EITF75 Systems and Signals

Lecture 2
LTI systems: convolutions, impulse responses (and more)

Fredrik Rusek

EITF75 Systems and Signals

LTI systems. recap

$$x(n) \rightarrow \boxed{\text{system}} \rightarrow y(n)$$

A system is LTI if-and-only if:

- It is linear
- It is time-invariant

Linear system

$$x(n) = \alpha x_1(n) + \beta x_2(n)$$
$$\Longleftrightarrow$$
$$y(n) = \alpha y_1(n) + \beta y_2(n)$$

Time invariant system

$$x(n) \text{ replaced by } x(n-D)$$
$$\Longleftrightarrow$$
$$y(n) \text{ replaced by } y(n-D)$$

Linear system $\Longleftrightarrow$ linear algebra

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LTI systems

Today we will show that these are equivalent

$x(n) \rightarrow \boxed{\text{LTI}} \rightarrow y(n)$

We start here

Alt method at the end

$$y(n) = \sum_k x(k) \cdot h(n-k)$$

For some h(k)

Convolution

"Faltning"

Rec. eq.

$$\sum_k a(k) y(n-k) = \sum_\ell b(\ell) x(n-\ell)$$

For some a(k), b(k)

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$x(n) \longrightarrow$ **LTI system** $\longrightarrow y(n)$

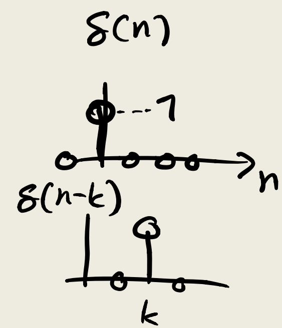
A system is LTI if-and-only if:

- It is linear
- It is time-invariant

Input signal	Output signal	
$\delta(n)$	$h(n)$	impulse response
$\delta(n-k)$	$h(n-k)$	$\underline{\underline{L}}TI$
$x(k)\cdot\delta(n-k)$	$x(k)h(n-k)$	$\underline{\underline{L}}TI$
$x(n)=\sum_k x(k)\delta(n-k)$	$\sum_k x(k)h(n-k)$	$\underline{\underline{L}}TI$
	$= y(n)$	

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)}$$

convolution



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Summary

LTI

$$x(t) \rightarrow \boxed{\text{Sampling}} \rightarrow x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) \rightarrow \boxed{\text{rec.}} \rightarrow y(t)$$

Input/Output relation
(Convolution)

$$y(n) = \sum_k x(k)h(n-k)$$

Short-hand notation

$$y(n) = x(n) \star h(n)$$

$$\boxed{y = x * h}$$

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Agenda

Next (Today)

Get familiar with $y(n) = x(n) \star h(n)$ through some examples

For what $h(n)$ do we have BIBO stability?

Prove equivalence between $h(n)$ and $\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$

Some notes on correlation functions

In the long run (Loosely speaking)

Study $\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$ in detail via z-transform, and 2 types of Fourier transforms

The sampling-reconstruction issues

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Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

"Home work 1"
Verify that causal x(n) and causal h(n) yields causal y(n)

"Home work 2"
If x(n) starts at -3, and h(n) at -4. When does y(n) start?

"Home work 3"
etc

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Example

Three more methods
Method 1

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

h(0)=3 h(1)=2 h(2)=1 h(3)=0

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

$$= 3x(n) + 2x(n-1) + x(n-2)$$

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Example

Three more methods
Method 1

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

$$= 3x(n) + 2x(n-1) + x(n-2)$$

y(0)

$n = 0$

n = 0

$h(n) = \{3\ \underline{2}\ 1\}$

1 2 3

1 2 3

2 4 6 4 2

4 12 Σ 16

6 Σ 6

n = 0
y(-1)

$h(0-k)$	1	2	$\underline{3}$	0	0	0	0		
$x(k)$	0	0	$\underline{2}$	4	6	4	2		
$h(0-k)x(k)$			$\underline{6}$						$\sum = \underline{6} = y(0)$

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Example

Three more methods
Method 1

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

$$= 3x(n) + 2x(n-1) + x(n-2)$$

$$n = 1$$

$h(1-k)$	1	2	$\underline{3}$			
$x(k)$		$\underline{2}$	4	6	4	2
$h(1-k)x(k)$		4	12			$\sum = 16 = y(1)$

