Problem 1

Initial consideration

Since we turn on the circuit at $n=0$, the impulse response must be causal.

Consider now the case of 2 zeros and 1 pole:

$$H(z) = b_0 \frac{(z-z_1)(z-z_2)}{z-p_1} = b_0 z \frac{(1-z^{-1}z_1)(1-z^{-1}z_2)}{1-z^{-1}p_1}$$

The corresponding $h(n)$ is not causal, due to the presence of the term z. There are two interpretations, both of which were accepted at the exam:

(i) 1 pole and 2 zeros is not a valid setup

(ii) The text refers to non-trivial zero/poles, so that

1 pole and 2 zeros mean

$$H(z) = b_0 \frac{(z-z_1)(z-z_2)}{z\cdot(z-p_1)} = b_0 \frac{(1-z_1^{-1}z_1)(1-z^{-1}z_2)}{1-z_1^{-1}p_1} = b_0 \frac{B(z)}{A(z)}$$

In what follows I take the approach (ii)

a/ output without input $\Longrightarrow$ system not at rest.

$$Y^+(z) = \frac{1}{A(z)} N_0(z) + \underbrace{b_0 \frac{B(z)}{A(z)}}_{H(z)} X^+(z)$$

where $N_0(z)$ reflects the initial conditions

Assume 1 pole

$$\frac{N_0(z)}{1-z^{-1}p_1} = \frac{1}{1+z^{-1}}$$

follows that $\boxed{p_1 = -1}$ and $N_0(z) = 1$.

Now, $N_0(z) = a_1 z^{-1} y(-1) z^1 = a_1 y(-1)$

but $a_1 = -p_1$ so $N_0(z) = -p_1 y(-1) = 1$

Thus, $\boxed{y(-1) = 1}$

Assume 2 poles

$$\frac{N_0(z)}{(1 - z^{-1} p_1)(1 - z^{-1} p_2)} = \frac{1}{1 + z^{-1}}$$

follows that $\boxed{p_1 = -1}$ and $N_0(z) = 1 - z^{-1} p_2$

Now,

$$N_0(z) = a_1 z^{-1} y(-1) z^1 + a_2 z^{-2} \left[y(-1) z + y(-2) z^2 \right]$$

$$= a_1 y(-1) + a_2 y(-2) + a_2 y(-1) z^{-1}$$

So $a_2 y(-1) = -1 \implies \boxed{y(-1) = -\frac{1}{a_2}}$

$$\underline{y(-2)} = \frac{1 - a_1 y(-1)}{a_2} = \frac{1 + \frac{a_1}{a_2}}{a_2} = \frac{a_2 + a_1}{a_2^2}$$

Answer : we can say that :

1. System not at rest

2. pole at $z=-1$

3. if only one pole, then $y(-1)=1$

4 if two poles, we dont know the second pole since $y(-1)$ and $y(-2)$ must have been selected such that $N_0(z)$ cancels $1-z^{-1}p_2$

b/ Since we dont get any Information about $B(z)$ when $x(n)=0$, this is not possible.

c/ $$Y^+(z) = \frac{1}{1+z^{-1}} + H(z)\frac{1}{1-z^{-1}}$$ From a/

The text implies that

$H(z)\frac{1}{1-z^{-1}}$ must be of the form

$$-\frac{1}{1+z^{-1}} + \sum_k \frac{A_k}{1-z^{-1}q_k}$$

where

- the # of terms in the sum depends on the # poles in $H(z)$
- $|q_k| < 1$

$|q_k| < 1$ ensures that the time-domain version of $\sum_k \frac{A_k}{1-z^{-1}q_k}$ becomes 0 as $n \to \infty$.

Assume 1 pole and 1 zero

$$H(z) = b_0 \frac{1 - z^{-1} z_1}{1 - z^{-1} p_1}$$

$$H(z) \frac{1}{1 - z^{-1}} = b_0 \frac{1 - z^{-1} z_1}{1 - z^{-1} p_1} \cdot \frac{1}{1 - z^{-1}}$$

After PFE we get $\frac{A_1}{1 - z^{-1} p_1} + \frac{A_2}{1 - z^{-1}}$

it follows that $p_1 = -1$, $A_1 = -1$ and $A_2 = 0$

Now, $A_2 = 0$ means that the term $1 - z^{-1}$ is being cancelled by a term in the numerator.

