EITF75 Systems and Signals

Summary of Course

Fredrik Rusek

EITF75 Systems and Signals

Sampling and reconstruction

EITF75 Systems and Signals

A/D and D/A

Signal $\xrightarrow{x(t)}$ sampling $\xrightarrow{x(n)}$

$X_a(F)$ $\qquad$ $X(f)$

Key step is to understand what X(f) looks like in terms of $X_a(F)$

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

If sampling is to sparse, there is aliasing.

We find X(f) by the "folding technique"

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Example: Folding

Signal

F_s = 4000 Hz

Folding

Step 1: Identify $F_s/2$

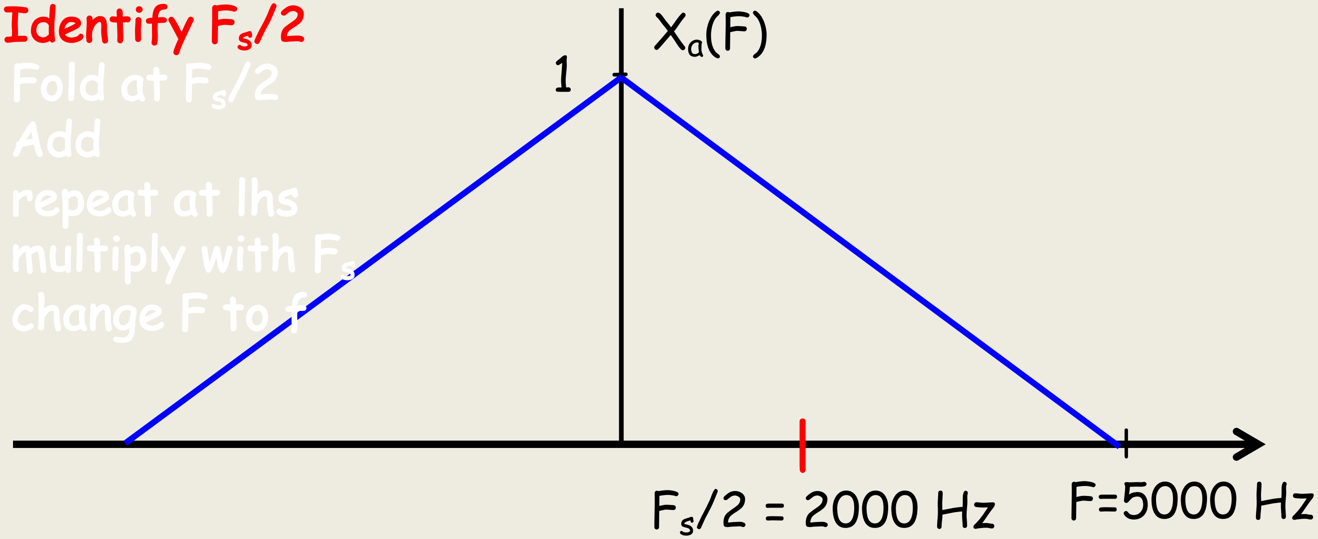
Step 2: Fold at $F_s/2$

Step 3: Add

Step 4: repeat at lhs

Step 5: multiply with F_s

Step 6: change F to f



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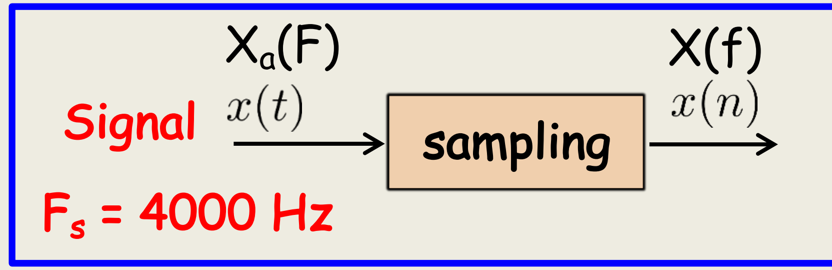

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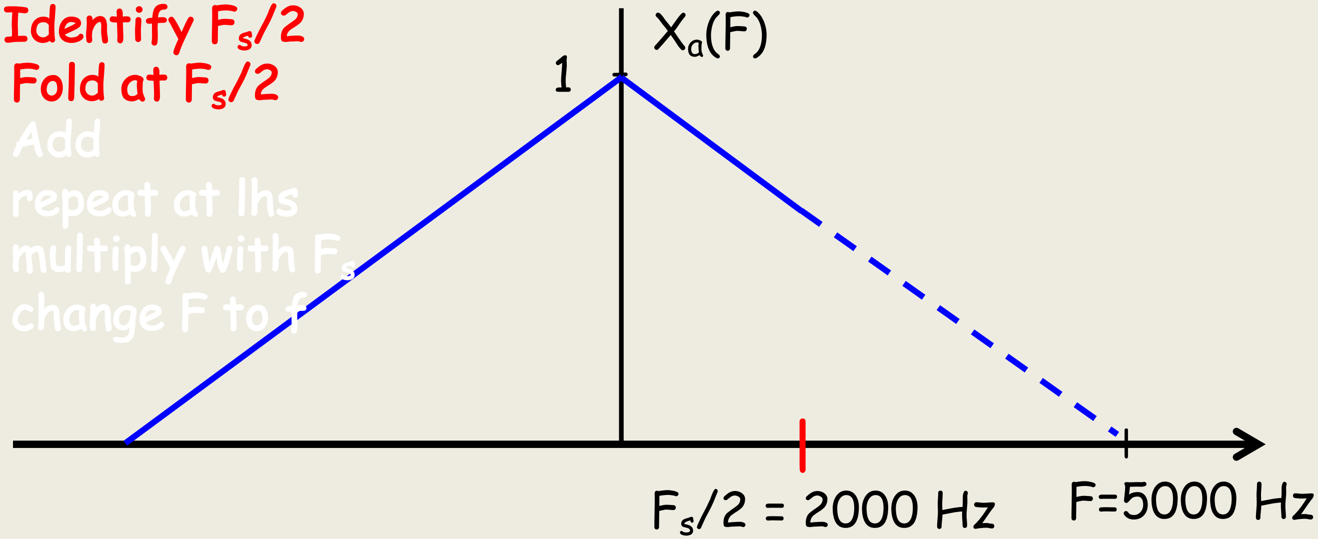

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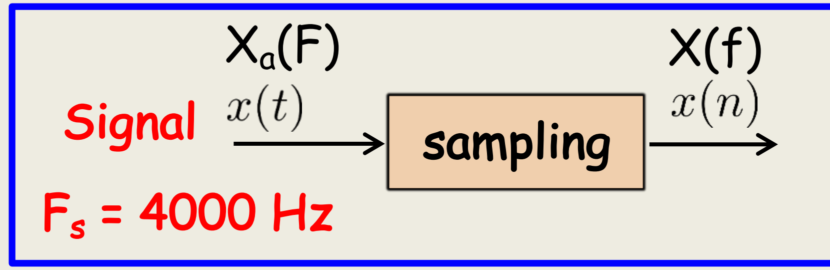

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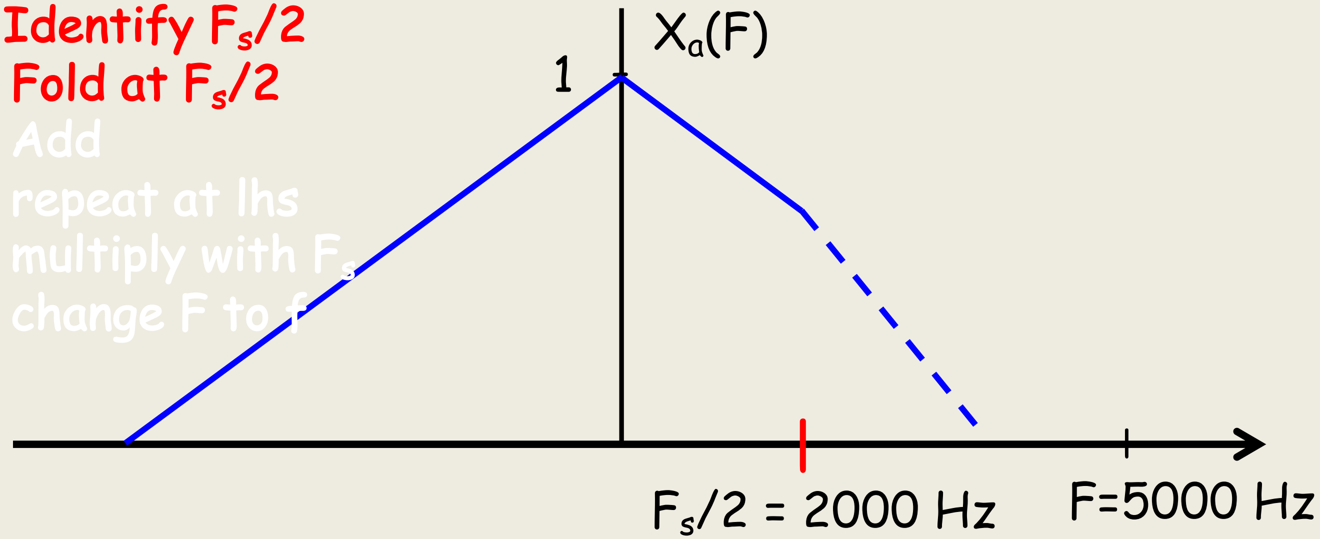

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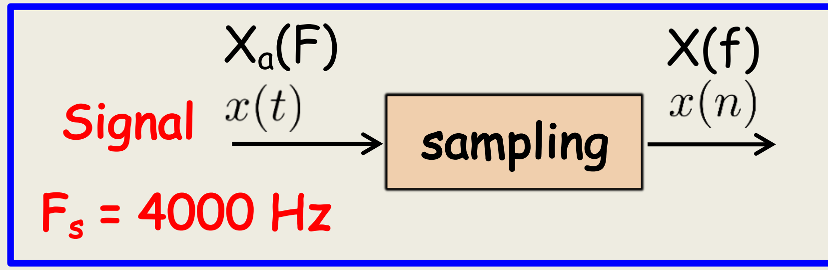

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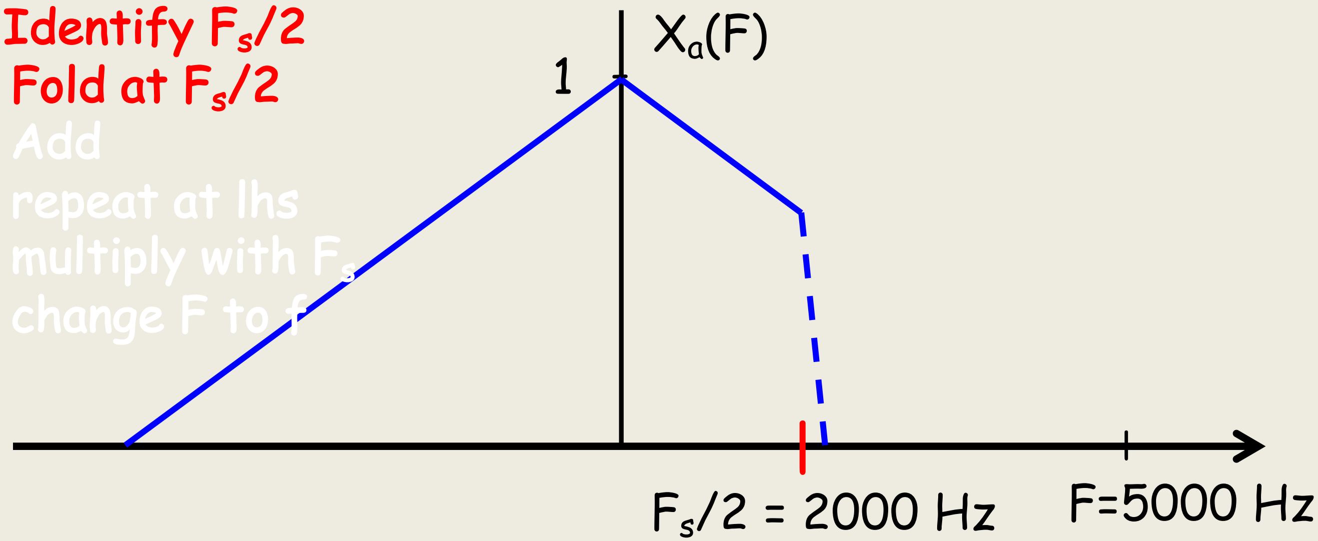

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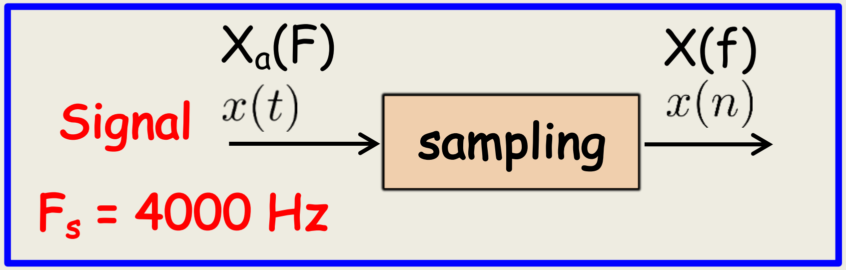

