## 1

$$y(n) + \frac{1}{2} y(n-1) = 3x(n) - x(n-1)$$

$$\Longleftrightarrow$$

1

$$Y(z)\left[1 + \frac{1}{2} z^{-1}\right] = X(z)\left[3 - z^{-1}\right]$$

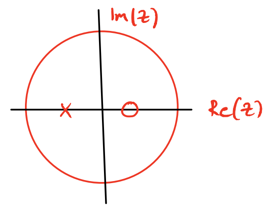
$$Y(z) = \underbrace{\frac{3 - z^{-1}}{1 + \frac{1}{2} z^{-1}}}_{H(z)} X(z)$$

2

$$H(z) = \frac{3z - 1}{z + \frac{1}{2}} = 3 \cdot \frac{z - \frac{1}{3}}{z + \frac{1}{2}}$$

zero: $z = \frac{1}{3}$

pole: $z = -\frac{1}{2}$




3

$$\frac{3 - z^{-1}}{1 + \frac{1}{2} z^{-1}} = \frac{3}{1 + \frac{1}{2} z^{-1}} - z^{-1} \frac{1}{1 + \frac{1}{2} z^{-1}}$$

$$\Longleftrightarrow$$

$$h(n) = 3 \cdot u(n) \left(-\frac{1}{2}\right)^n - u(n-1) \cdot \left(-\frac{1}{2}\right)^{n-1} = \ldots = \underline{5\left(-\frac{1}{2}\right)^n u(n) - 2\,\delta(n)}$$

4

$$H(f) = H(z)\Big|_{z = e^{i2\pi f}} = \frac{3 - e^{-i2\pi f}}{1 + \frac{1}{2} e^{-i2\pi f}} = \frac{3 - e^{-i2\pi f}}{1 + \frac{1}{2} e^{-i2\pi f}} \cdot \frac{1 + \frac{1}{2} e^{i2\pi f}}{1 + \frac{1}{2} e^{i2\pi f}}$$

$$= \frac{2.5 + \frac{3}{2} e^{i2\pi f} - e^{-i2\pi f}}{\frac{5}{4} + \frac{1}{2}\left[e^{i2\pi f} + e^{-i2\pi f}\right]} = \frac{2.5 + \frac{3}{2}\cos(2\pi f) + \frac{3}{2} i \sin(2\pi f) - \cos(2\pi f) + i \sin(2\pi f)}{\frac{5}{4} + \cos(2\pi f)}$$

$$= \boxed{\frac{2.5 + \frac{1}{2}\cos(2\pi f)}{\frac{5}{4} + \cos(2\pi f)} + i \, \frac{\frac{5}{2}\sin(2\pi f)}{\frac{5}{4} + \cos(2\pi f)}}$$

5 $\quad x(n) = [1\ 1\ {-1}\ 1] \longleftrightarrow X(z) = 1 + z^{-1} - z^{-2} + z^{-3}$

$$Y(z) = \frac{(3-z^{-1})(1+z^{-1}-z^{-2}+z^{-3})}{1+\frac{1}{2}z^{-1}} \cdot \underbrace{\frac{z^4}{z^4}}_{=1} = \frac{3z^4+2z^3-4z^2+4z-1}{z+\frac{1}{2}} \cdot z^{-3}$$

Long divison

$$\begin{array}{l} 3z^3 + \frac{1}{2}z^2 - \frac{17}{4}z + \frac{49}{8} \\ \hline 3z^4 + 2z^3 - 4z^2 + 4z - 1 \;\big|\; z+\frac{1}{2} \\ -\ 3z^4 + \frac{3}{2}z^3 \\ \hline \quad \frac{1}{2}z^3 - 4z^2 + 4z - 1 \\ \quad -\ \frac{1}{2}z^3 + \frac{1}{4}z^2 \\ \hline \qquad -\frac{17}{4}z^2 + 4z - 1 \\ \qquad -\ -\frac{17}{4}z^2 - \frac{17}{8}z \\ \hline \qquad\quad \frac{49}{8}z - 1 \\ \qquad\quad -\ \frac{49}{8} + \frac{49}{16} \\ \hline \qquad\qquad -1 - \frac{49}{16} = -\frac{65}{16} \end{array}$$