Therefore, it follows that $\boxed{p_1 = -1}$

and $b_0 (1 - z^{-1} z_1) = -(1 - z^{-1})$

so $\boxed{z_1 = 1}$ and $\boxed{b_0 = -1}$

Assume 1 pole and 2 zeros

$$H(z) = b_0 \frac{(1 - z^{-1} z_1)(1 - z^{-1} z_2)}{1 - z^{-1} p_1}$$

Again, $p_1 = -1$, $z_1 = 1$, $b_0 = -1$. Then $z_2 = 0$

so that $H(z) = b_0 \frac{(1 - z^{-1} z_1)}{1 - z^{-1} p_1}$ which has 1 pole and 1 poles

Therefore 1 pole and $\underbrace{2}_{\text{non-trivial}}$ zeros Not possible

Assume 2 poles and 1 zero

we have $H(z) = \dfrac{(z-z_1)}{(z-p_1)(z-p_2)} b_0$

$$= b_0 \cdot z^{-1} \frac{1-z^{-1}z_1}{(1-z^{-1}p_1)(1-z^{-1}p_2)}$$

So $H(z) \dfrac{1}{1-z^{-1}} = z^{-1} b_0 \dfrac{1-z^{-1}z_1}{(1-z^{-1}p_1)(1-z^{-1}p_2)} \cdot \dfrac{1}{1-z^{-1}}$

After PFE, we get

$$H(z) = b_0 \left[\frac{A_1}{1-z^{-1}p_1} + \frac{A_2}{1-z^{-1}p_2} + \frac{A_3}{1-z^{-1}} \right]$$

Again we must have

$p_1 = -1$, $b_0 A_1 = -1$, $A_3 = 0$ and $|p_2| < 1$

Then A_2 is arbitrary

$A_3 = 0 \implies z_1 = 1$

After the PFE, we can adjust b_0 so that $b_0 A_1 = -1$

Summary: one pole at $z=-1$
one zero at $z=1$
The other pole arbitrary, but $|p_2|<1$
b_0 adjusted so that $b_0 A_1 = -1$

Assume 2 poles and 2 zeros

Again, after PFE, we get

$$H(z) = b_0 \left[\frac{A_1}{1-z^{-1}p_1} + \frac{A_2}{1-z^{-1}p_2} + \frac{A_3}{1-z^{-1}} \right]$$

one pole at $z=-1$
one zero at $z=1$
The other pole arbitrary, but $|p_2|<1$
The other zero arbitrary, since
b_0 can be adjusted to satisfy $b_0 A_1 = -1$

Answer :

- with 1 pole, 2 non-trivial zeros is the same case as 1 zero.

- In all cases we must have 1 pole at $z=-1$ and 1 zero at $z=1$

- for 2 poles, the other pole must satisfy $|P_2|<1$

- for 2 poles and 2 zeros the other zero is arbitrary

- In all cases b_0 has been adjusted so that $y(n) \to 0$, $n \to \infty$

d/ The text implies that

$$H(z)\frac{1}{1-z^{-1}} = -\frac{1}{1+z^{-1}}$$

Assume 1 pole and 1 zero

by inspection, the answer in c/ results in $y(n)=0 \quad n\geq 0$

so 1 pole and 1 zero is an option for the circuit

Assume 1 pole and 2 zeros

The same case as 1 pole and 1 zero

Assume 2 poles and 1 zero

From c/

$$H(z)\frac{1}{1-z^{-1}} = z^{-1} b_0 \frac{1-z^{-1}z_1}{(1-z^{-1}p_1)(1-z^{-1}p_2)} \cdot \frac{1}{1-z^{-1}}$$

After PFE, we get

$$H(z) = b_0 \left[\frac{A_1}{1-z^{-1}p_1} + \frac{A_2}{1-z^{-1}p_2} + \frac{A_3}{1-z^{-1}} \right]$$

Now, both A_2 and A_3 must be 0, and this is not possible ($[1-z^{-1}z_1]$ can cancel either $[1-z^{-1}p_2]$ or $[1-z^{-1}]$)

Assume 2 poles and 1 zero

From c/

$$H(z)\frac{1}{1-z^{-1}} = z^{-1} b_0 \frac{(1-z^{-1}z_1)(1-z^{-1}z_2)}{(1-z^{-1}p_1)(1-z^{-1}p_2)} \cdot \frac{1}{1-z^{-1}}$$