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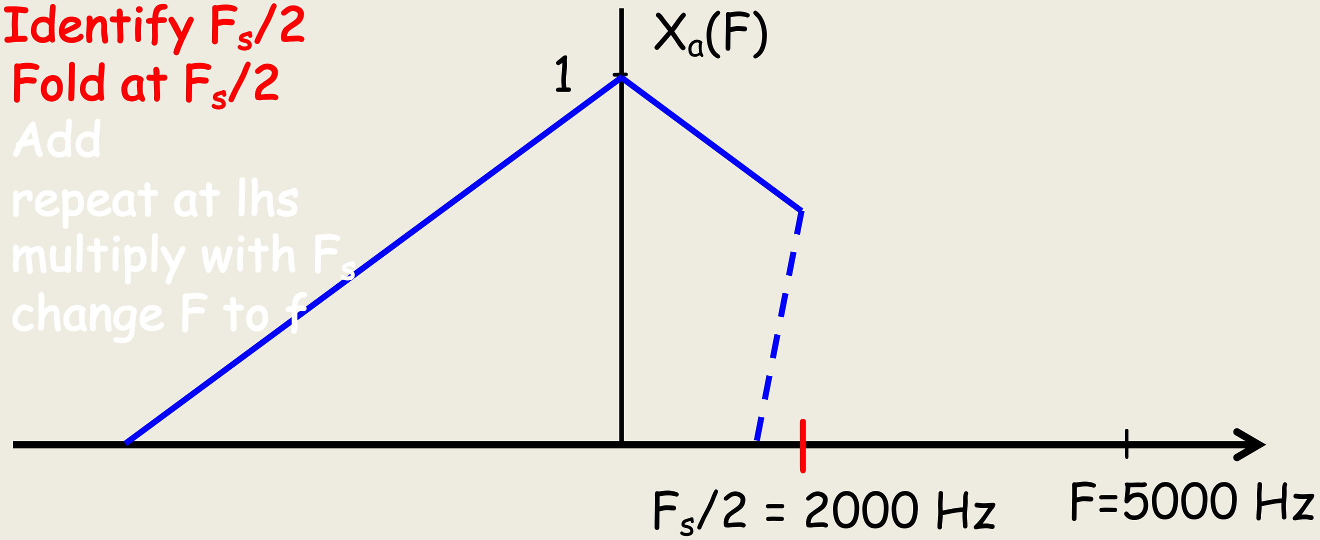

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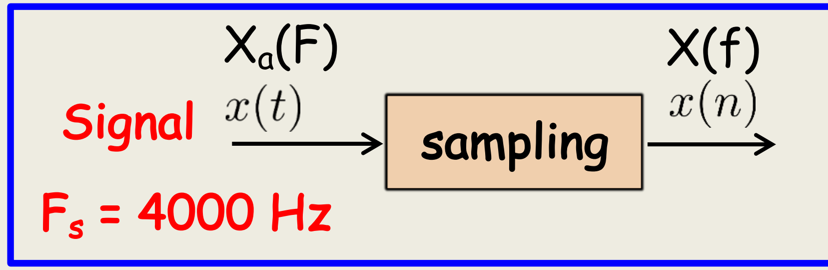

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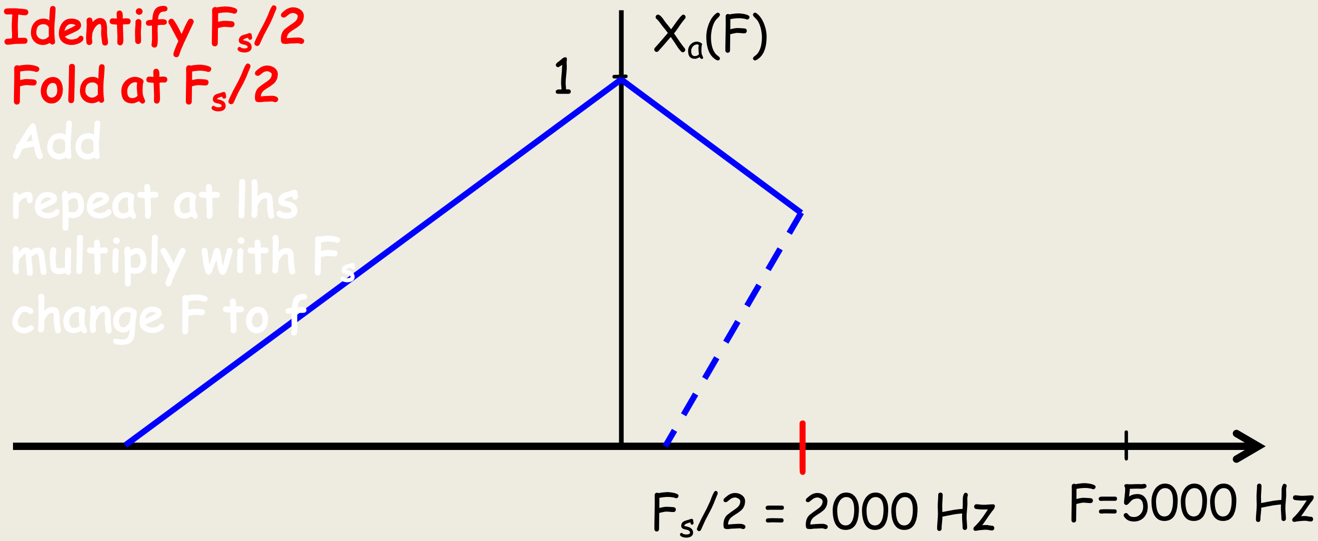

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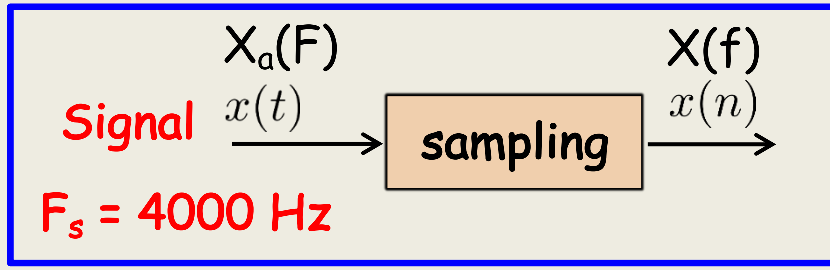

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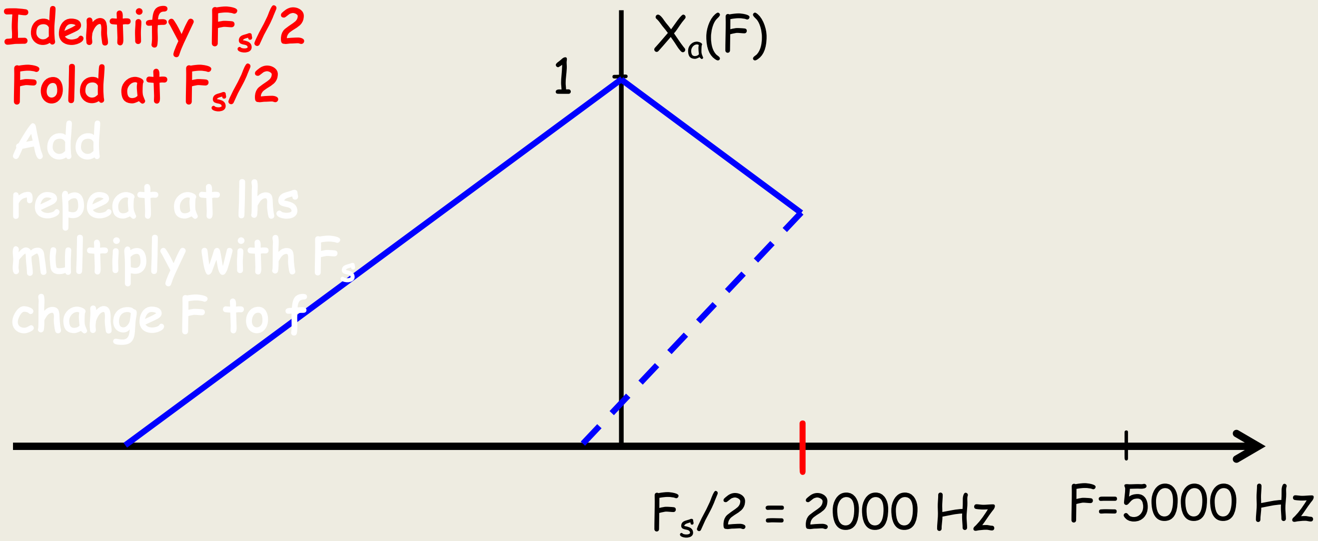

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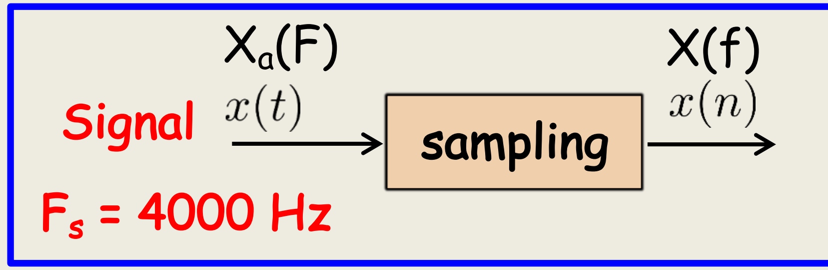

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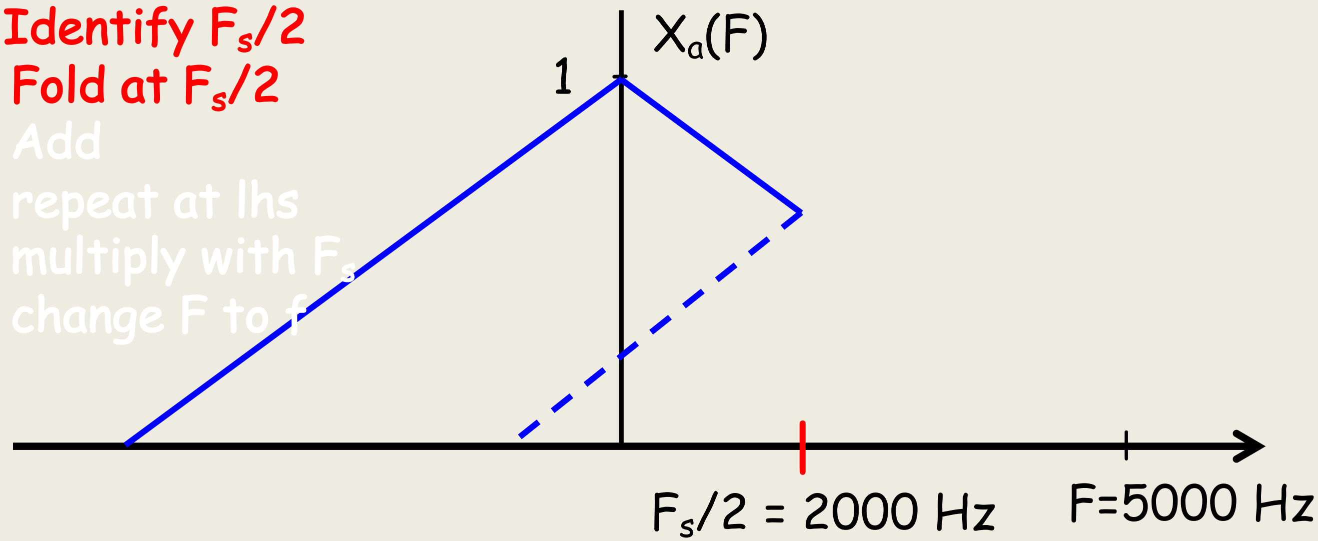

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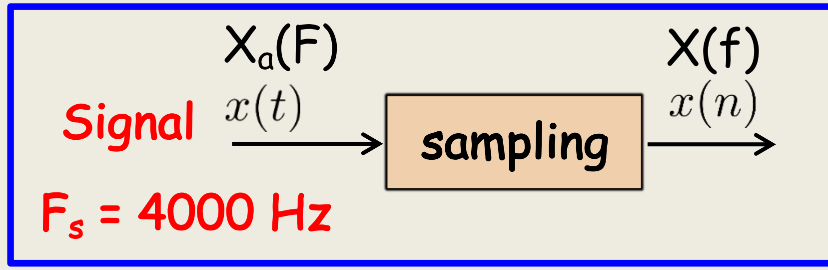