$\frac{8}{8}4 + \frac{17}{8}$

$\frac{32+17}{8} =$

$$\Longrightarrow \frac{3z^4+2z^3-4z^2+4z-1}{z+\frac{1}{2}} = 3z^3 + \frac{1}{2}z^2 - \frac{17}{4}z + \frac{49}{8} - \frac{65}{16} \cdot \frac{1}{z+\frac{1}{2}}$$

$$\Longrightarrow Y(z) = 3 + \frac{1}{2}z^{-1} - \frac{17}{4}z^{-2} + \frac{49}{8}z^{-3} - \frac{65}{16}z^{-3}\frac{1}{z+\frac{1}{2}}$$

only changes

$$= 3 + \frac{1}{2}z^{-1} - \frac{17}{4}z^{-2} + \frac{49}{8}z^{-3} - \frac{65}{16}\frac{z^{-4}}{z^{-1}}\frac{1}{z+\frac{1}{2}}$$

$$= 3 + \frac{1}{2}z^{-1} - \frac{17}{4}z^{-2} + \frac{49}{8}z^{-3} - \frac{65}{16}z^{-4}\frac{1}{1+\frac{1}{2}z^{-1}}$$

$$\Longrightarrow \boxed{y(n) = 3\delta(n) + \frac{1}{2}\delta(n-1) - \frac{17}{4}\delta(n-2) + \frac{49}{8}\delta(n-3) - \frac{65}{16}\left(-\frac{1}{2}\right)^{n-4}u(n-4)}$$

other solutions also possible

6 $\quad X(z) = \frac{1}{1-z^{-1}}$ step in $\longrightarrow$ steady-state = scaled step

scaling = $H(0) = H(z)\big|_{z=1} = \frac{3-z^{-1}}{1+\frac{1}{2}z^{-1}}\bigg|_{z=1} = \frac{2}{\frac{3}{2}} = \frac{4}{3}$

$\uparrow$ DTFT

steady-state output : $u(n) \cdot \frac{4}{3}$

7 $\quad X(z) = \frac{1}{1-z^{-1}} \longrightarrow \quad Y(z) = \frac{3-z^{-1}}{1+\frac{1}{2}z^{-1}} \cdot \frac{1}{1-z^{-1}} \cdot \underbrace{\frac{z^2}{z^2}}_{=1}$

$$= z \, \frac{3z-1}{(z+\frac{1}{2})(z-1)}$$

Perform PFE

$$\frac{3z-1}{(z+\frac{1}{2})(z-1)} = \underbrace{\frac{A}{(z+\frac{1}{2})}}_{\text{Transient}} + \underbrace{\frac{B}{z-1}}_{\text{steady-state}}$$

Side comments

B must be 4/3 from 6

B not needed in 7

$$A = \frac{3z-1}{z-1}\bigg|_{z=-\frac{1}{2}} = \frac{-\frac{5}{2}}{-\frac{3}{2}} = \frac{5}{3}$$

$$Y_{tr.}(z) = \frac{5}{3} z \, \frac{1}{z+\frac{1}{2}} = \frac{5}{3} \frac{1}{z^{-1}} \frac{1}{z+\frac{1}{2}} = \frac{5}{3} \frac{1}{1+\frac{1}{2}z^{-1}}$$

$$y_{tr}(n) = \frac{5}{3} u(n) \left(-\frac{1}{2}\right)^n$$

**2** Since we know $Y(z) = G(z)X(z)$

it follows that $y(n) = g(n) * x(n)$.

To find $g(n)$ we input $x(n) = \delta(n)$.

$$\delta(n) \longrightarrow \boxed{\uparrow} \xrightarrow[\delta(n)]{} \boxed{h(n)} \xrightarrow[h(n)]{} \boxed{\downarrow} \longrightarrow g(n)$$

$g(n)$ is every second sample of $h(n)$, i.e.

$$g(n) = h(2n) = u(2n)\left(-\tfrac{1}{2}\right)^{2n} = u(n)\cdot\left(\tfrac{1}{4}\right)^{n}.$$

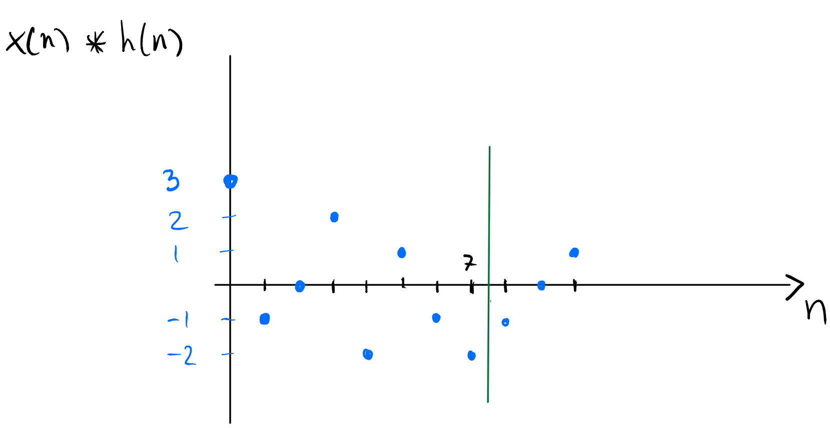
This means that $G(z) = \dfrac{1}{1-\frac{1}{4}z^{-1}}$

**3**

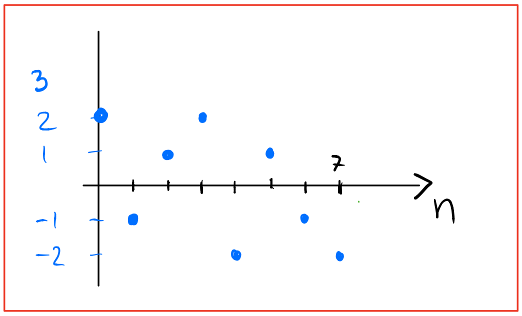
part A

The operations imply that $y(n)$ is a circular convolution.

we first find the normal convolution.