After PFE, we get

$$H(z) = b_0 \left[\frac{A_1}{1-z^{-1}p_1} + \frac{A_2}{1-z^{-1}p_2} + \frac{A_3}{1-z^{-1}} \right]$$

Again $A_2 = A_3 = 0 \Rightarrow$

$z_1 = 1$, $z_2 = p_2$, $p_1 = -1$

But $z_2 = p_2$ means that there is only 1 pole and 1 zero

Answer

The circuit has
1 pole and 1 zero
(at $z=-1$) (at $z=1$)

Problem 2

$$Y(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{2}z^{-1}} + z^{-1}\frac{1}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{z+1}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{2}}$$

$$= \frac{(z+1)(z+\frac{1}{2}) + z(z-\frac{1}{2})}{(z-\frac{1}{2})(z+\frac{1}{2})}$$

$$= \frac{z^2 + \frac{3}{2}z + \frac{1}{2} + z^2 - z/2}{(z-\frac{1}{2})(z+\frac{1}{2})}$$

$$= \frac{2z^2 + z + \frac{1}{2}}{(z-\frac{1}{2})(z+\frac{1}{2})}$$

$$H(z) = \frac{2z^2 + z + \frac{1}{2}}{z^2 + \frac{1}{2}z - \frac{1}{4}}$$

$$X(z) = \frac{Y(z)}{H(z)} = \frac{\frac{2z^2 + z + \frac{1}{2}}{(z - \frac{1}{2})(z + \frac{1}{2})}}{\frac{2z^2 + z + \frac{1}{2}}{z^2 + \frac{1}{2}z - \frac{1}{4}}}$$

$$= \frac{z^2 + \frac{1}{2}z - \frac{1}{4}}{(z - \frac{1}{2})(z + \frac{1}{2})} = \frac{z^2 + \frac{1}{2}z - \frac{1}{4}}{z^2 - \frac{1}{4}} =$$

$$= 1 + \frac{1}{2}\frac{z}{(z - \frac{1}{2})(z + \frac{1}{2})}$$

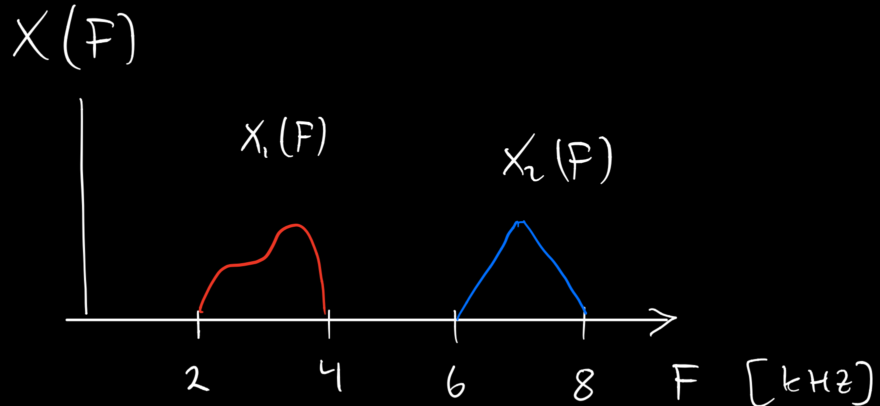
$$= 1 + \frac{1}{2}\left[\frac{A}{z - \frac{1}{2}} + \frac{B}{z + \frac{1}{2}}\right] = \left[\begin{matrix} A = \frac{1}{2} \\ B = \frac{1}{2} \end{matrix}\right]$$

$$= 1 + \frac{1}{4}z\left[\frac{z^{-1}}{1 - z^{-1}\frac{1}{2}} + \frac{z^{-1}}{1 + z^{-1}\frac{1}{2}}\right]$$

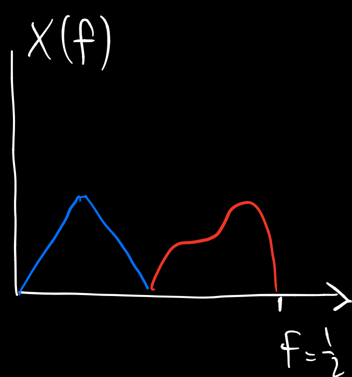
$$\boxed{x(n) = \delta(n) + \frac{1}{4}\left(-\frac{1}{2}\right)^{n-1} u(n-1) + \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u(n-1)}$$