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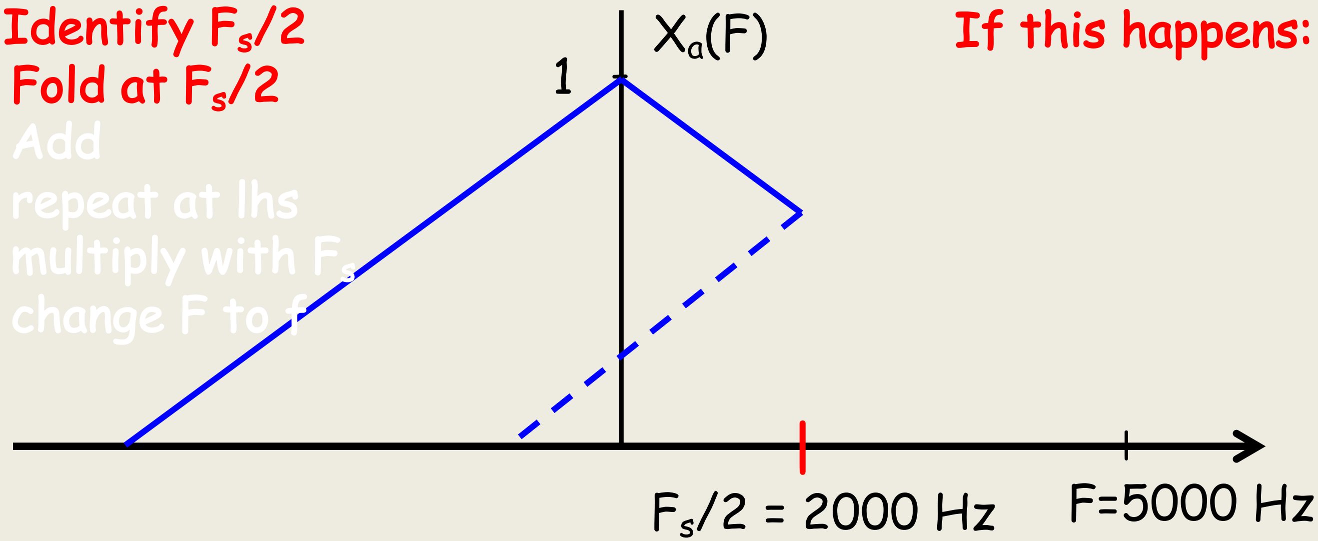

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Example: Folding

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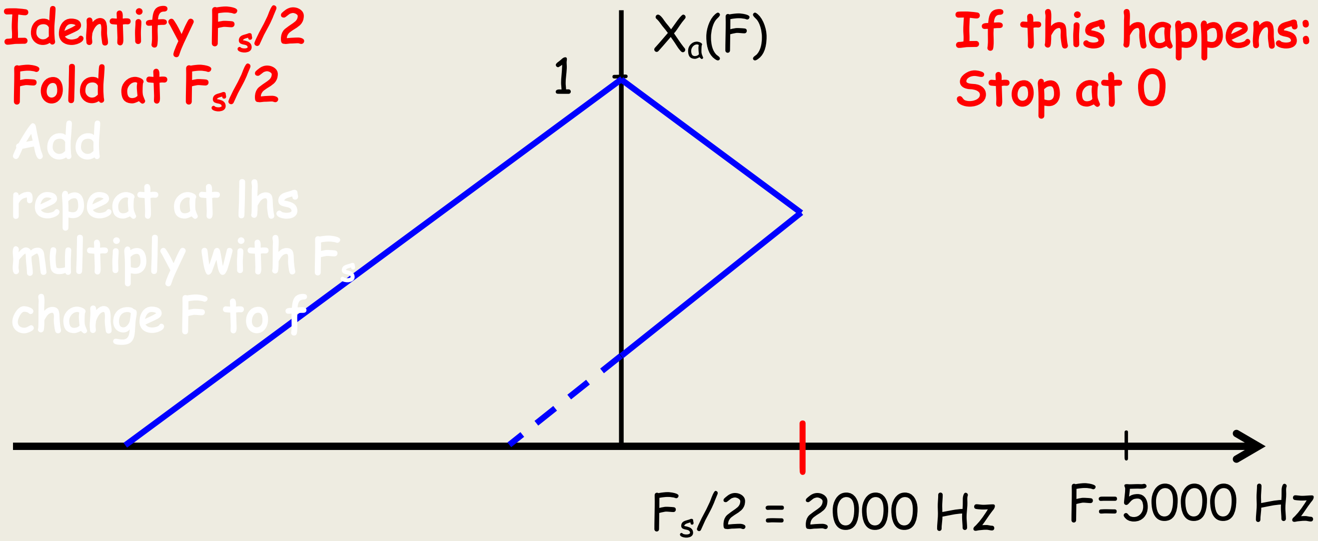

If this happens:
Stop at 0

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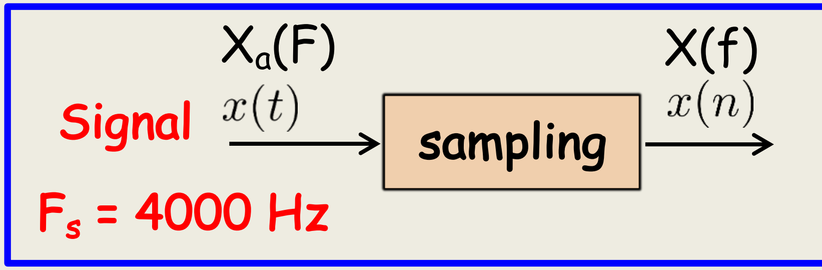
Example: Folding



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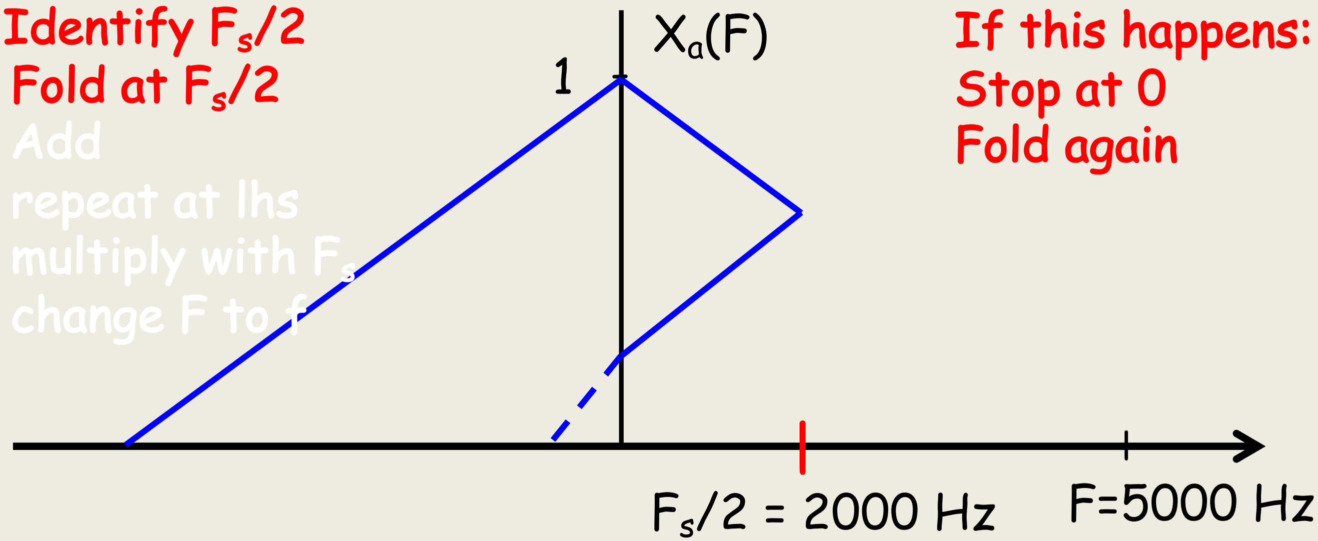

If this happens:
Stop at 0
Fold again

EITF75 Systems and Signals

Example: Folding

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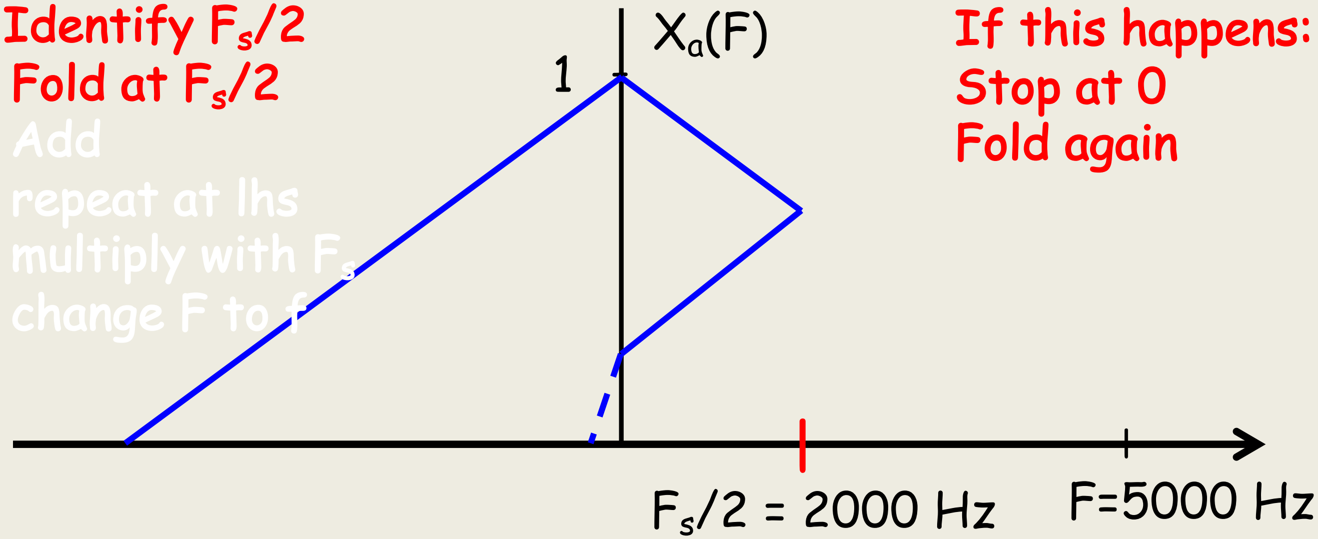

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EITF75 Systems and Signals

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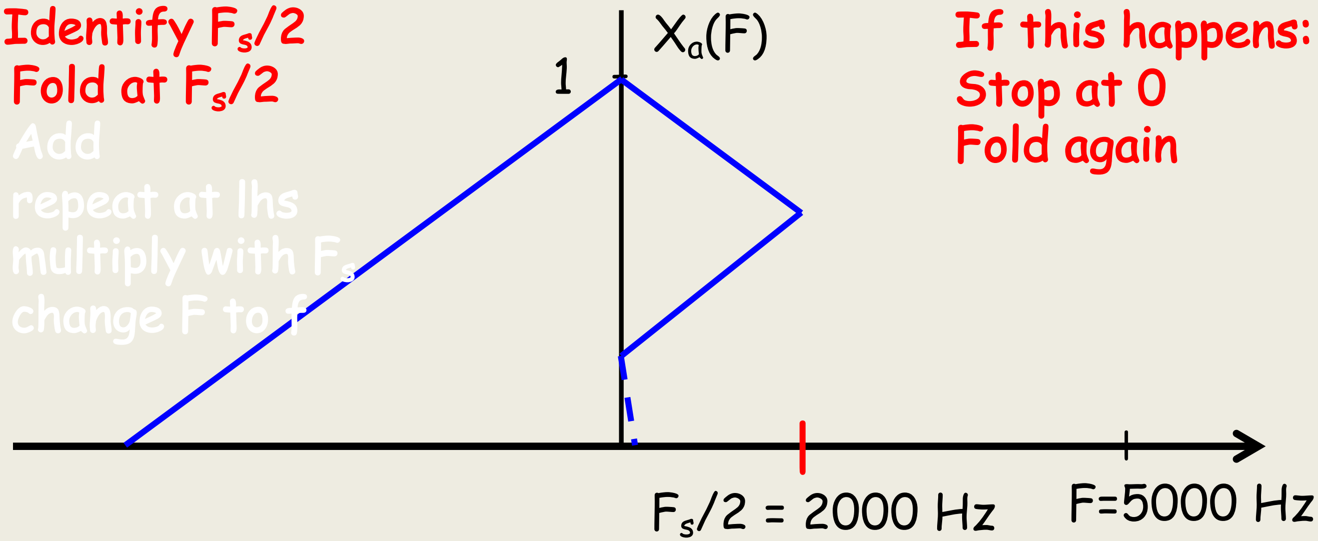

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EITF75 Systems and Signals

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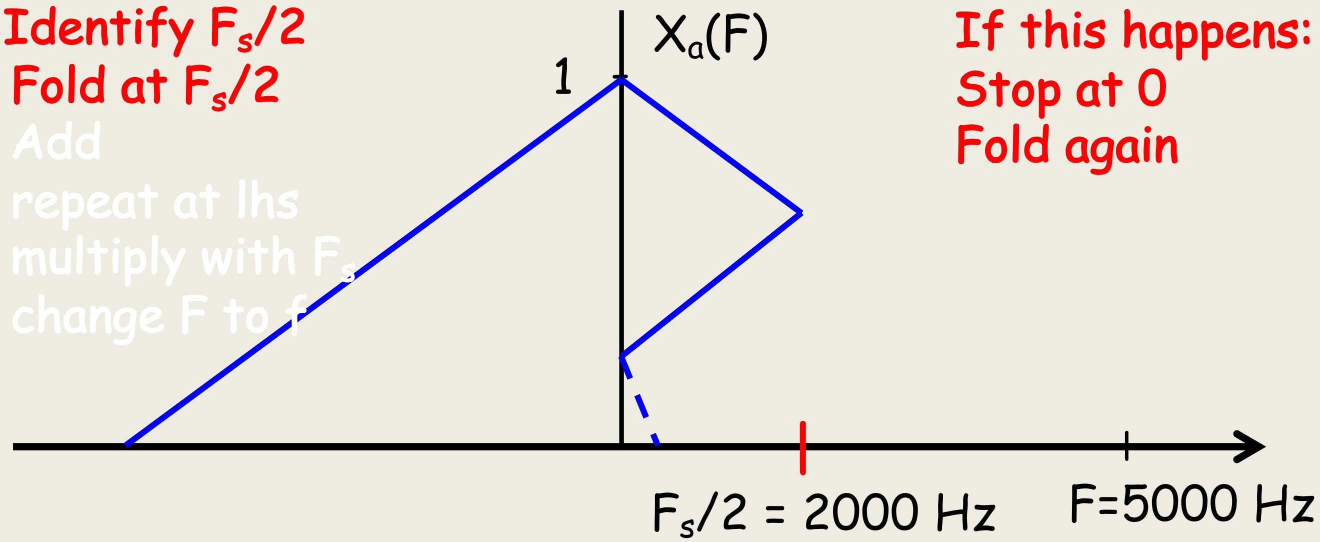