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Example

Three more methods
Method 1

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $$y(n) = \sum_k x(k) \cdot h(n-k)$$

$$= 3x(n) + 2x(n-1) + x(n-2)$$

$n = 2$

$h(2-k)$	1	2	$\underline{3}$			
$x(k)$	$\underline{2}$	4	6	4	2	
$h(2-k)x(k)$	2	8	18			$\sum = 28 = y(2)$

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Example

Three more methods

Method 1

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

$$= 3x(n) + 2x(n-1) + x(n-2)$$

$n=3$
$y(3)=28$

$n=4$
$y(4)=20$

$n=5$

$n=6$
$y(6)\neq 0$

$n=7$
$y(7)=0$

					1	2	3	
						1	2	3
				1	2	3		
			1	2	3			
		1	2	3				
$h(3-k)$		1	2	3				
$x(k)$	$\underline{2}$	4	6	4	2			
$h(3-k)x(k)$		4	12	12			Σ 28	
			6	9	6		Σ 20	

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Example

Three more methods
Method 2

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Put numbers in a table and multiply

	$\underline{2}$	4	6	4	2
$\underline{3}$	$\underline{6}$				
2					
1					

x(n)

h(n)

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Example

Three more methods
Method 2

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Put numbers in a table and multiply

	$\underline{2}$	4	6	4	2
$\underline{3}$	$\underline{6}$	12	18	12	6
2	4	8	12	8	4
1	2	4	6	4	2

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Example

Three more methods
Method 2

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Sum the diagonals

	$\underline{2}$	4	6	4	2
$\underline{3}$	$\underline{6}$	12	18	12	6
2	4	8	12	8	4
1	2	4	6	4	2

$\underline{6}$ 16 28 28 20 8 2

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Example

Three more methods
Method 2

Given: Input signal and impulse response

$5+3-1=7$

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\} \quad 5$$

$$h(n) = \{\underline{3}\ 2\ 1\} \quad 3$$

2

$$y(n) = \{\underline{6}\ 16\ 28\ 28\ 20\ 8\ 2\}$$

	$\underline{2}$	4	6	4	2
$\underline{3}$	$\underline{6}$	12	18	12	6
2	4	8	12	8	4
1	2	4	6	4	2

Result: $\underline{6}$ 16 28 28 20 8 2

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Make sure that you understand why a convolution of a length K signal with a length L signal has length K+L-1

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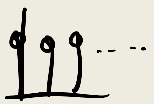
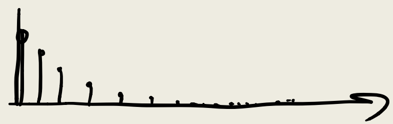
Example

Three more methods

Method 3: Analytical solution

$$x(n) = u(n) \qquad h(n) = \left(\frac{1}{2}\right)^n \cdot u(n)$$





$h(n) > 0 \ , \ \forall n \geq 0$

$$x(n) \longrightarrow \boxed{\begin{array}{l}\text{LTI} \\ h(n) = \left(\frac{1}{2}\right)^n u(n)\end{array}} \longrightarrow y(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) = \sum_{k=-\infty}^{\infty} h(k) x(n-k)$$

var. change

$n-k = k' \ ; \quad k' = k$

$$\boxed{\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}}$$

geom. series

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k)\, u(n-k) = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = \begin{cases} \dfrac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = \ldots = 2 - \left(\frac{1}{2}\right)^n & , n \geq 0 \\ 0 & , n < 0 \end{cases}$$

$$u(n-k) = \begin{cases} 0 & k > n \\ 1 & \text{otherw.} \end{cases}$$

$$y(n) = \begin{cases} 2 - \left(\frac{1}{2}\right)^n & , \ n \geq 0 \\ 0 & , \ n < 0 \end{cases} \iff \boxed{y(n) = \left[2 - \left(\frac{1}{2}\right)^n\right] u(n)}$$

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Standard Properties

Commutativity

$$\boxed{x_1(n) \star x_2(n) = x_2(n) \star x_1(n)}$$

Associativity

$$\boxed{x_1(n) \star [x_2(n) \star x_3(n)] = [x_1(n) \star x_2(n)] \star x_3(n)}$$

