$x(t)$ cont. a-periodic

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi F t}\, dt$$

Transform pair

$(F\ Hz)$

verify to prove transform pair

$$x(t) = \int_{-\infty}^{\infty} X(F)\, e^{i2\pi F t}\, dF = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\, e^{-i2\pi F \tau}\, d\tau\ e^{i2\pi F t}\, dF$$

def $\delta(\tau)$

$$= \int_{-\infty}^{\infty} x(\tau) \underbrace{\int_{-\infty} e^{i2\pi F(t-\tau)}\, dF}_{\delta(t-\tau)}\, d\tau = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau = x(t)$$

change order of integration

correct

Recall

Cont. - periodic

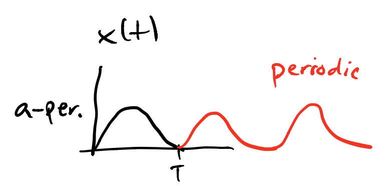
$$c_k = \frac{1}{T} \int_{0}^{T} x(t)\, e^{-i2\pi k \frac{t}{T}}$$

cont - aperiodic

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi F t}\, dt$$

$$X\left(k \cdot \frac{1}{T}\right) = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi k \frac{t}{T}}\, dt = \int_{0}^{T} x(t)\, e^{-i2\pi k \frac{t}{T}}\, dt$$

$$= T c_k$$



## disc. periodic $\qquad x(n) \qquad$ period $N$

$$\boxed{\begin{aligned} c_\ell &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-\frac{i2\pi \ell n}{N}} \\ x(n) &= \sum_{k=0}^{N-1} c_k\, e^{\frac{i2\pi k n}{N}} \end{aligned}}$$

$c_\ell$ is periodic with period $N$

$\Leftrightarrow$

$$c_{\ell+Nk} = c_\ell \quad \forall k$$

$\infty$

verify to prove transform pair

$$\frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-\frac{i2\pi \ell n}{N}} \overset{?}{=} c_\ell$$

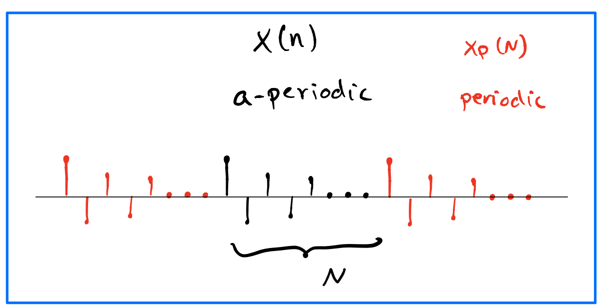
$$\frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-\frac{i2\pi \ell n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c_k\, e^{\frac{i2\pi (k-\ell) n}{N}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} c_k \underbrace{\sum_{n=0}^{N-1} e^{\frac{i2\pi (k-\ell) n}{N}}}_{\text{geom series}} \qquad \text{geom series} = \begin{cases} N, & k-\ell = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$$

$$= \frac{1}{N} c_k$$

Correct!

## aper. disc. (Main case)




$$\left.\begin{aligned} X(f) &= \sum_{n=0}^{N-1} x(n)\, e^{-i2\pi n f} \\ c_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-\frac{i2\pi n k}{N}} \end{aligned}\right\} \quad X(k/N) = N c_k$$

$X(f)$ periodic with period 1

$$x_p(n) = \frac{1}{N} \sum x\left(\frac{k}{N}\right) e^{\frac{i2\pi n k}{N}}$$

$$= \left[\frac{1}{N} = \Delta f\right] = \Delta f \sum x(k\Delta f)\, e^{i2\pi n k \Delta f}$$

$$x(n) = \lim_{N \to \infty} x_p(n) = \lim_{\Delta f \to 0} x_p(n) = \lim_{\Delta f \to 0} \underbrace{\Delta f \sum_{k=0}^{N-1} x(k\Delta f)\, e^{i2\pi n k \Delta f}}_{\text{def of an integral}}$$

$$= \int_0^1 x(f)\, e^{i2\pi n f}\, df$$

Conclusion

$$\boxed{\begin{aligned} X(f) &= \sum_{n=0}^{N-1} x(n)\, e^{-i2\pi n f} \\ x(n) &= \int_0^1 X(f)\, e^{i2\pi n f}\, df \end{aligned}}$$