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EITF75 Systems and Signals

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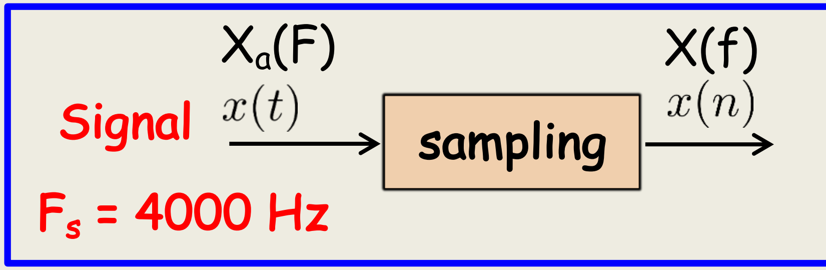

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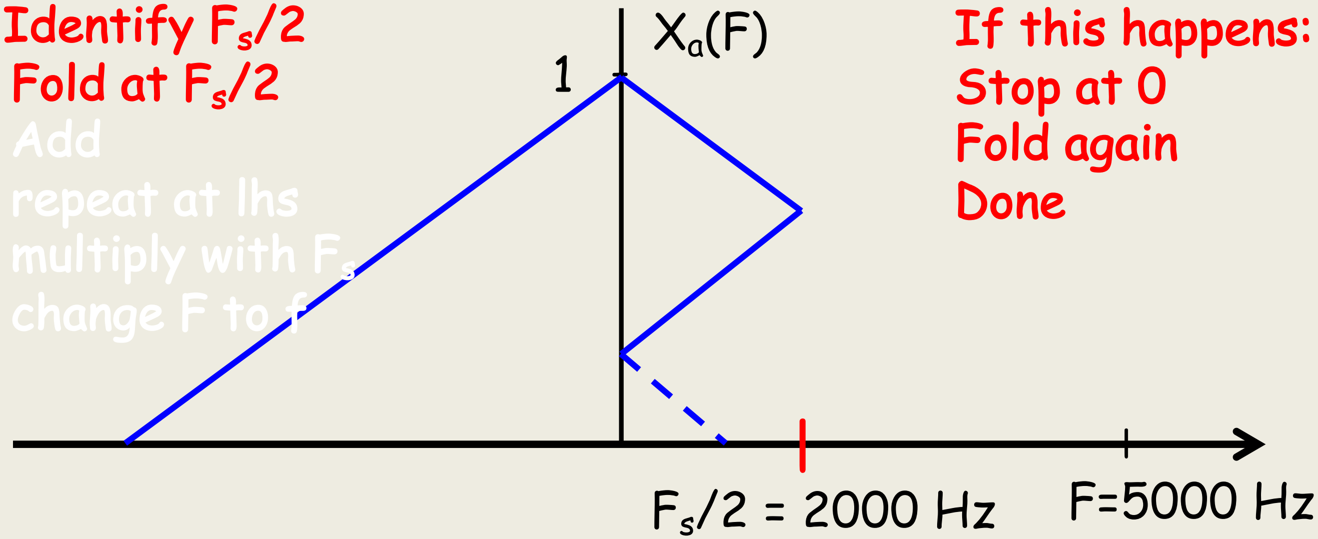

If this happens:
Stop at 0
Fold again
Done

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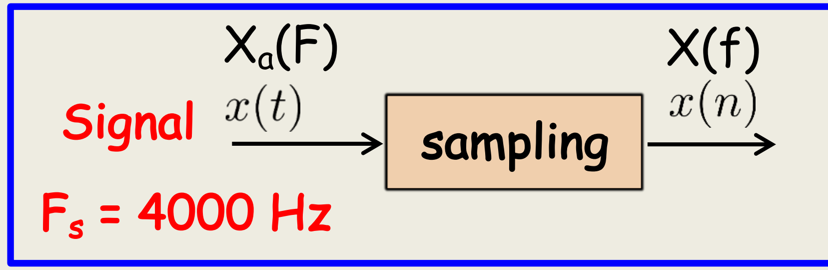
Example: Folding



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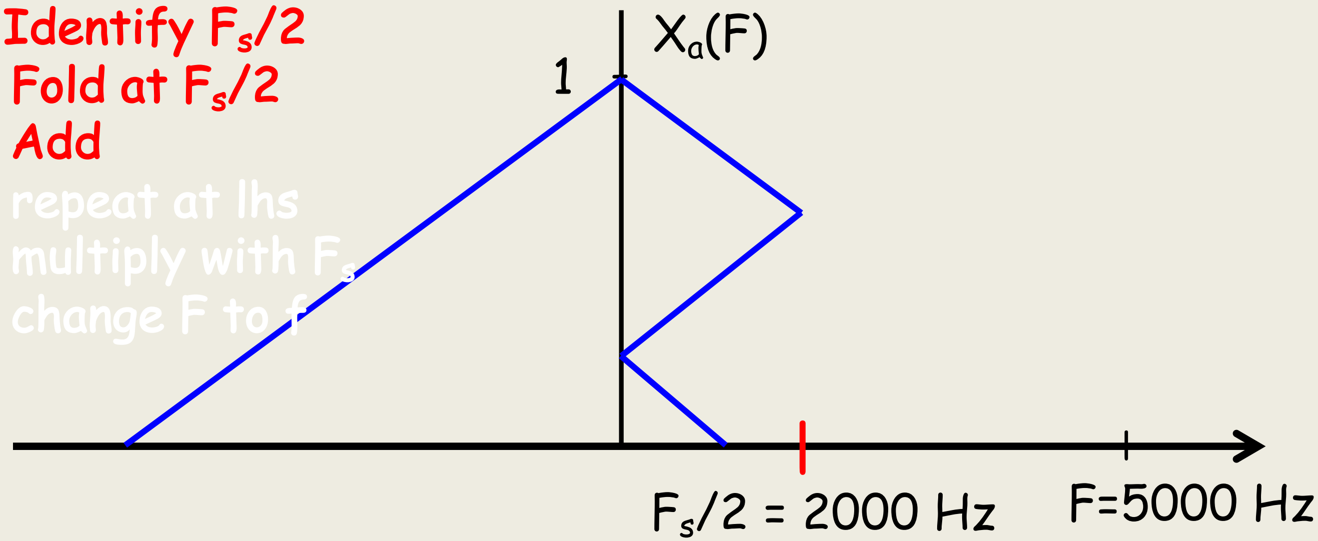

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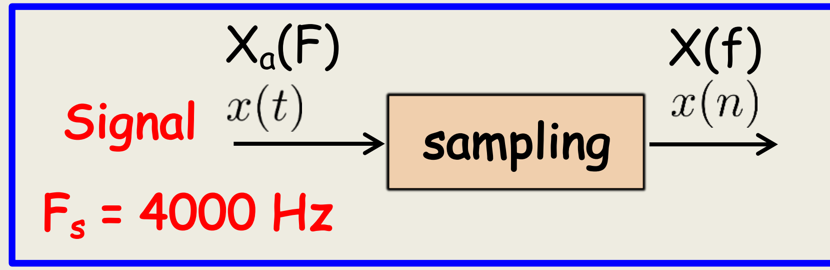

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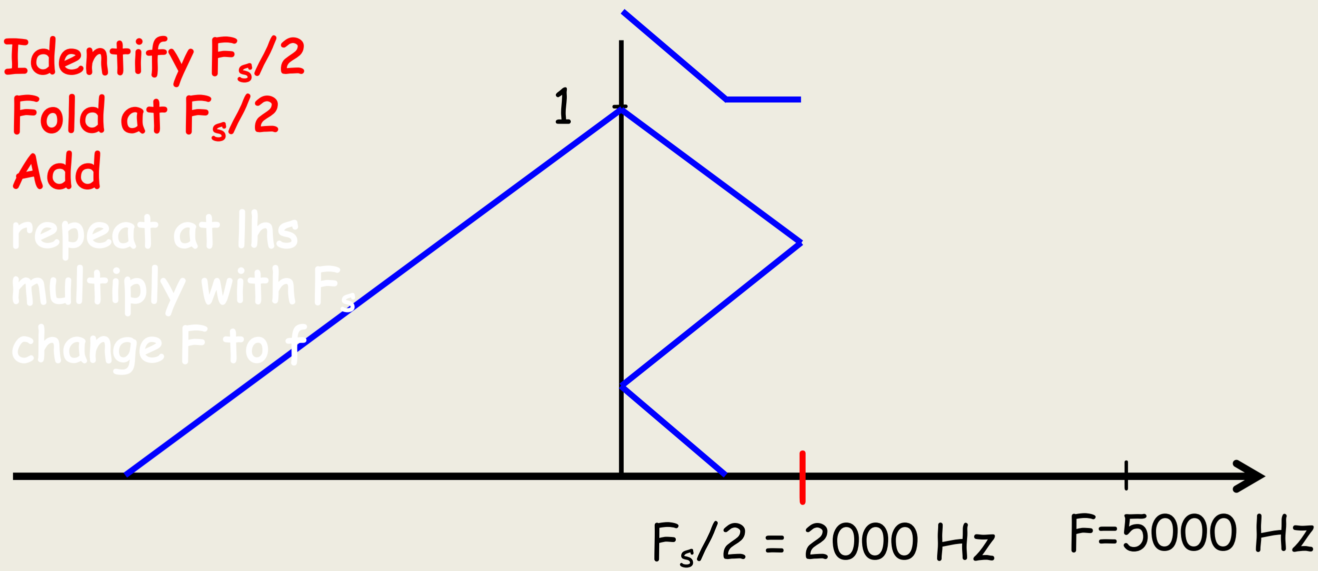

EITF75 Systems and Signals

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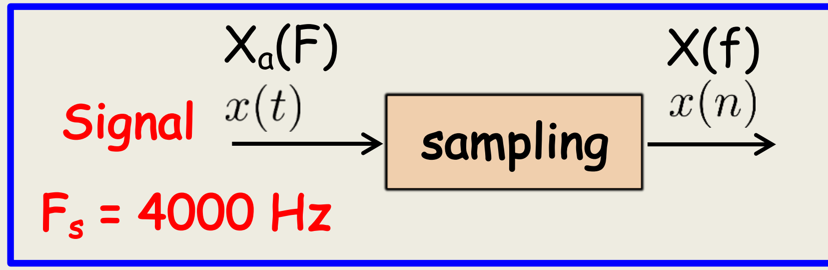

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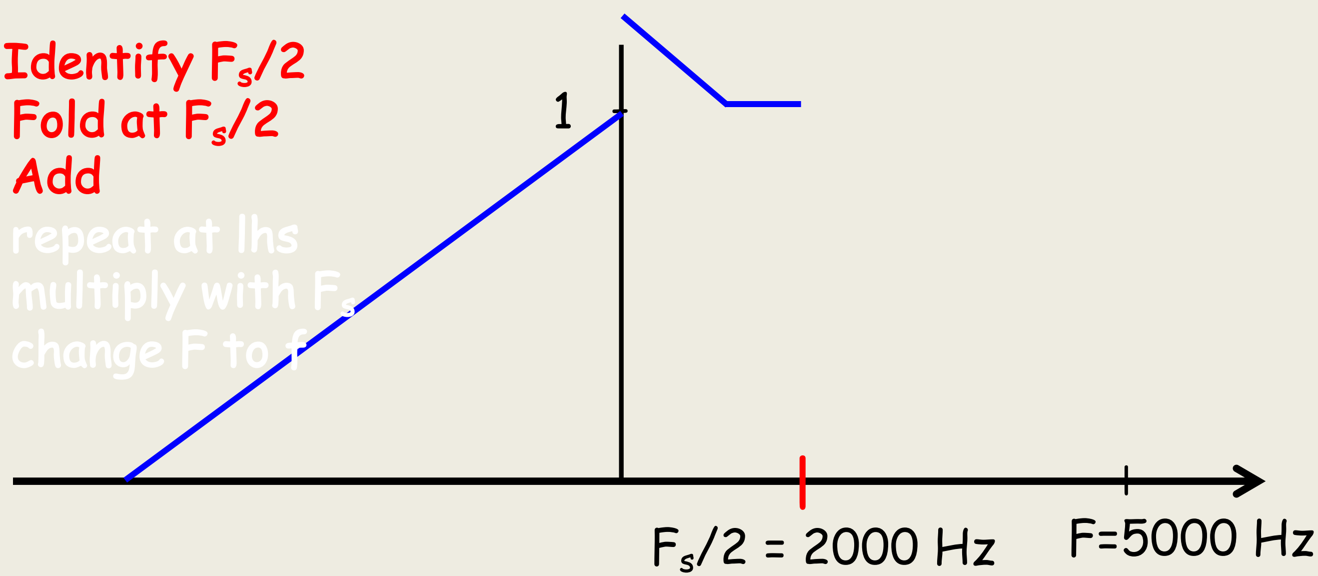