Distributivity

$$\boxed{x_1(n) \star [x_2(n) + x_3(n)] = x_1(n) \star x_2(n) + x_1(n) \star x_3(n)}$$

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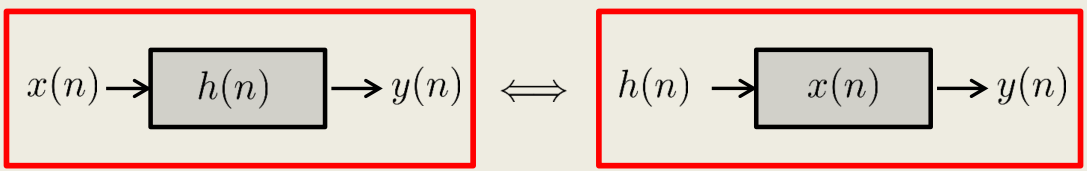
$$x_1(n) \star x_2(n) = x_2(n) \star x_1(n)$$ **Commutativity**

$$x_1(n) \star [x_2(n) \star x_3(n)] = [x_1(n) \star x_2(n)] \star x_3(n)$$ **Associativity**

$$x_1(n) \star [x_2(n) + x_3(n)] = x_1(n) \star x_2(n) + x_1(n) \star x_3(n)$$ **Distributivity**

Some consequences



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$$x_1(n) \star x_2(n) = x_2(n) \star x_1(n)$$ Commutativity

$$x_1(n) \star [x_2(n) \star x_3(n)] = [x_1(n) \star x_2(n)] \star x_3(n)$$ Associativity

$$x_1(n) \star [x_2(n) + x_3(n)] = x_1(n) \star x_2(n) + x_1(n) \star x_3(n)$$ Distributivity

Some consequences

$$x(n) \rightarrow \boxed{h_1(n)} \rightarrow \boxed{h_2(n)} \rightarrow y(n) \quad \Longleftrightarrow \quad x(n) \rightarrow \boxed{h(n) = h_1(n) \star h_2(n)} \rightarrow y(n)$$

$$\Longleftrightarrow$$

$$x(n) \rightarrow \boxed{h_2(n)} \rightarrow \boxed{h_1(n)} \rightarrow y(n)$$

Try to figure out which properties which were used

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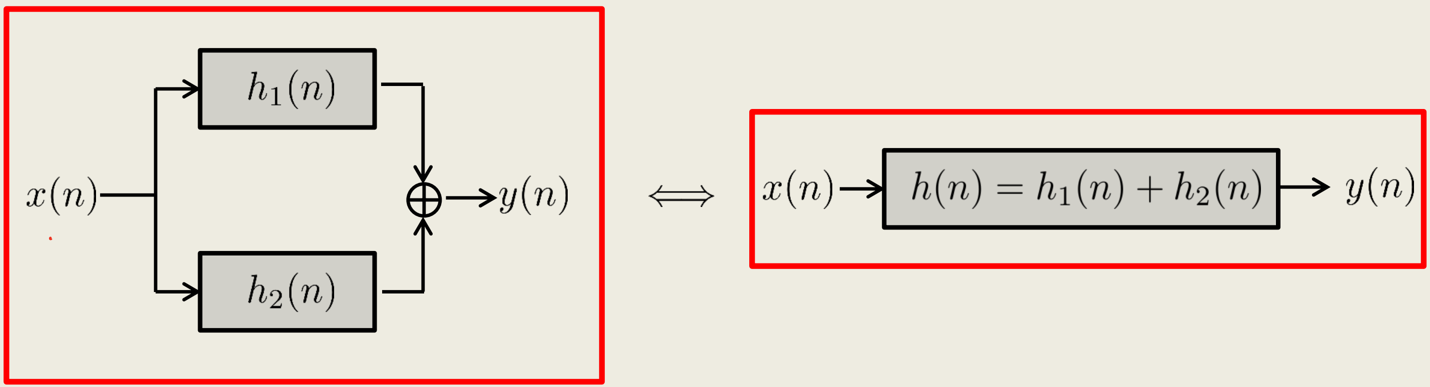
$$\boxed{x_1(n) \star x_2(n) = x_2(n) \star x_1(n)}$$ Commutativity

$$\boxed{x_1(n) \star [x_2(n) \star x_3(n)] = [x_1(n) \star x_2(n)] \star x_3(n)}$$ Associativity

$$\boxed{x_1(n) \star [x_2(n) + x_3(n)] = x_1(n) \star x_2(n) + x_1(n) \star x_3(n)}$$ Distributivity