Fourier transform pair

## Properties of discrete-time a-periodic Fourier transforms

$$\boxed{X(f) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-i2\pi n f}}$$

periodic period 1

Discrete-time-Fourier-Transform (DTFT)

z- transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$X(f) = X(z \,|\, z = e^{i2\pi f})$$

Important

DTFT is a z-transform evaluated at the unit circle

c.f. Laplace - Fourier

## Time delay

① $\begin{cases} x(n) \longleftrightarrow X(f) \\ x(n-n_0) \longleftrightarrow e^{-i2\pi f n_0} X(f) \end{cases}$

c.f. $x(n) \longleftrightarrow X(z)$

$x(n-n_0) \longleftrightarrow z^{-n_0} X(z)$

## convolution

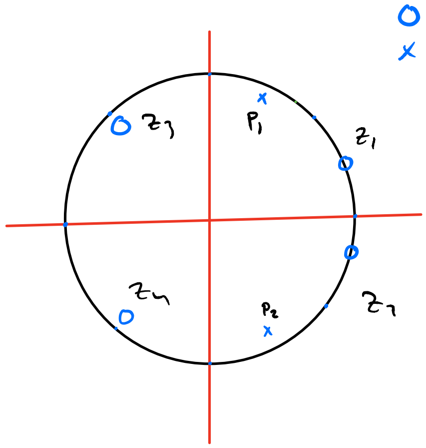
② $y(n) = x(n) * h(n) \longleftrightarrow Y(f) = X(f) \cdot H(f)$

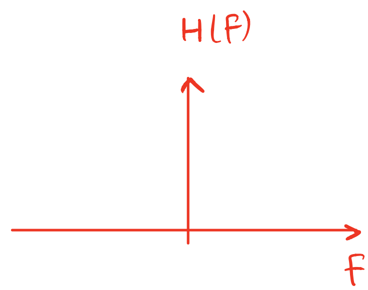
# Ex

Given: pole-zero diagram of $H(z)$

To find: DTFT $H(f)$

O : zeros
x : poles




$$\underset{\text{DTFT}}{X(f)} = \underset{\text{z-transf.}}{X}\left(e^{i2\pi f}\right)$$

| f | z |
|---|---|
| $-\frac{1}{2}$ | $-1$ |
| $-\frac{1}{4}$ | $-i$ |
| $0$ | $1$ |
| $1/8$ | $\frac{1+i}{\sqrt{2}}$ |
| $\frac{1}{4}$ | $i$ |
| $3/8$ | $\frac{-1+i}{\sqrt{2}}$ |
| $\frac{1}{2}$ | $-1$ |
| $\frac{3}{4}$ | $-i$ |

etc

$$H(z) = \frac{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}{(z-p_1)(z-p_2)}$$

$$H(f=0) = H(z=1) = \frac{(1-z_1)(1-z_2)(1-z_3)(1-z_4)}{(1-p_1)(1-p_2)}$$

... unclear value ...

$$H(f\approx \tfrac{1}{16}) = H(z=z_1) = \frac{\overbrace{(z_1-z_1)}^{=0}(z_1-z_2)(z_1-z_3)(z_1-z_4)}{(z_1-p_1)(z_1-p_2)} = 0$$

from figure

$$H(f\approx \tfrac{3}{16}) = H(\underbrace{z=e^{\frac{i2\pi 3}{16}}}_{\tilde{z}}) = \frac{(\tilde{z}-z_1)(\tilde{z}-z_2)(\tilde{z}-z_3)(\tilde{z}-z_4)}{\underbrace{(\tilde{z}-p_1)}_{\approx 0}(\tilde{z}-p_2)} \gg 1$$

from figure

Big!