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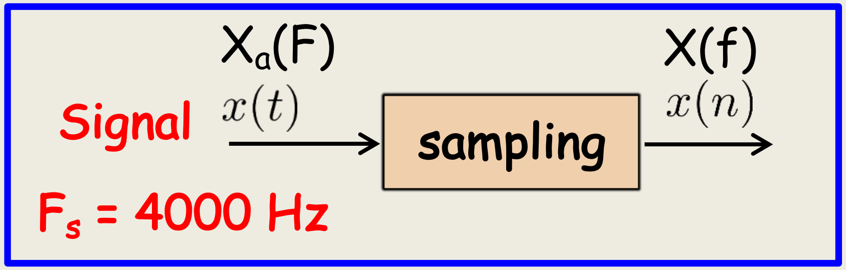

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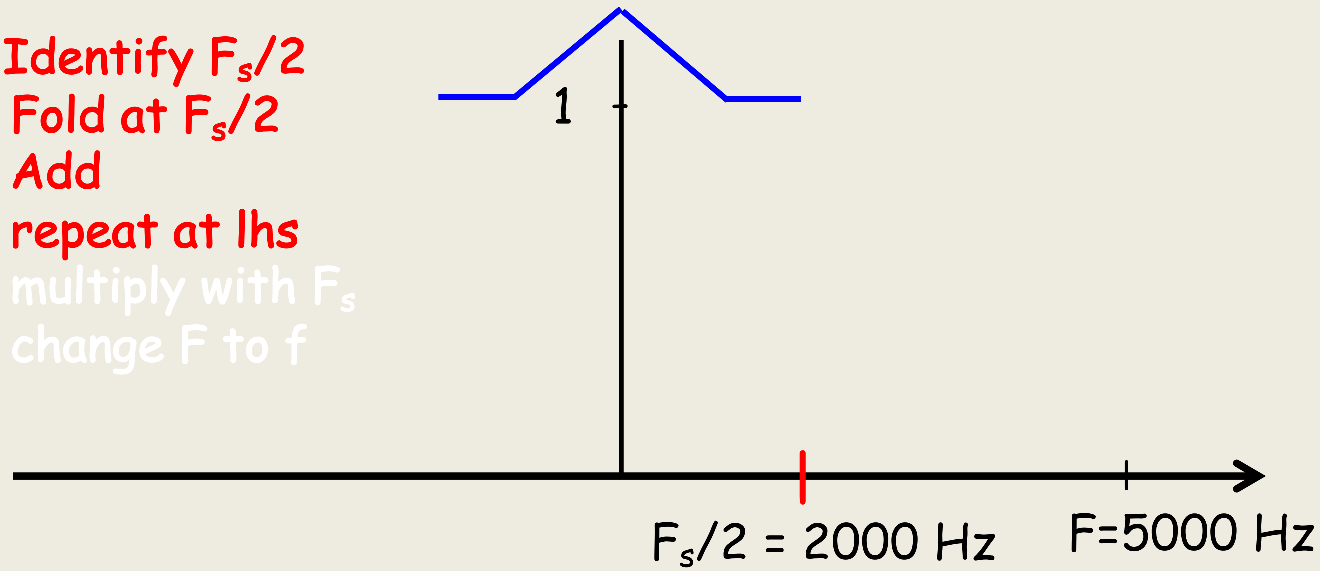

EITF75 Systems and Signals

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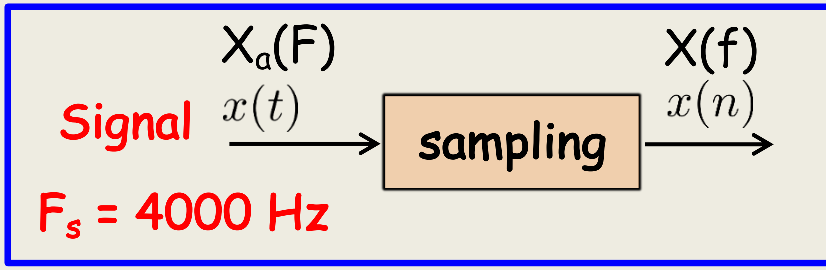

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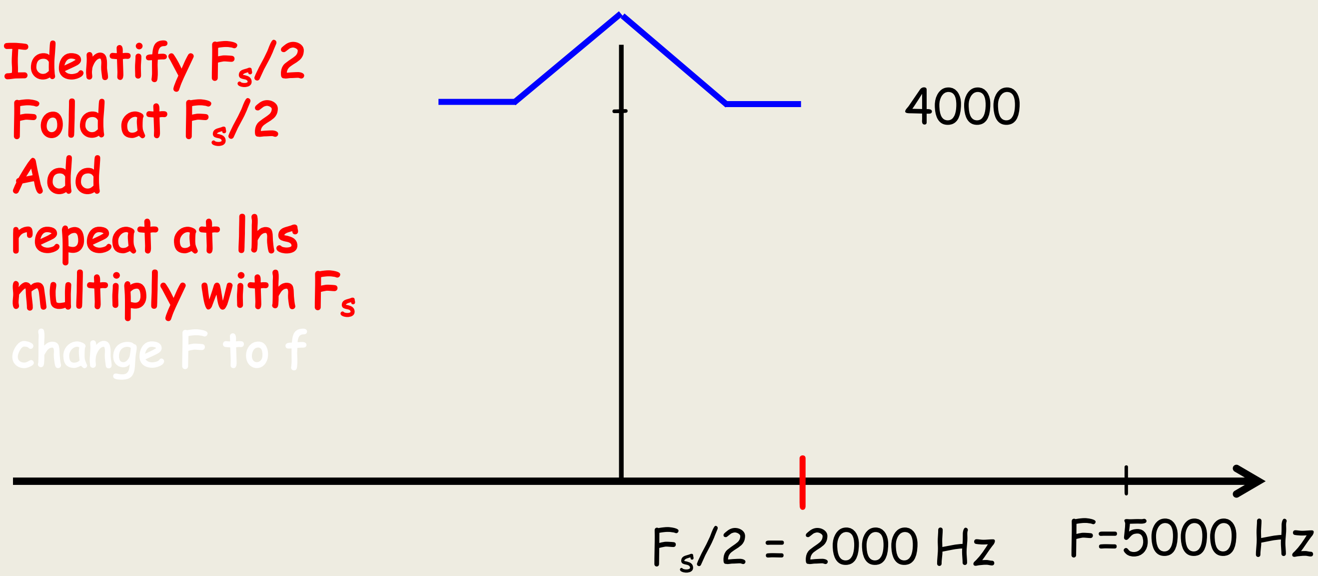

EITF75 Systems and Signals

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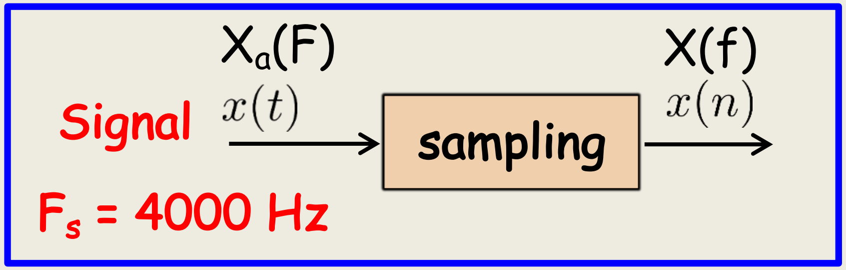

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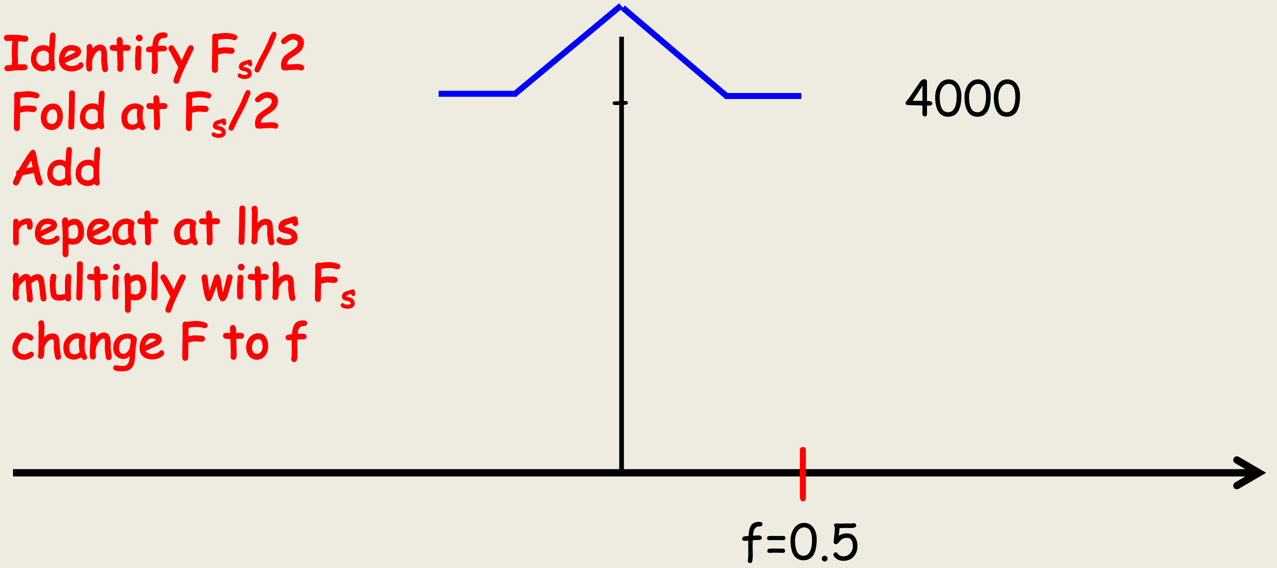

EITF75 Systems and Signals

A/D and D/A

Signal $\quad \xrightarrow[x(t)]{X_a(F)} \boxed{\text{sampling}} \xrightarrow[x(n)]{X(f)}$

Key step is to understand what X(f) looks like in terms of $X_a(F)$

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(f) = F_s X_a(f F_s)$$

k=0

If no aliasing
(e.g., sampling
Theorem fulfilled)

Sampling Theorem (Shannon 1948)

If $F_s > 2B$, where B is the highest frequency of the analog signal, then the analog signal can be recovered from its sampled version

If aliasing: In general not possible to revocer x(t) from x(n)

EITF75 Systems and Signals

A/D and D/A

$X_a(F)$ $\quad$ $X(f)$

Signal $x(t) \longrightarrow$ **sampling** (F_1 Hz) $\xrightarrow{x(n)}$ **Reconstruction** (F_2 Hz) $\longrightarrow \hat{x}(t)$

Note: sampling and reconstruction frequencies can differ. See lecture 9-10

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(f) = F_s X_a(f F_s)$$

k=0

Reconstruction.

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n)\operatorname{sinc}(F_s(t - n/F_s))$$

No aliasing

$$\hat{x}(t) = x(t)$$

Aliasing

$$\hat{x}(t) \neq x(t)$$

EITF75 Systems and Signals

LTI systems and z-transforms

EITF75 Systems and Signals

LTI systems

$x(n) \longrightarrow$ **system** $\longrightarrow y(n)$

A system is LTI if-and-only if:

- It is linear
- It is time-invariant

Linear system

$$x(n) = \alpha x_1(n) + \beta x_2(n)$$
$$\Longleftrightarrow$$
$$y(n) = \alpha y_1(n) + \beta y_2(n)$$

Time invariant system

$$x(n) \text{ replaced by } x(n-D)$$
$$\Longleftrightarrow$$
$$y(n) \text{ replaced by } y(n-D)$$

An LTI system is fully characterized by a difference equation ....or... an Impulse response $h(n)$

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Output if input is
$x(n) = \delta(n) = [1 \;\; 0 \;\; 0 \;\; \cdots]$
and system at rest

EITF75 Systems and Signals

LTI systems

$$x(n) \rightarrow \boxed{\text{system}} \rightarrow y(n)$$

A system is LTI if-and-only if:

- It is linear
- It is time-invariant

Assume that we turn on the circuit at n=0

System at rest if

$$y(-k) = 0, \; 1 \leq k \leq N$$

Not at rest if (has initial conditions)

$$\exists k, \; 1 \leq k \leq N, \; : \; y(-k) \neq 0$$

An LTI system is fully characterized by a difference equation ....or... an Impulse response $h(n)$

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EITF75 Systems and Signals

$x(n) \rightarrow \boxed{\text{system}} \rightarrow y(n)$

What is output for a given input

Found by z-transform

The z-transform of $h(n)$ is defined as

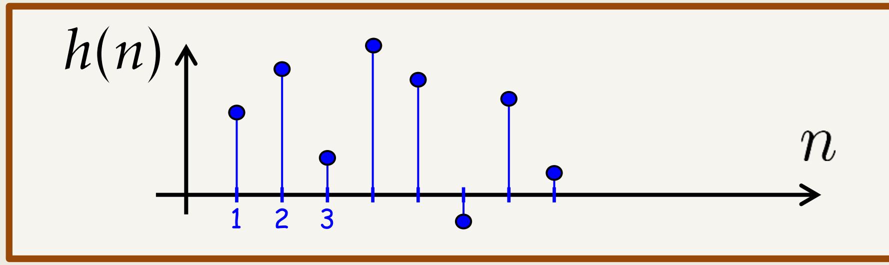
$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

What is the z-transform?