Some consequences



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BIBO stability

A system is BIBO stable if $|x(n)| < M_x \implies |y(n)| < M_y < \infty$

For LTI

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \right|$$

$|a+b| \leq |a| + |b|$

$$\leq \sum_{k=-\infty}^{\infty} |x(k)\,h(n-k)| = \sum_{k=-\infty}^{\infty} \underbrace{|x(k)|}_{\leq M_x}\,|h(n-k)|$$

$$\leq M_x \sum_{k=-\infty}^{\infty} |h(n-k)| = M_x \sum_{k=-\infty}^{\infty} |h(k)|$$

$$\boxed{\text{An LTI system is stable iff } \sum_{k=-\infty}^{\infty} |h(k)| < \infty}$$

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LTI systems

Today we will show that these are equivalent

$$x(n) \rightarrow \boxed{\text{LTI}} \rightarrow y(n)$$

$$y(n) = \sum_k x(k) \cdot h(n-k)$$

For some h(k)

We now show this

$$\sum_k a(k) y(n-k) = \sum_\ell b(\ell) x(n-\ell)$$

For some a(k), b(k)

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Relation to difference equations

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

assume:

$$a(k) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases} \qquad b(\ell) = 0 \quad \ell > L,\ \ell < 0$$

Result:

$$y(n) = \sum_{\ell=0}^{L} b(\ell)\,x(n-\ell)$$

def. convolution

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Relation to difference equations

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

Consider now $a(0) = 1,\ a(1) = a_1 \quad b(0) = b_0$

We then get $y(n) = -a_1 y(n-1) + b_0 x(n)$

$$y(0) = -a_1 y(-1) + b_0 x(0)$$

$$y(1) = -a_1 \underline{y(0)} + b_0 x(1) = \ldots = (-a_1)^2 y(-1) + b_0 x(1) + (-a_1) b_0 x(0)$$

$$y(2) = -a_1 y(1) + b_0 x(2) = \ldots = (-a_1)^3 y(-1) + b_0 x(2) + (-a_1) b_0 x(1) +$$
$$+ (-a_1)^2 b_0 x(0)$$

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Relation to difference equations

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

IIR filter : imp. resp $h(n)$ of inf. duration

FIR filter: $h(n)$ of finite duration

Consider now $a(0) = 1,\ a(1) = a_1 \quad b(0) = b_0$

$y(-1) = 0$: No initial condition / system at rest

We then get $y(n) = -a_1 y(n-1) + b_0 x(n)$

$y(-1) \neq 0$: Has initial cond./ not at rest

Pattern recognition, suitably done at home, gives

$$y(n) = \sum_{k=0}^{n} (-a_1)^k b_0 x(n-k) + \underbrace{(-a_1)^{n+1} y(-1)}$$

$\not\sim x(n)$

$y(-1)$: initial state

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k), \quad h(k) = (-a_1)^k b_0\, u(k)$$

IIR

Note: $h(k) > 0 \ \forall k \geq 0$

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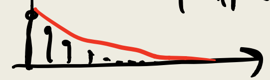
Summary

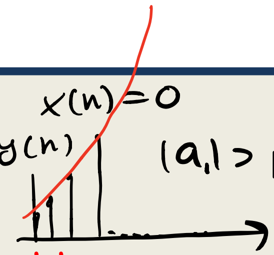
$x(n) = 0$

$y(n)$ $|a_1| < 1$



$x(n) = 0$

$y(n)$ $|a_1| > 1$



AN LTI SYSTEM "A" is equivalent to
A CONVOLUTION WITH IMPULSE RESPONSE "h(n)" which is equivalent to
A RECURSION EQUATION WITH COEFFICIENTS "a(k), b(k)"