Therefore $y(n) = x(n) \circledast h(n)$




**part B** The operations imply aliasing in the time domain. (Lecture 10)

So, $$r(n) = \sum_{m=-\infty}^{\infty} x(n-4m)$$

$r(0) = x(0) + x(4) + x(8) = 3 + 1 + 0 = 4$

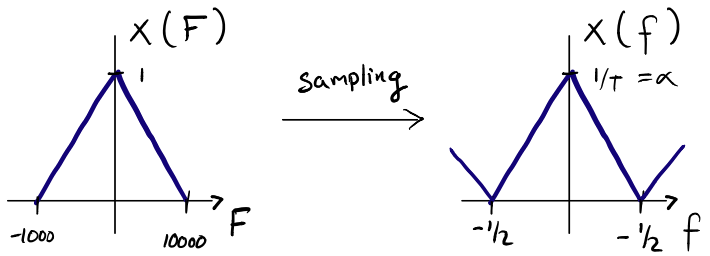
$r(1) = x(1) + x(5) = -1 + 0 = -1$

$r(2) = x(2) + x(6) = 0 - 1 = -1$

$r(3) = x(3) + x(7) = 2 + 0 = 2$

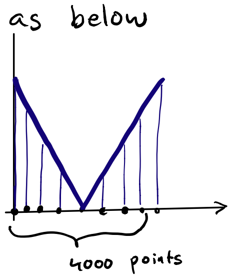
## 4

Assume

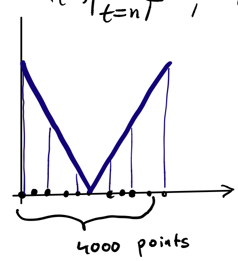



$x(t) \Longleftrightarrow X(F)$ $\quad$ $x(nT) = x(n) \Longleftrightarrow X(f)$

we would like to find a sequence with DFT as below




But $z(n)$ as proposed is equivalent to an expansion by 2 of the 2000-point DFT of $x(n) = x(t)\big|_{t=nT}$, i.e.,




$$z[k] = \begin{cases} X[k], & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

So, no $z(n)$ does <u>not</u> lead to $Z[k]$ being equal to $X[k]$ in (1)

5

$\boxed{\text{General principle: same frequency out as in.}}$

Therefore, $x_1(n) \leftrightarrow y_1(n)$

$x_2(n) \leftrightarrow y_2(n)$

$x_3(n) \leftrightarrow y_3(n)$ <u>or</u> $y_4(n)$

$y_5(n)$ not output for $x_1(n)$, $x_2(n)$ or $x_3(n)$

$$H(z) = \frac{1 + \frac{1}{2} z^{-1}}{1 + 1.12 z^{-1} + 0.64 z^{-2}}$$

$$x(n) = u(n) \cos(2\pi f n) \longrightarrow y(n) = |H(f)| \cos(2\pi f n + \Phi(f))$$

<u>$A_1$</u>

$$\boxed{A_1 =} \; H(z)\Big|_{z = e^{j2\pi \cdot 0}} = H(1) =$$

$$= \frac{\frac{3}{2}}{1 + 1.12 + 0.64} = \boxed{\frac{3}{2} \; \frac{1}{2.76}}$$

$\underline{A_2}$ $\boxed{A_2} = \left| H(z)\Big|_{z=e^{i2\pi\frac{1}{4}}} \right| =$

$= |H(i)| = \left| \dfrac{1+\frac{1}{2}i^{-1}}{1+1.12i^{-1}+0.64i^{-2}} \right| =$

$= \ldots \boxed{\approx 0.95}$

$\underline{y_3(n) \text{ or } y_4(n) \text{ ?}}$

Must check $\Phi(\frac{1}{2})$

$H(f=\frac{1}{2}) = H(z)\Big|_{z=e^{i\pi}} = H(z=-1) =$

$= \dfrac{1-\frac{1}{2}}{1-1.12+0.64} = \dfrac{1/2}{0.52} \approx 0.96$

so, $H(f=\frac{1}{2}) = 0.96\,e^{i2\pi\cdot 0}$

$\rightarrow \Phi(\frac{1}{2}) = 0 \rightarrow \boxed{y_4(n) \text{ cannot occur}}$

$\rightarrow \underline{A_3 = 0.96}$