Problem 3

a)



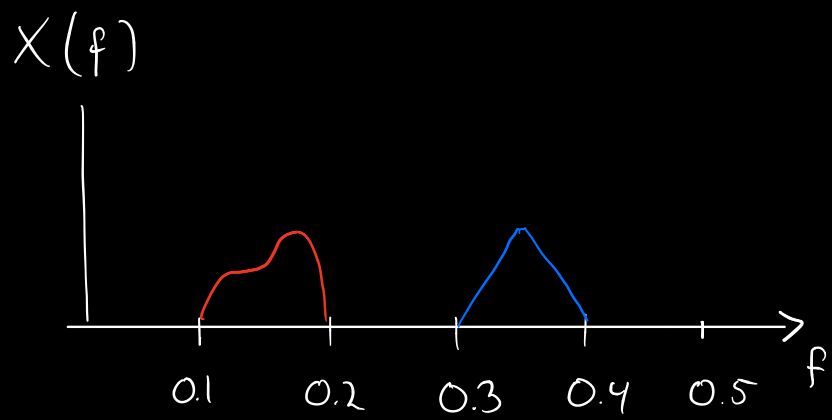
For $F_s = 8000$ Hz we get



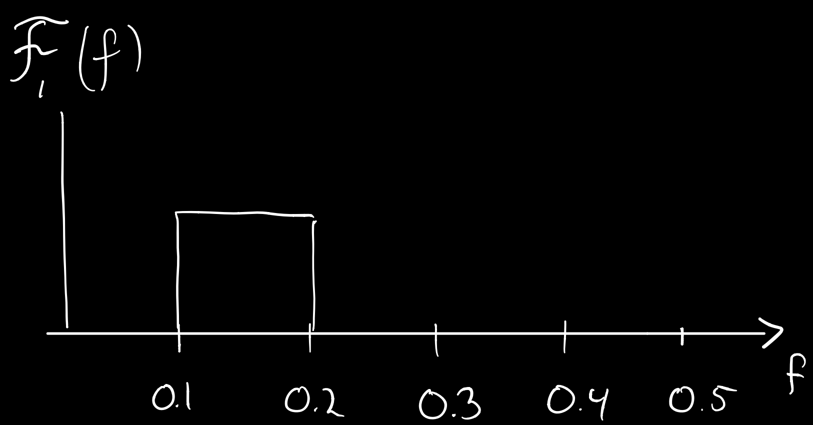
Possible to recover both $x_1(n)$ and $x_2(n)$

No smaller F_s exists.

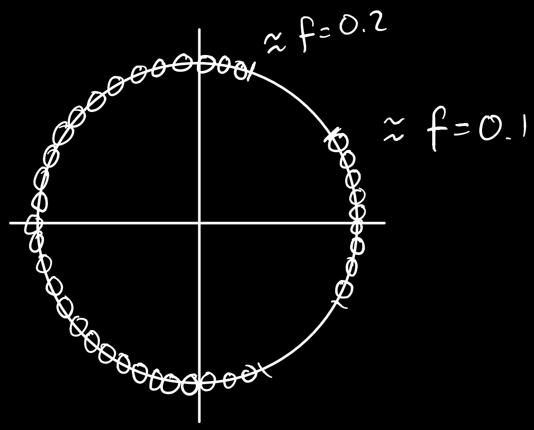
b/ $F_s = 20000\ Hz \implies$



$\implies$

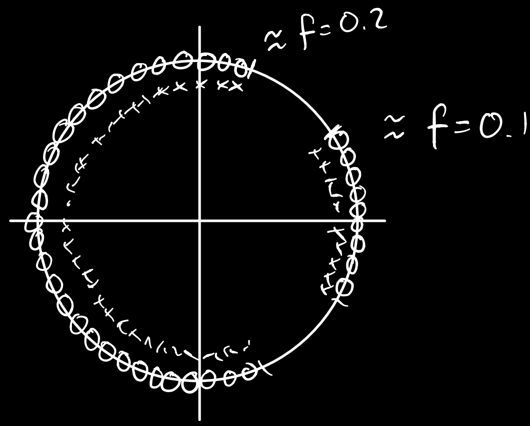


Alt 1



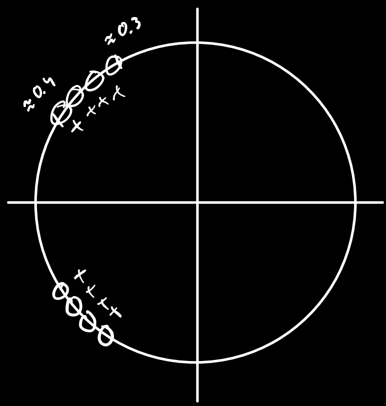
As we have seen, this is not very accurate <u>but sufficient for full points</u>

