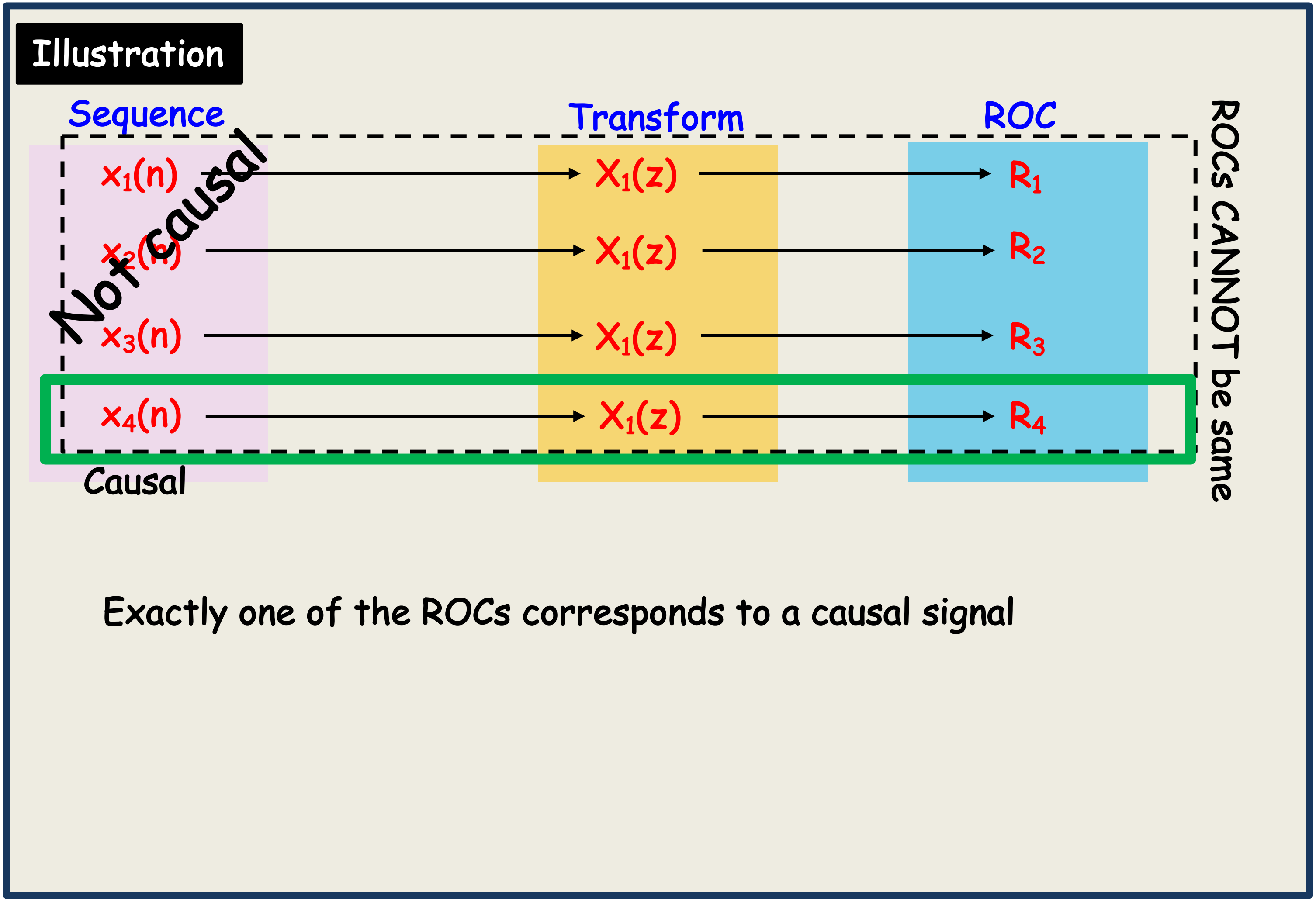
EITF75, z-transform



EITF75, z-transform

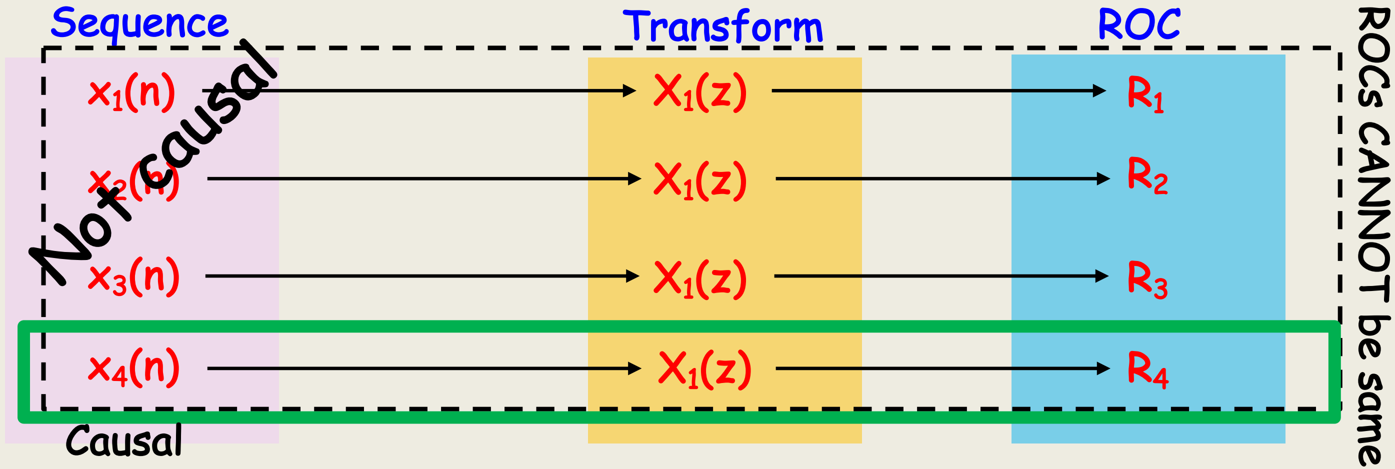


EITF75, z-transform



EITF75, z-transform

Illustration



Exactly one of the ROCs corresponds to a causal signal

So, if we know $X_1(z)$ and that we work with causal $x(n)$, we can establish $x_4(n)$ without knowing the ROC

$$\boxed{y(n) = h(n) * x(n)}$$

$\underline{H(z)}$, $\underline{X(z)}$

$x(n) \longrightarrow \boxed{LTI} \longrightarrow y(n)$ ←

$$\boxed{Y(z)} = \sum_{n=-\infty}^{\infty} y(n) z^{-n} = \sum_{n=-\infty}^{\infty} \underbrace{\sum_{k=-\infty}^{\infty} h(k) x(n-k)}_{\triangleq\, y(n)} z^{-n}$$

$$= \sum_{n} \sum_{k} h(k) x(n-k) z^{-(n-k)} z^{-k}$$

$$= \sum_{k} h(k) z^{-k} \cdot \underbrace{\sum_{n=-\infty}^{\infty} x(n-k) z^{-(n-k)}}_{} = \sum h(k) z^{-k} \sum_{m} x(m) z^{-m}$$

$$\boxed{= H(z) X(z)}$$

$h(n) * x(n)$

$\downarrow \qquad \downarrow$

$H(z) \qquad X(z)$

$Y(z) = H(z) X(z) \rightarrow \underline{\underline{y(n)}}$

Summary