FIR: $a(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$

IIR: $\exists \ell : a(\ell) \neq 0$, $\ell \neq 0$

$x(n) \rightarrow \boxed{\text{LTI}} \rightarrow y(n)$

$$y(n) = \sum_k x(k) \cdot h(n-k)$$

$$\sum_k a(k) y(n-k) = \sum_\ell b(\ell) x(n-\ell)$$

$y(n) = \dots + (-a_1)^{n+1} y(-1)$

$0 \rightarrow \boxed{\text{LTI}} \rightarrow 0$

$0 \rightarrow \boxed{} \rightarrow (-a_1)^{n+1} y(-1)$

Not strictly LTI

Brief info on correlation

Not focal point of course, but highly important in signal processing

Correlation measures similarity between two signals

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Brief info on correlation

Not focal point of course, but highly important in signal processing

Correlation measures similarity between two signals

Auto correlation	Cross correlation
$r_{xx}(k) = \sum_{n=-\infty}^{\infty} x(n)x(n-k) = x(n) \star x(-n)$	$r_{yx}(k) = \sum_{n=-\infty}^{\infty} y(n)x(n-k) = y(n) \star x(-n)$
Measures similarity between time shifted versions of the same signal	Measures similarity between time shifted versions of different signals

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Brief info on correlation

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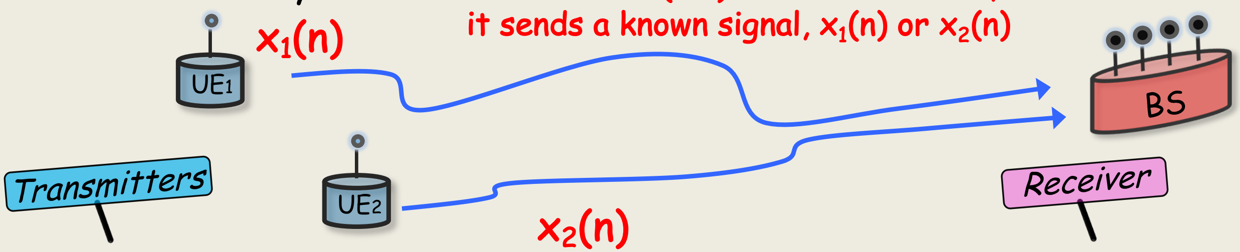
Correlation measures similarity between two signals

Auto correlation	Cross correlation
$r_{xx}(k) = \sum_{n=-\infty}^{\infty} x(n)x(n-k) = x(n) \star x(-n)$	$r_{yx}(k) = \sum_{n=-\infty}^{\infty} y(n)x(n-k) = y(n) \star x(-n)$
Measures similarity between time shifted versions of the same signal	Measures similarity between time shifted versions of different signals

Example: 5G communication system

When a user (UE) wants to connect, it sends a known signal, $x_1(n)$ or $x_2(n)$



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Auto correlation	Cross correlation
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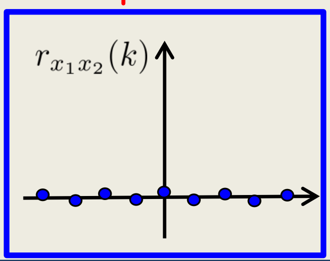
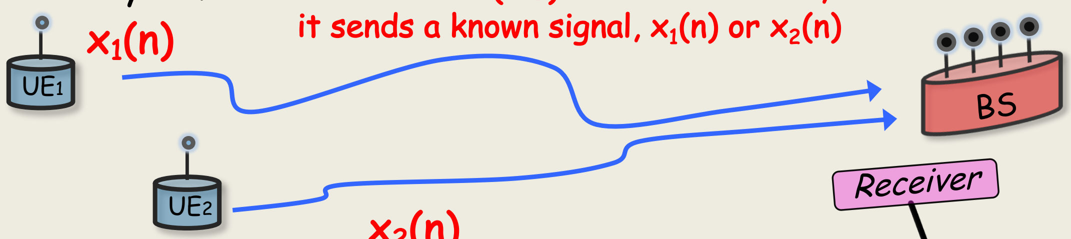
Measures similarity between time shifted versions of the same signal	Measures similarity between time shifted versions of different signals

Example: 5G communication system

When a user (UE) wants to connect, it sends a known signal, $x_1(n)$ or $x_2(n)$





Cross correlation between $x_1(n)$ and $x_2(n)$ should be small (to know who is connecting)