Alt 2

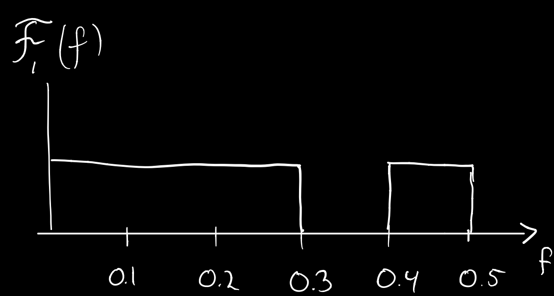


We have seen that this is <u>much</u> better

Alt 3



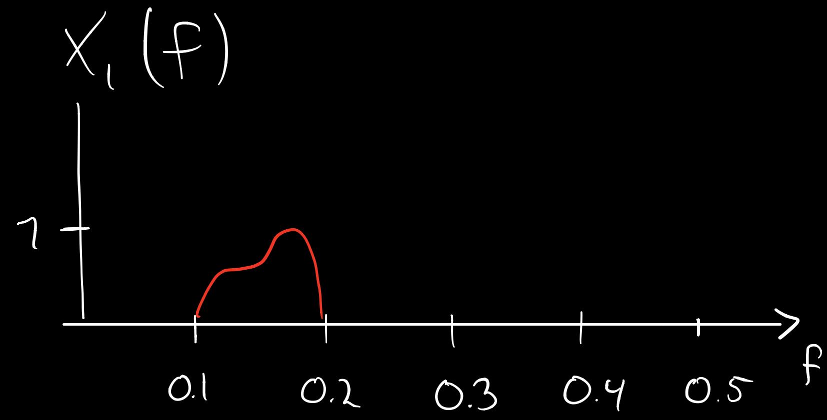
Actually enough to filter away $X_2(f)$



c/

$$x(t) \longrightarrow \boxed{\text{sampling}} \longrightarrow \boxed{\mathcal{F}_1(f)} \longrightarrow x_1(n)$$