$$\boxed{y(n) = h(n) * x(n)}$$

$$\boxed{Y(z) = H(z) X(z)}$$

Saving 100 kr / month 5% interest

start at 0 kr

Let :

$y(n)$ = money at month n

$x(n)$ = deposit at month n

$$y(-1) = 0$$

$$x(n) = u(n) \cdot 100 \leftrightarrow \frac{100}{1-z^{-1}} = X(z)$$

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

← make z-transform

$y(n) = \ldots$

$y(n) \leftrightarrow Y(z)$

$y(n-1) \leftrightarrow z^{-1} Y(z)$

$x(n) \leftrightarrow X(z)$

$$Y(z) = 1.05 \cdot z^{-1} Y(z) + X(z)$$

$$X(z) = 100 \cdot \frac{1}{1-z^{-1}}$$

$$Y(z) - 1.05 z^{-1} Y(z) = X(z)$$

$$Y(z)\left[1 - 1.05 z^{-1}\right] = X(z)$$

$$Y(z) = \frac{X(z)}{1-1.05z^{-1}} = \frac{1}{1-1.05z^{-1}} \cdot \frac{100}{1-z^{-1}}$$

$$Y(z) = \frac{X(z)}{1-1.05z^{-1}} = \frac{1}{1-1.05z^{-1}} \cdot \frac{100}{1-z^{-1}}$$

$y(n) = \ldots$?

$$\frac{1}{1-1.05z^{-1}} \rightarrow 1.05^n u(n)$$

$$\frac{1}{1-z^{-1}} \rightarrow u(n)$$

$$\frac{1}{1-1.05z^{-1}} \cdot \frac{100}{1-z^{-1}} = \frac{A}{1-1.05z^{-1}} + \frac{B}{1-z^{-1}}$$