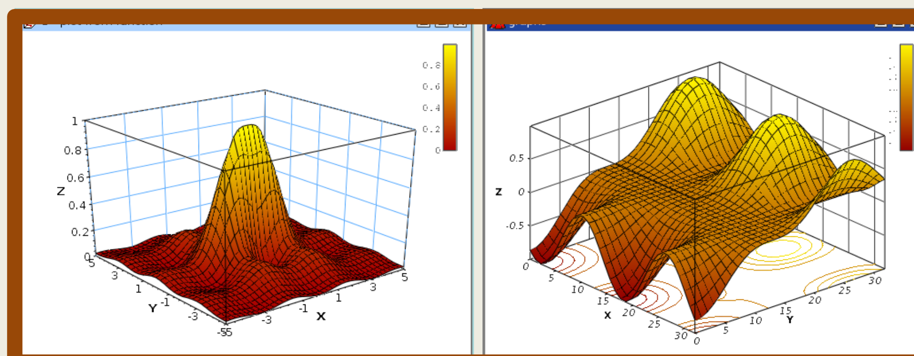
- A map from sequences to complex valued functions

What is H(z)?

- A complex function of a complex number



z-transform



If we want to plot H(z), we need 2 plots, one for the real part, one for the imaginary

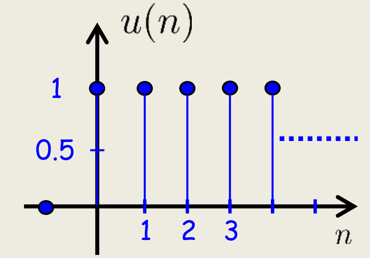
Z-transforms are not meant for "plotting and obtaining insights"

EITF75 Systems and Signals

An *important* example

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1 - z^{-1}}$$

$$\text{ROC} \quad |z| > 1$$



$$H(z) = \sum_{n=-\infty}^{\infty} -u(-n-1) z^{-n}$$

Let's specify the ROC

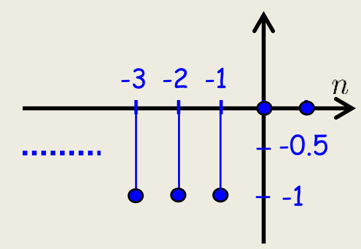
$$= -\sum_{n=-\infty}^{-1} z^{-n} = -\sum_{n=0}^{\infty} z^{n} + 1$$

$$= -\frac{1}{1-z} + 1 = \ldots = \frac{-z}{1-z}$$

$$= \frac{1}{1 - z^{-1}} \qquad \text{ROC} \quad |z| < 1$$

Anti-causal step

$$h(n) = -u(-n-1)$$



Different signals,
Same z-transform
Different ROC

EITF75, z-transform

Convention

If we are given an $X(z)$, and assume that the signal $x(n)$ is causal, then we can be sloppy with the ROC

There are many $x(n)$ for the same $X(z)$, and the ROC specifies the particular one. However, there is only one that is causal.

EITF75 Systems and Signals

LTI systems

$$x(n) \longrightarrow \boxed{\text{system}} \longrightarrow y(n)$$

What is output for a given input

Found by z-transform

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Expression for general difference equation

$$y(n)+\sum_{k=1}^{N} a_k y(n-k)=\sum_{k=0}^{M} b_k x(n-k)$$

Solution for general difference equation (at rest)

$$Y(z)+\sum_{k=1}^{N} a_k z^{-k} Y(z)=\sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z)=\underbrace{\boxed{\frac{b_0+b_1 z^{-1}+\cdots+b_M z^{-M}}{1+a_1 z^{-1}+\cdots+a_N z^{-N}}}}_{H(z)}\cdot X(z)$$

Step 1:

Change y(n-k) to z^{-k} Y(z)

Step 2:

Change x(n-k) to z^{-k} X(z)

Step 3:

Express Y(z) as H(z)X(z)

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Expression for general difference equation

$$y(n)+\sum_{k=1}^{N} a_k y(n-k)=\sum_{k=0}^{M} b_k x(n-k)$$

Step 4:

Find the roots of the denominator and nominator of H(z). Roots should be in terms of z, not z^{-1}

Solution for general difference equation (at rest)

$$Y(z)+\sum_{k=1}^{N} a_k z^{-k} Y(z)=\sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z)=\boxed{\frac{b_0+b_1 z^{-1}+\cdots+b_M z^{-M}}{1+a_1 z^{-1}+\cdots+a_N z^{-N}}}\cdot X(z)$$

$$Y(z)=\frac{z^{-M}}{z^{-N}}\,\frac{b_0 z^M+b_1 z^{M-1}+\cdots+b_M}{z^N+a_1 z^{N-1}+\cdots+a_N}X(z)$$

zeros

$$=b_0\cdot\frac{z^{-M}}{z^{-N}}\cdot\frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)}\cdot X(z)$$

poles

$H(z)$

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

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$H(z)$

$$Y(z)=\frac{z^{-M}}{z^{-N}}\frac{b_0 z^{M}+b_1 z^{M-1}+\cdots+b_M}{z^{N}+a_1 z^{N-1}+\cdots+a_N}X(z)$$

zeros

$$=b_0\cdot\frac{z^{-M}}{z^{-N}}\cdot\frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)}\cdot X(z)$$

poles

If degree of numerator >= degree of denominator. Perform polynomial division

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Expression for general difference equation

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zeros

$$=b_0\cdot\frac{z^{-M}}{z^{-N}}\cdot\frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)}\cdot X(z)$$

poles

$H(z)$

Will turn up in the time-domain as a delay
(can be negative delay)

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Expression for general difference equation

$$\boxed{y(n)+\sum_{k=1}^{N}a_k y(n-k)=\sum_{k=0}^{M}b_k x(n-k)}$$

Assume $\boxed{X(z)=\frac{N(z)}{Q(z)}}$

$$Y(z)=\frac{\text{Some polynomial in } z}{(z-p_1)\cdots(z-p_N)\cdot(z-q_1)\cdots(z-q_L)}$$

Assuming all poles are real and distinct

Assuming deg(num) < deg(denom)

$$Y(z)=\sum_{k=1}^{N}\frac{A_k}{1-z^{-1}p_k}+\sum_{k=1}^{L}\frac{Q_k}{1-z^{-1}q_k}$$

$$Y(z)=\frac{z^{-M}}{z^{-N}}\frac{b_0z^M+b_1z^{M-1}+\cdots+b_M}{z^N+a_1z^{N-1}+\cdots+a_N}X(z)$$

$$=b_0\cdot\frac{z^{-M}}{z^{-N}}\cdot\frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)}\cdot X(z)$$

Perform partial fraction expansion

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Expression for general difference equation

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$Y(z) = \frac{z^{-M}}{z^{-N}} \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{z^N + a_1 z^{N-1} + \cdots + a_N} X(z)$$

$$= b_0 \cdot \frac{z^{-M}}{z^{-N}} \cdot \frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)} \cdot X(z)$$

Assume $\boxed{X(z) = \frac{N(z)}{Q(z)}}$

$$Y(z) = \frac{\textit{Some polynomial in z}}{(z-p_1)\cdots(z-p_N)\cdot(z-q_1)\cdots(z-q_L)}$$

Assuming all poles are real and distinct

Assuming deg(num) < deg(denom)

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - z^{-1} p_k} + \sum_{k=1}^{L} \frac{Q_k}{1 - z^{-1} q_k}$$

Invert

$$y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Expression for general difference equation

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - z^{-1} p_k} + \sum_{k=1}^{L} \frac{Q_k}{1 - z^{-1} q_k}$$

This...

...generates that

$$y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Important: To get stable output, all poles must be inside the unit circle

Assume $X(z) = \frac{N(z)}{Q(z)}$

$$Y(z) = \frac{1}{(z-p_1)\cdots(z-p_N)} \cdot \frac{\text{Some polynomial in } z}{(z-q_1)\cdots(z-q_L)}$$

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1-z^{-1}p_k} + \sum_{k=1}^{L} \frac{Q_k}{1-z^{-1}q_k}$$

Important: poles in H(z) and in X(z) determines the output structure:

"You can never get a term in y(n) that doesn't exist in either X(z) or H(z)"

$$Y(z) = \frac{z^{-M}}{z^{-N}} \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{z^N + a_1 z^{N-1} + \cdots + a_N} X(z)$$

$$= b_0 \cdot \frac{z^{-M}}{z^{-N}} \cdot \frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)} \cdot X(z)$$

$$y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

EITF75 Systems and Signals

A complex conjugated pair of poles

$$h(n) = r^n \cdot \sin(\omega n)u(n)$$

$$h(n) = r^n \cdot \cos(\omega n)u(n)$$

$$H(z) = \frac{r\sin(\omega)z^{-1}}{1 - 2r\cos(\omega)z^{-1} + r^2 z^{-2}}$$

$$H(z) = \frac{1 - r\cos(\omega)z^{-1}}{1 - 2r\cos(\omega)z^{-1} + r^2 z^{-2}}$$

Polar coordinates: r is "length" and w is angle of the pole.
To get stable output: r<1 (poles inside the unit circle)

Example

Quite messy to invert a mixture of the two above: Make sure you know how to do that.

Invert $$H(z) = z^{-1} \cdot \frac{1 - z^{-1}}{1 - 1.27z^{-1} + 0.81z^{-2}}$$

EITF75 Systems and Signals

Systems not at rest

Use the one-sided z-transform

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

EITF75 Systems and Signals

Systems not at rest

Use the one-sided z-transform

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

End result: The solution at rest + contribution from initial conditions

$$Y(z) = H(z)X(z) + \frac{N_0(z)}{A(z)} = \frac{B(z)}{A(z)} X(z) + \frac{N_0(z)}{A(z)}$$

$$N_0(z) = -\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(-n) z^n$$

N: highest power of z^{-1} in A(z)

EITF75 Systems and Signals

Fourier analysis. 4 cases

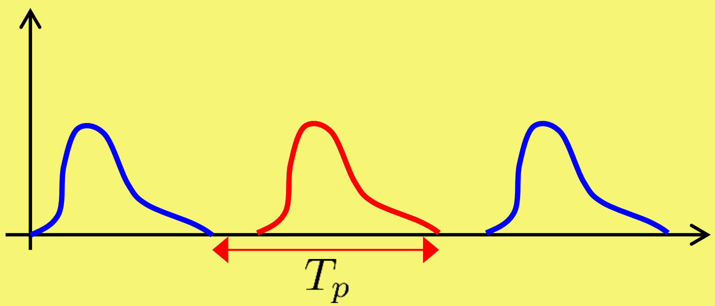
Periodic/aperiodic signal
Continuous/discrete signal