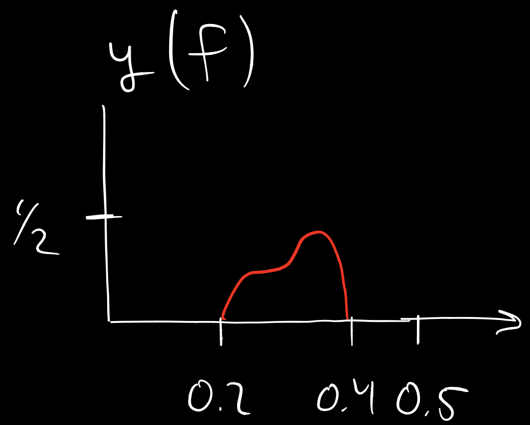
$F_s = 20000$



Down sample with a factor 2

$$x_1(n) \longrightarrow \boxed{\downarrow 2} \longrightarrow y(n)$$

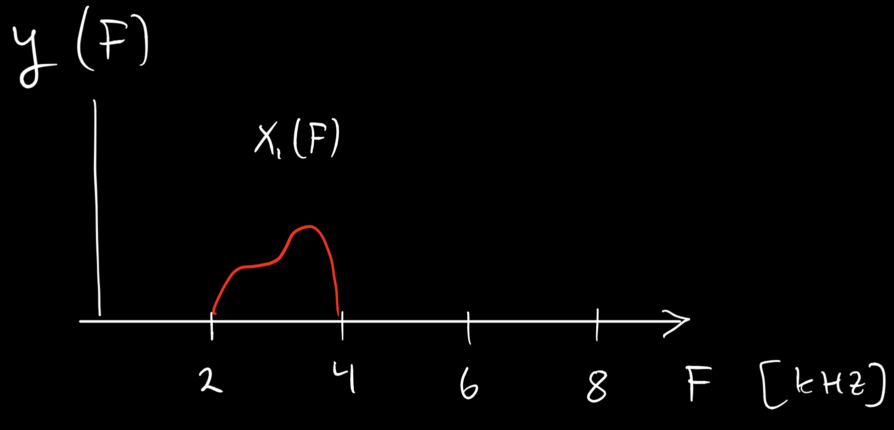
Follows that



Reconstruct with $F_s = 10000$

$$y(n) \longrightarrow \boxed{\text{reconst}} \longrightarrow y(F)$$

$F_s = 10000$



$$y(F) = X_1(F)$$

Problem 4

a/ $\underline{H_1(z)} = 1 + z^{-D} + z^{-2D}$

$$= \frac{1}{z^{2D}} \left[z^{2D} + z^{D} + 1 \right]$$

$$x = z^D \longrightarrow \frac{1}{x^2} \left[x^2 + x + 1 \right]$$

zeros (in terms of x):

$$x_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i = e^{\pm i \frac{2\pi}{3}}$$

zeros in z:

$$z_{1,2,\dots,D} = e^{i\left[\frac{2\pi}{D} + \frac{2\pi}{D} k\right]} \qquad 0 \le k \le D-1$$

$$z_{D+1,\dots,2D} = e^{i\left[-\frac{2\pi}{D} + \frac{2\pi}{D} k\right]} \qquad 0 \le k \le D-1$$

poles : $2D$ poles at $z=0$

$$\underline{H_2(z)} = \frac{1}{1 - z^{-D}\alpha} = \frac{z^D}{z^D - \alpha} = \left[x = z^D \right] = \frac{x}{x - \alpha}$$

poles (in x) : $x = \alpha$

poles (in z): $z_{1,\dots,D} = \sqrt[D]{\alpha} \cdot e^{i \frac{2\pi}{D} k} \qquad 0 \le k \le D-1$

D zeros at $z=0$

b/ $\frac{D}{F_s} = \frac{1}{2} \iff 1000 = \frac{F_s}{2}$

$\Rightarrow F_s = 2000$ Hz

c/ All poles inside the unit circle

$\iff |\alpha| < 1$

d/ To invert $\frac{1}{1-z^{-D}\alpha}$ we can invert

$\frac{1}{1-z^{-1}\alpha}$ and then plug in $D-1$

zeros between the samples. However,

the number of echos stays the same

<u>Inversion of</u> $\frac{1}{1-z^{-1}\alpha} \rightarrow \alpha^n u(n)$

$10\log_{10}(\alpha^{2n}) > -20 \iff 2n\log_{10}(\alpha) > -2$

$n > -1/\log_{10}(\alpha)$

So, we can hear $\left\lfloor \frac{-1}{\log_{10}(\alpha)} \right\rfloor$ echos.

Problem 5

a/ This is true if the circulant convolution of $x(n)$ and $h(n)$ equals the linear one.

The length of the linear convolution is $L + L_h - 1$ samples, where L_h is the length of $h(n)$

Thus $\underline{L + L_h - 1 \leq N}$

b/ essentially the same problem.

Length of the linear convolution is $2L-1$, so $2L-1 \leq N$

c/ $X_k = X\left(\omega \mid \omega = \frac{2\pi}{5} k\right)$

Therefore, $X\left(\frac{3\pi}{5}\right)$ is not a value of the DTFT that has been computed by the DFT.

However, since $L = N$, the DFT contains as much information as the DTFT.

Since X_k is a sampled version of $X(\omega)$, and the sampling rate is sufficient, we can reconstrut $X(\omega)$ through normal sinc-based reconstruction. So, yes possible.

d/ The DFT is

1/ discrete and therefore preferred by a computer

2/ Fast to compute

3/ Contains as much information as the DTFT (if $L = N$)

e/ Multiplication in the DFT domain means circular convolution in the time domain. Question is to find what the sequence $h(n)$ is.

we can recognize H_k

as the samples of the DTFT of $h(n) = \alpha^n \cos(\omega_0 n)\, u(n)$.

this sequence is however not of finite length.

We have seen that sampling in frequency implies aliasing in time.

So, the effect on $y(n)$ is

$$y(n) \circledast \tilde{h}(n)$$

where $\tilde{h}(n) = \sum_{k=-\infty}^{\infty} h(n-kN)$