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Brief info on correlation

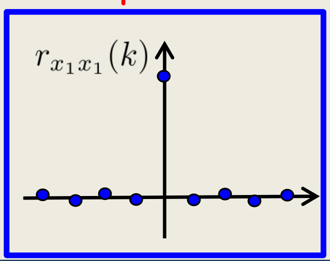
Not focal point of course, but highly important in signal processing

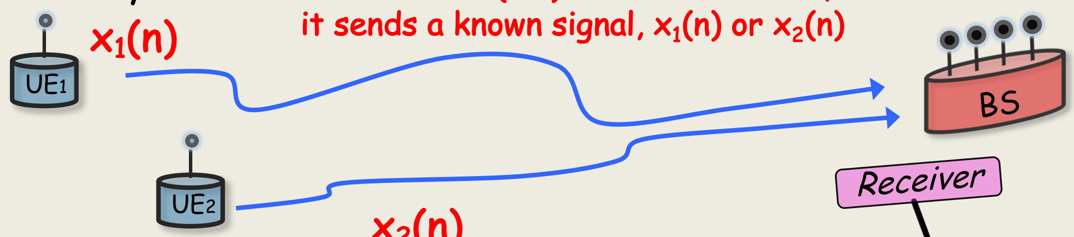
Correlation measures similarity between two signals

Auto correlation	Cross correlation
$r_{xx}(k) = \sum_{n=-\infty}^{\infty} x(n)x(n-k) = x(n) \star x(-n)$	$r_{yx}(k) = \sum_{n=-\infty}^{\infty} y(n)x(n-k) = y(n) \star x(-n)$
Measures similarity between time shifted versions of the same signal	Measures similarity between time shifted versions of different signals

Example: 5G communication system

When a user (UE) wants to connect, it sends a known signal, $x_1(n)$ or $x_2(n)$





Auto correlation of $x_1(n)$ (and $x_2(n)$) should be delta (to know when a user is connecting)

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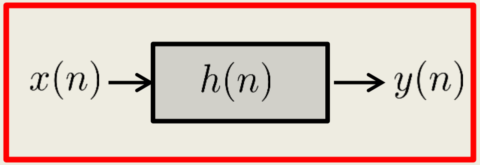
Brief info on correlation

Cross correlation for input and output signals



$$
\begin{aligned}
r_{yx}(k) &= y(k) \star x(-k) \\
&= x(k) \star h(k) \star x(-k) \\
&= h(k) \star x(k) \star x(-k) \\
&= h(k) \star r_{xx}(k)
\end{aligned}
$$

$$
\begin{aligned}
r_{yy}(k) &= y(k) \star y(-k) \\
&= x(k) \star h(k) \star x(-k) \star h(-k) \\
&= r_{hh}(k) \star r_{xx}(k)
\end{aligned}
$$

Appendix

$x(n) \rightarrow$ LTI system $\rightarrow y(n)$

A system is LTI if-and-only if:

- It is linear
- It is time-invariant

Method II (not in book, but I find it illuminating)

Input x(n): A sequence of numbers

$$\ldots,\ 0.4,\ -0.3,\ 1.2,\ 0.8,\ \ldots$$

Output y(n): A sequence of numbers

$$\ldots,\ -0.6,\ -0.34,\ 3.8,\ -1.8,\ \ldots$$

In the Linear algebra course, how did we represent a sequence of numbers?

Linear algebra = " EITF75 – TI "

EITF75 Systems and Signals

A system is LTI if-and-only if:

- It is linear
- It is time-invariant

Method II (not in book, but I find it illuminating)

Input x(n): A sequence of numbers

$$\dots,\ 0.4,\ -0.3,\ 1.2,\ 0.8,\ \dots$$

Output y(n): A sequence of numbers

$$\dots,\ -0.6,\ -0.34,\ 3.8,\ -1.8,\ \dots$$

In the Linear algebra course, how did we represent a sequence of numbers? With a vector

$$\begin{array}{ccc} \text{OUT} & & \text{IN} \\ \begin{bmatrix} \dots \\ -0.6 \\ -0.34 \\ 3.8 \\ -1.8 \\ \dots \end{bmatrix} & = & \begin{bmatrix} \dots \\ 0.4 \\ -0.3 \\ 1.2 \\ 0.8 \\ \dots \end{bmatrix} \end{array}$$

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Why is the linear algebra course dealing so much with matrices?