EITF75, Fourier transforms

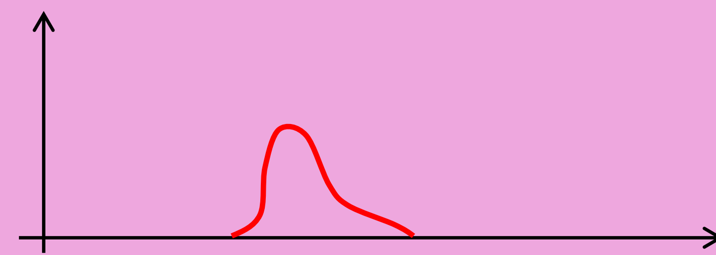
4 different type of signals

Time signals shown, not Fourier transforms

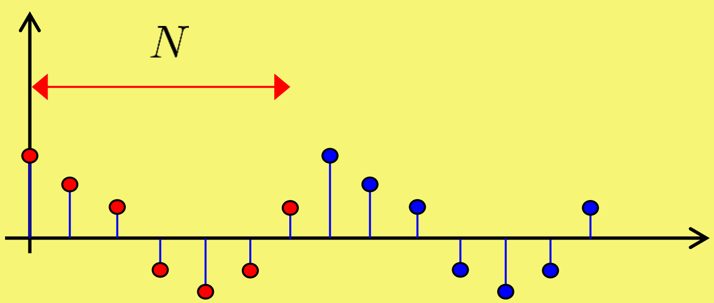
Continuous and periodic



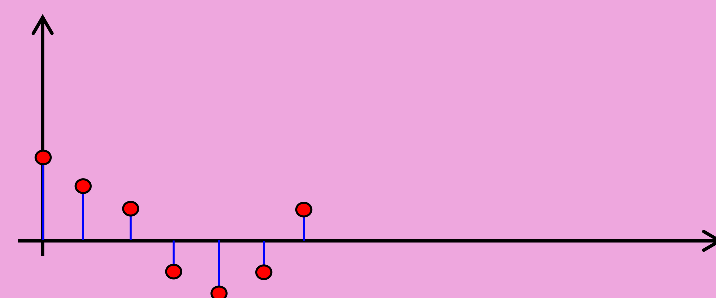
Continuous and aperiodic



Discrete and periodic



Discrete and aperiodic



Discrete spectra

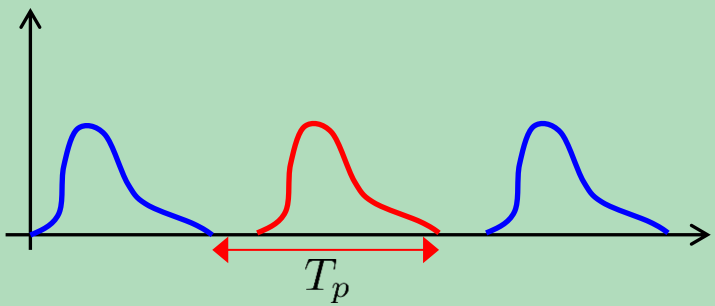
Continuous spectra

EITF75, Fourier transforms

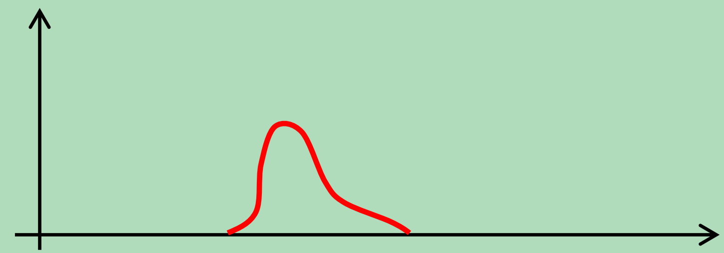
4 different type of signals

Aperiodic spectra

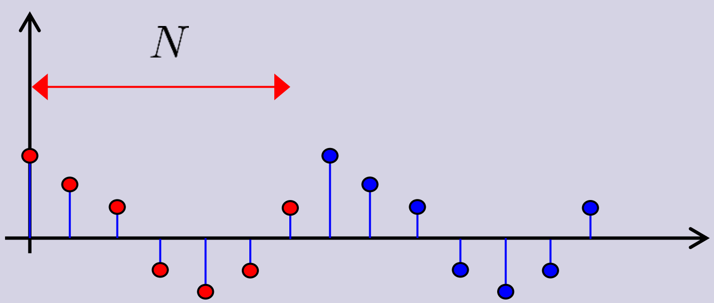
Continuous and periodic



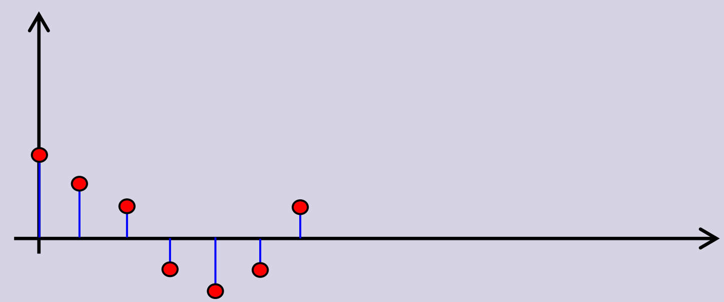
Continuous and aperiodic



Discrete and periodic

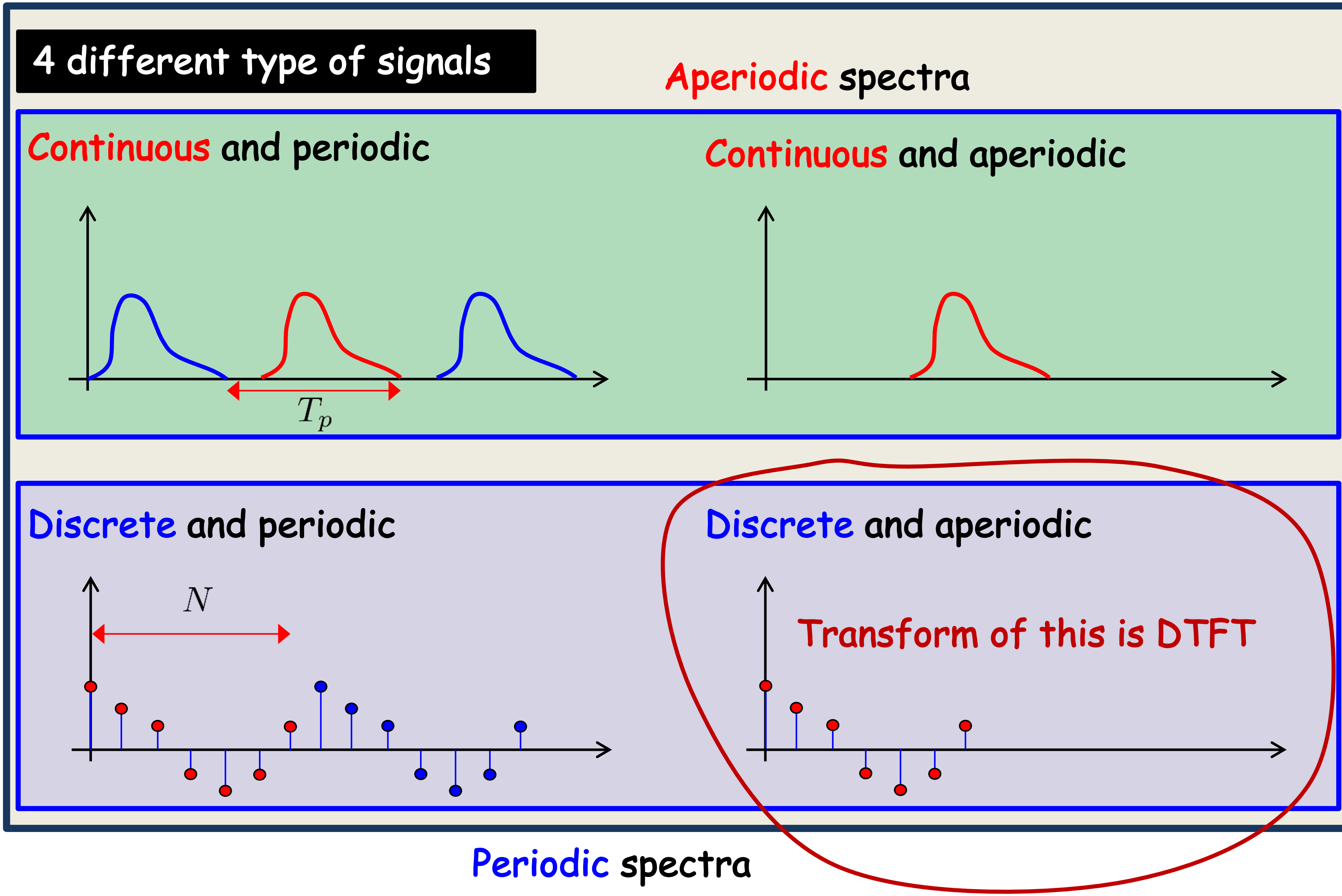


Discrete and aperiodic



Periodic spectra

EITF75, Fourier transforms



EITF75 Systems and Signals

Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

DTFT

(discrete time Fourier transform)

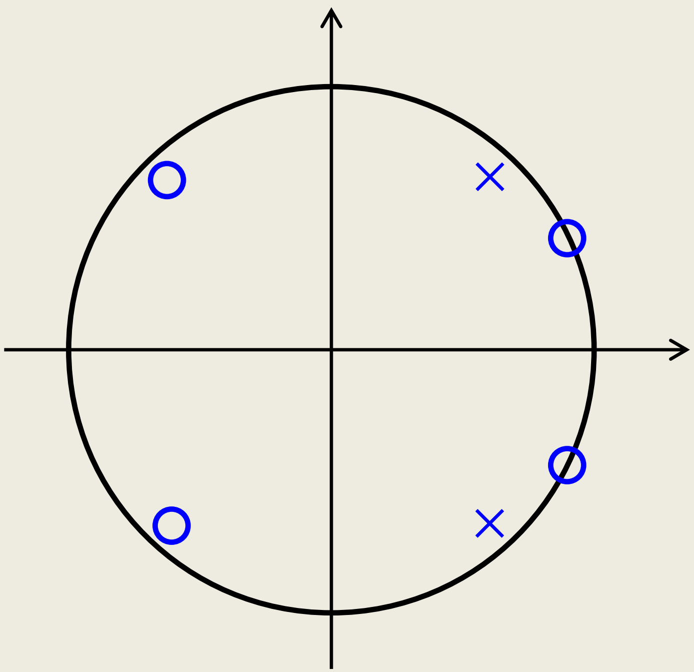
$$X(f) = \sum_{n=-\infty}^{\infty} x(n) \exp(-i2\pi n f)$$

$$= X(z|z = \exp(i2\pi f))$$

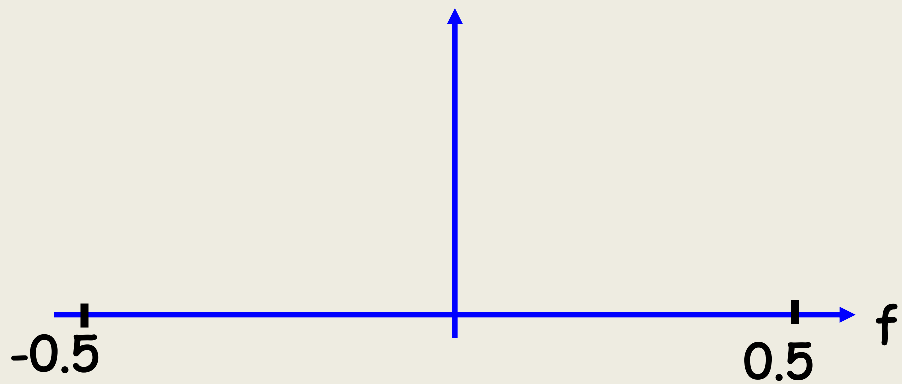
Important: DTFT is z-transform evaluated at unit circle

EITF75, DTFT

Pole-zero plot



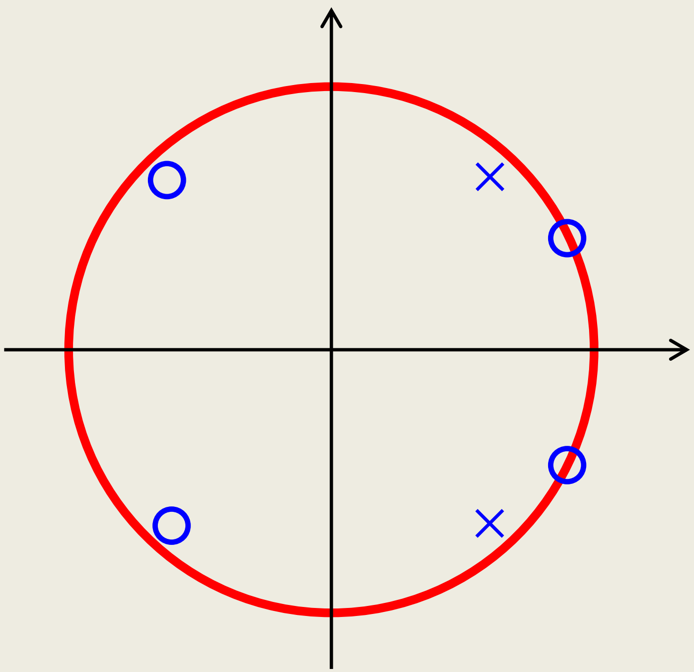
DTFT



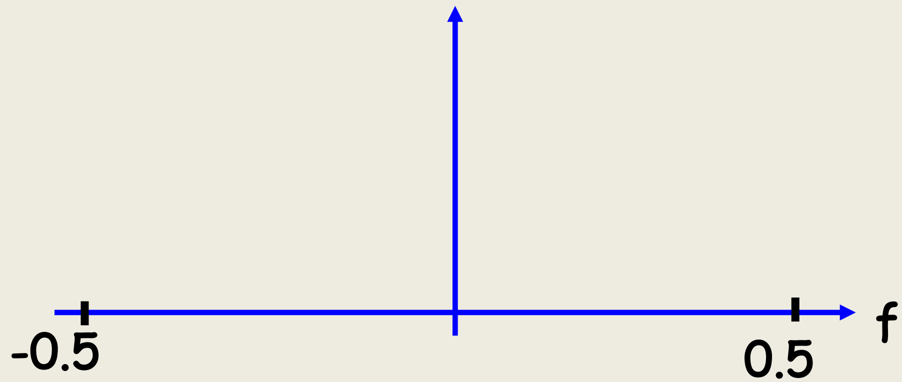
Book makes a big deal out of this. But quite easy....