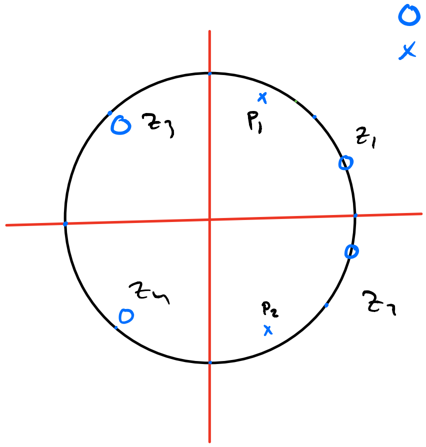
$$H(f\approx \tfrac{3}{8}) = H(\underbrace{z=e^{\frac{i2\pi 3}{8}}}_{\tilde{z}}) = \frac{(\tilde{z}-z_1)(\tilde{z}-z_2)\overbrace{(\tilde{z}-z_3)}^{\approx 0}(\tilde{z}-z_4)}{(\tilde{z}-p_1)(\tilde{z}-p_2)} \approx 0$$

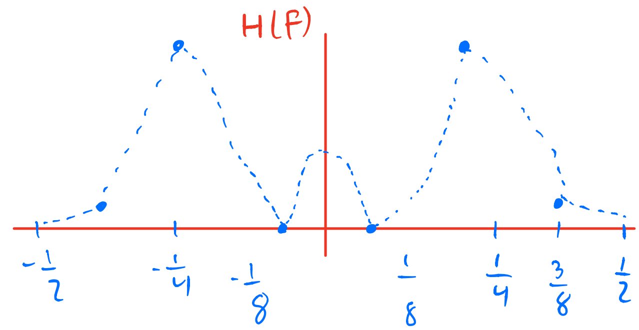
from figure

small but not zero

## Conclusion

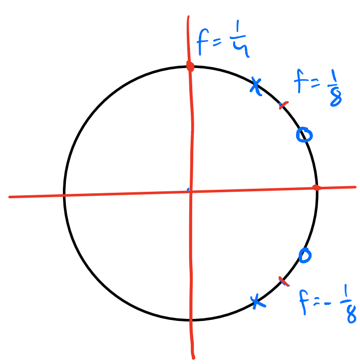
O : zeros
x : poles

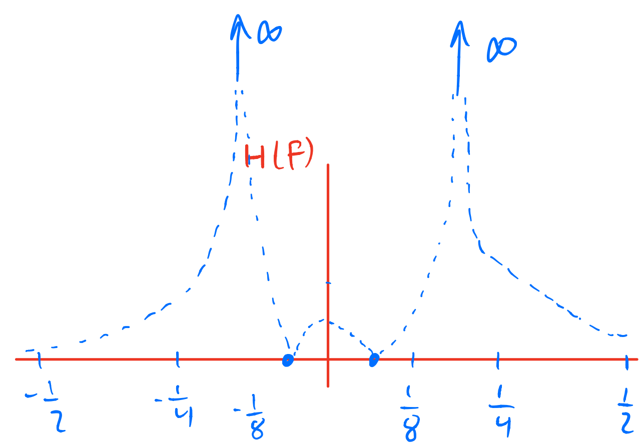



periodically extended outside $-\frac{1}{2} \leq f \leq \frac{1}{2}$

## Ex



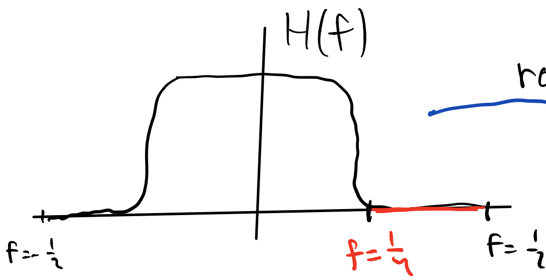




periodically extended outside $-\frac{1}{2} \leq f \leq \frac{1}{2}$

# Filters

$x(n) \longrightarrow \boxed{h(n)} \longrightarrow$ $y(n) = x(n) * h(n)$

$X(f) \longrightarrow \boxed{H(f)} \longrightarrow$ $Y(f) = X(f) \cdot H(f)$




roughly

Low-pass filter