PARTIAL FRACTION EXPANSION

1 Roots in denominator are real and distinct

2 — " ——— are distinct, but complex conjugated pairs

~~3 1 + multiple root~~

~~4 2 + — " ——~~

$$\frac{1}{1-1.05z^{-1}} \cdot \frac{100}{1-z^{-1}} = \frac{A}{1-1.05z^{-1}} + \frac{B}{1-z^{-1}}$$

$$\left.\begin{aligned} A &= 21 \cdot 100 \\ B &= -20 \cdot 100 \end{aligned}\right\} \text{PFE gives}$$

$$Y(z) = 100\left[\frac{21}{1-1.05z^{-1}} - \frac{20}{1-z^{-1}}\right]$$

$$\frac{1}{1-az^{-1}} \leftrightarrow a^n u(n)$$

$$\boxed{y(n) = 100 \cdot 21 \cdot 1.05^n u(n) - 100 \cdot 20 \cdot u(n)}$$

PFE: Assume you can handle roots of single multiplities

EITF75, z-transform

(a.k.a. recursion)

Summary: How to solve a difference equation in 6 simple steps

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

EITF75, z-transform

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$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

Step 1:

Change y(n-k) to $z^{-k}Y(z)$

EITF75, z-transform

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$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

Step 1:

Change y(n-k) to $z^{-k}Y(z)$

Step 2:

Change x(n-k) to $z^{-k}X(z)$

EITF75, z-transform

Summary: How to solve a difference equation in 6 simple steps

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

Step 1:

Change y(n-k) to $z^{-k}Y(z)$

Step 2:

Change x(n-k) to $z^{-k}X(z)$

Step 3:

Express Y(z) as H(z)X(z)

EITF75, z-transform

Summary: How to solve a difference equation in 6 simple steps

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Change y(n-k) to z^{-k} Y(z)

Step 2:

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Step 4:

Find the roots of the denominator of H(z)

EITF75, z-transform

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Find the roots of the denominator of H(z)

Step 5:

Perform PFE (you may need your calculus book)

EITF75, z-transform

Summary: How to solve a difference equation in 6 simple steps

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

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Step 4:

Find the roots of the denominator of H(z)

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Step 6:

Invert each term of the FPE

For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

Summary: How to solve a difference equation in 6 simple steps

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Step 1:

Change y(n-k) to $z^{-k}Y(z)$

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Invert each term of the FPE

For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

One further example: $y(n)-1.27y(n-1)+0.81y(n-2)=x(n-1)-x(n-2)$

All signals causal

Step 1:

Change y(n-k) to $z^{-k}Y(z)$

Step 2:

Change x(n-k) to $z^{-k}X(z)$

Step 3:

Express Y(z) as H(z)X(z)

Step 4:

Find the roots of the denominator of H(z)

Step 5:

Perform PFE (you may need your calculus book)

Step 6:
Invert each term of the FPE
For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

One further example: $Y(z) - 1.27z^{-1}Y(z) + 0.81z^{-2}Y(z) = z^{-1}X(z) - z^{-2}X(z)$

All signals causal

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Step 2:

Change x(n-k) to $z^{-k}X(z)$

Step 3:

Express Y(z) as H(z)X(z)

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Perform PFE (you may need your calculus book)

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Invert each term of the FPE
For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

One further example:

$$Y(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}} \cdot X(z)$$

All signals causal

Step 1:

Change y(n-k) to $z^{-k}Y(z)$

Step 2:

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Step 3:

Express Y(z) as H(z)X(z)

Step 4:

Find the roots of the denominator of H(z)

Step 5:

Perform PFE (you may need your calculus book)

Step 6:

Invert each term of the FPE

For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

One further example:

All signals causal

$$z_{1,2} = 0.9e^{i\pm\pi/4}$$

Step 1:

Change y(n-k) to z^{-k} Y(z)

Step 2:

Change x(n-k) to z^{-k} X(z)

Step 3:

Express Y(z) as H(z)X(z)

Step 4:

Find the roots of the denominator of H(z)

Step 5:

Perform PFE (you may need your calculus book)

Step 6:

Invert each term of the FPE

For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

One further example:

All signals causal

$$z_{1,2} = 0.9e^{i\pm\pi/4}$$

Type II. PFE already done, 1 complex conjugated pair

Step 1:

Change y(n-k) to z^{-k} Y(z)

Step 2:

Change x(n-k) to z^{-k} X(z)

Step 3:

Express Y(z) as H(z)X(z)

Step 4:

Find the roots of the denominator of H(z)

Step 5:

Perform PFE (you may need your calculus book)

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Invert each term of the FPE
For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

One further example:

All signals causal

Step 1:

Change y(n-k) to $z^{-k}Y(z)$

Step 2:

Change x(n-k) to $z^{-k}X(z)$

Step 3:

Express Y(z) as H(z)X(z)

Next lecture…

Step 4:

Find the roots of the denominator of H(z)

Step 5:

Perform PFE (you may need your calculus book)

Step 6:
Invert each term of the FPE
For FPE-type II, we haven't seen the formula yet