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$$\begin{array}{c} \text{OUT} \\ \begin{bmatrix} \ldots \\ -0.6 \\ -0.34 \\ 3.8 \\ -1.8 \\ \ldots \end{bmatrix} \end{array} = A \begin{array}{c} \text{IN} \\ \begin{bmatrix} \ldots \\ 0.4 \\ -0.3 \\ 1.2 \\ 0.8 \\ \ldots \end{bmatrix} \end{array}$$

Why is the linear algebra course dealing so much with matrices?
Because every linear function can be represented by a matrix

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Summary so far: A linear system can be represented as $\mathbf{y} = A\mathbf{x}$

where $\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \end{bmatrix}$ $\mathbf{y} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{bmatrix}$

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But, our system is LTI, not only linear,
so this imposes restrictions on A
i.e., A must have a special structure

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A linear system can be represented as $\mathbf{y} = A\mathbf{x}$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \end{bmatrix}$$

Let us understand this special structure

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Assume $x(n) = \delta(n)$

Let us understand this special structure

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Assume $x(n) = \delta(n)$

The output must be the first column of A

Let us understand this special structure

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Assume $x(n) = \delta(n-1)$

Let us understand this special structure

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A linear system can be represented as $\mathbf{y} = A\mathbf{x}$

$$\begin{bmatrix} A_{12} \\ A_{22} \\ A_{32} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ \boxed{1} \\ 0 \\ \vdots \end{bmatrix}$$

Assume $x(n) = \delta(n-1)$

The output must be the second column of A

Let us understand this special structure

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$$\begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} A_{12} \\ A_{22} \\ A_{32} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

Now recall that system is time-invariant
Implication?

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$$\begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

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Now recall that system is time-invariant

Implication?

The outputs should be The same, but one step delayed

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$$\begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Now recall that system is time-invariant

Implication?

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$$\boxed{\begin{bmatrix} 0 \\ A_{11} \\ A_{21} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{12} \\ A_{22} \\ A_{32} \\ \vdots \end{bmatrix}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

i.e.

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$$\begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Now recall that system is time-invariant

Implication?

The outputs should be The same, but one step delayed

$$\begin{bmatrix} 0 \\ A_{11} \\ A_{21} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{12} \\ A_{22} \\ A_{32} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

i.e. equal values along all diagonals

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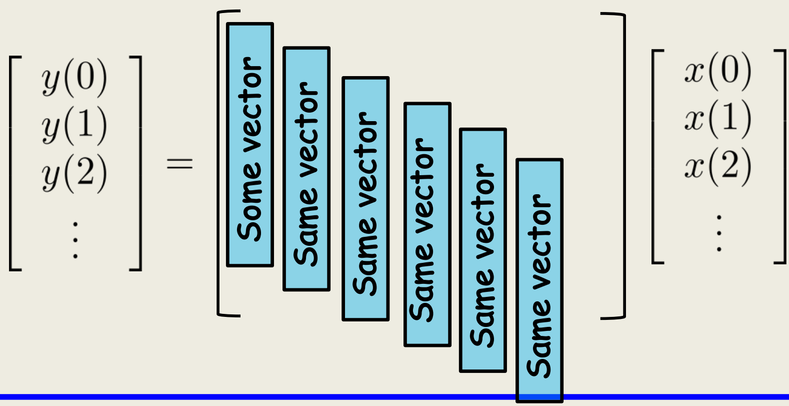
Summary. An LTI system is any discrete-time system that can be described by



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Summary. An LTI system is any discrete-time system that can be described by

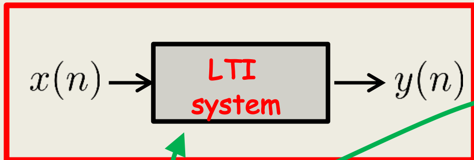
$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} \text{Some vector} & \text{Same vector} & \text{Same vector} & \text{Same vector} & \text{Same vector} & \text{Same vector} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \end{bmatrix}$$

Alternative formulation of course-goal:

Understand properties of a matrix of the form

$$\begin{bmatrix} \text{Some vector} & \text{Same vector} & \text{Same vector} & \text{Same vector} & \text{Same vector} & \text{Same vector} \end{bmatrix}$$

EITF75 Systems and Signals



The LTI system is FULLY characterized by one vector/sequence of numbers/discrete signal