EITF75, DTFT

Pole-zero plot



DTFT is H(z) at unit circle

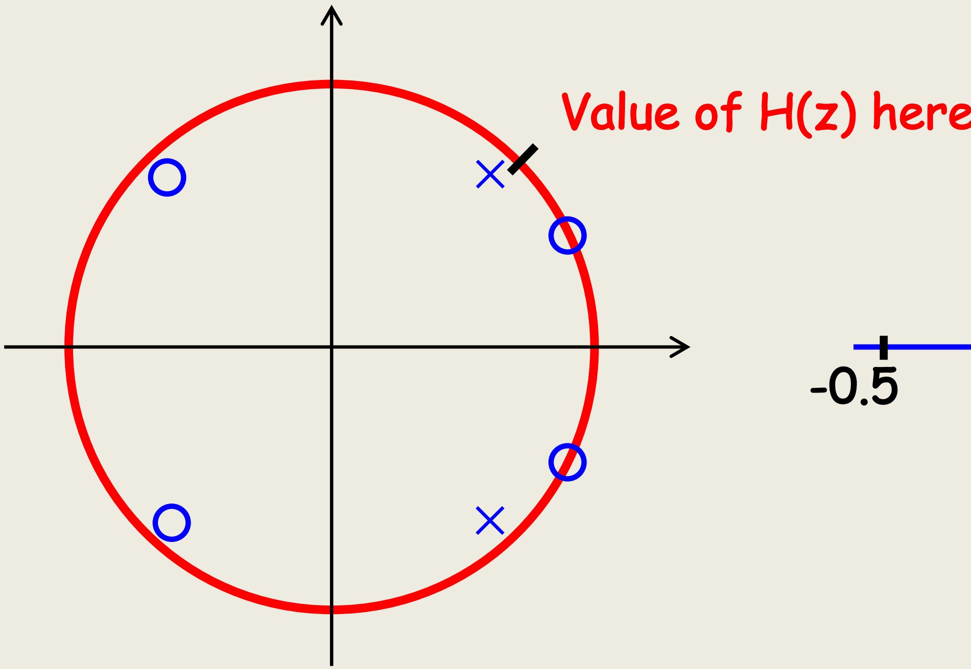


Recall

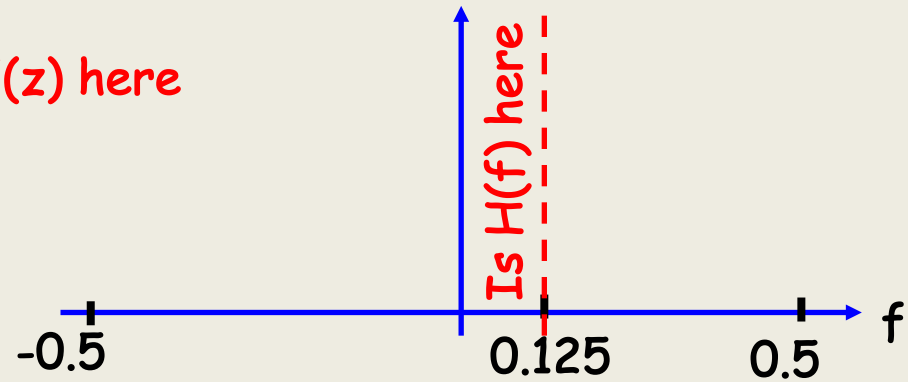
$$H(z) = \frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)}$$

EITF75, DTFT

Pole-zero plot



DTFT is H(z) at unit circle



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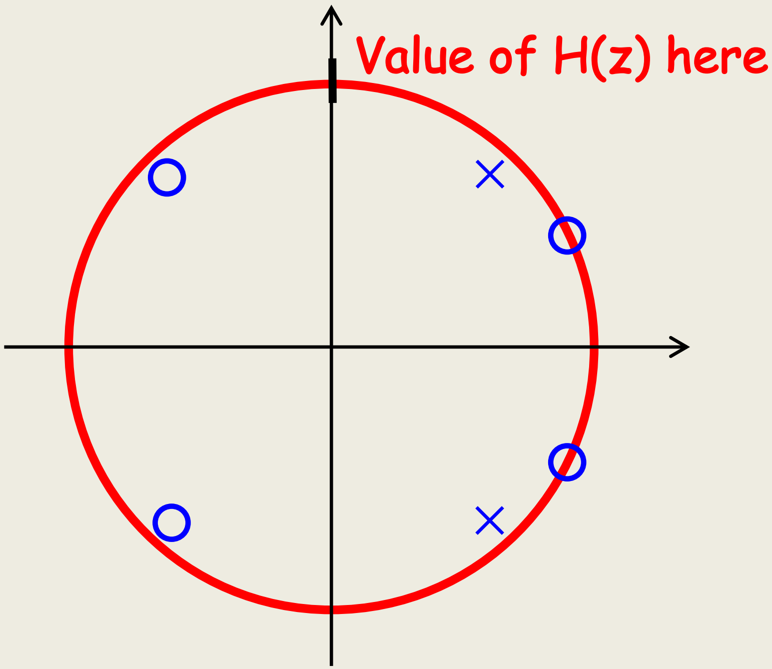
EITF75, DTFT

Pole-zero plot



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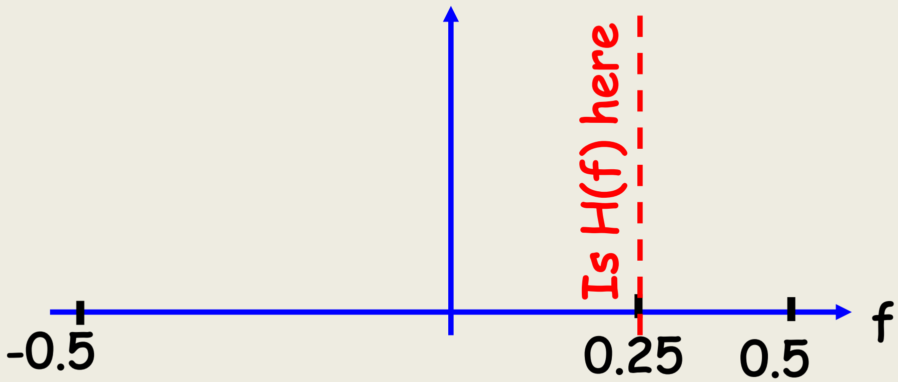

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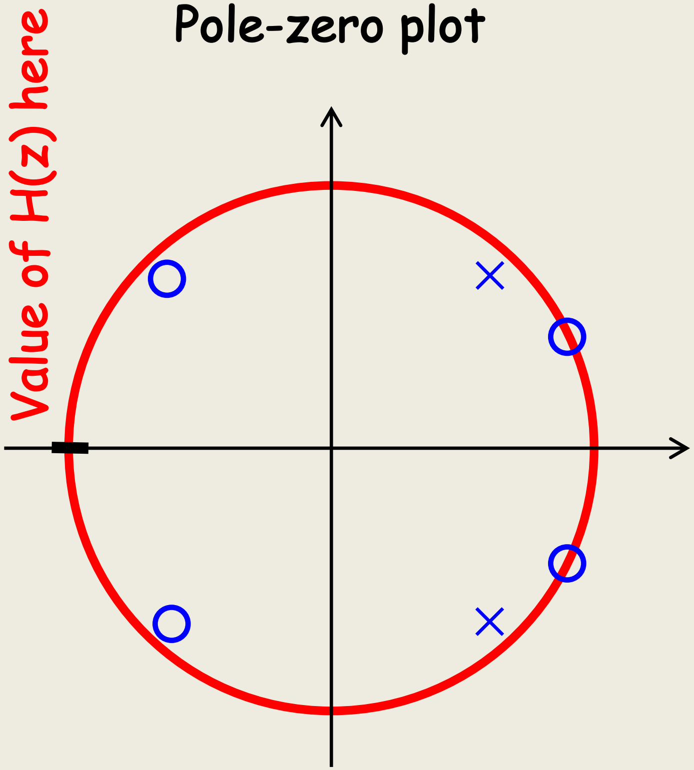
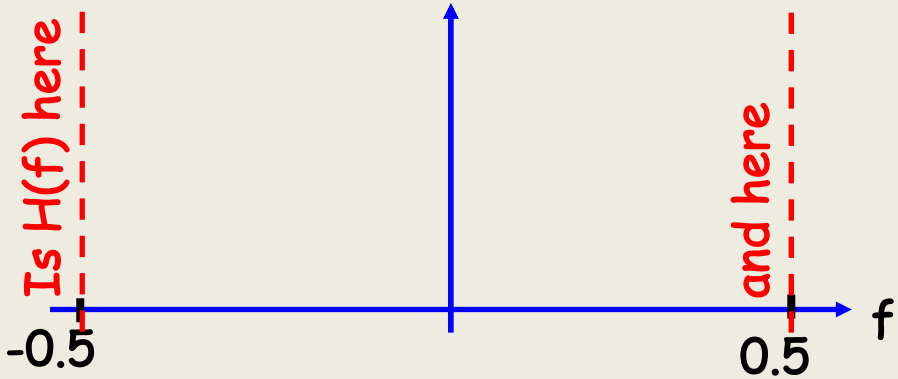
EITF75, DTFT

Pole-zero plot



DTFT is H(z) at unit circle



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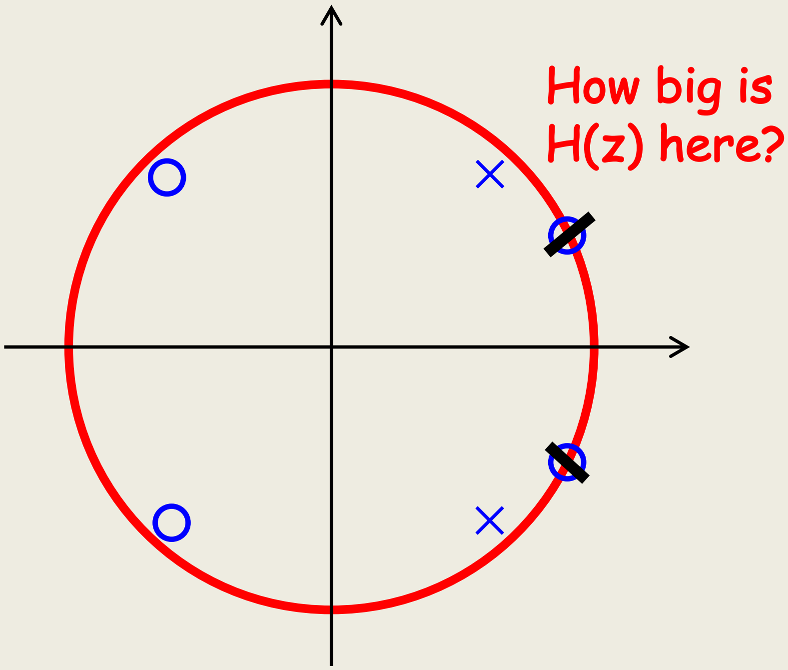
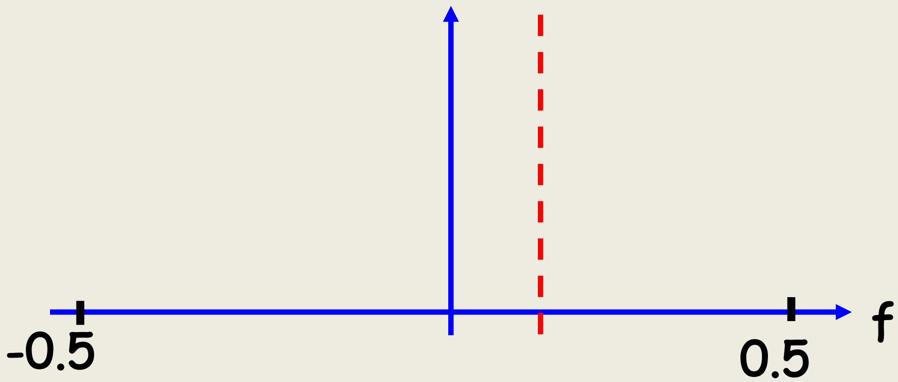
EITF75, DTFT

Pole-zero plot



DTFT is H(z) at unit circle



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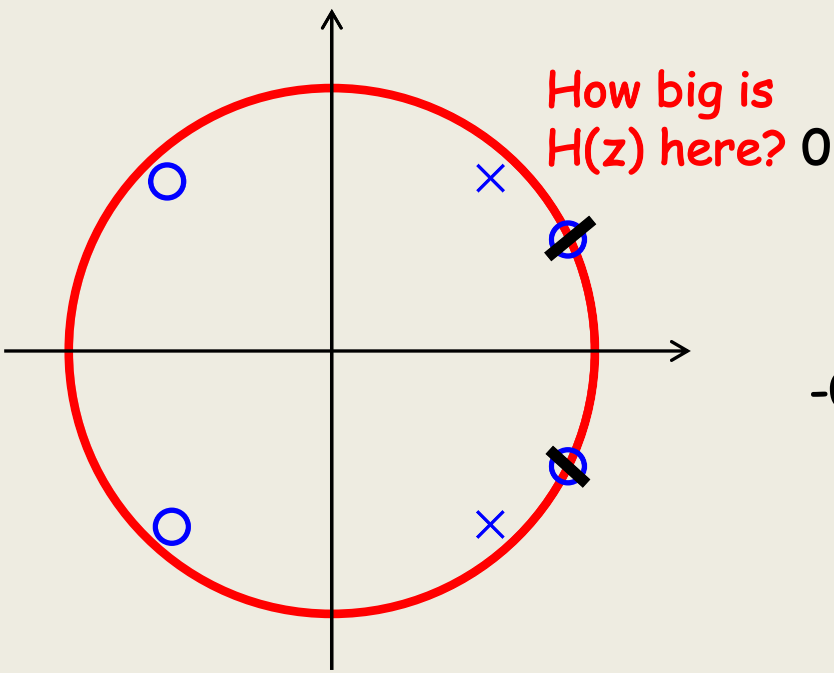
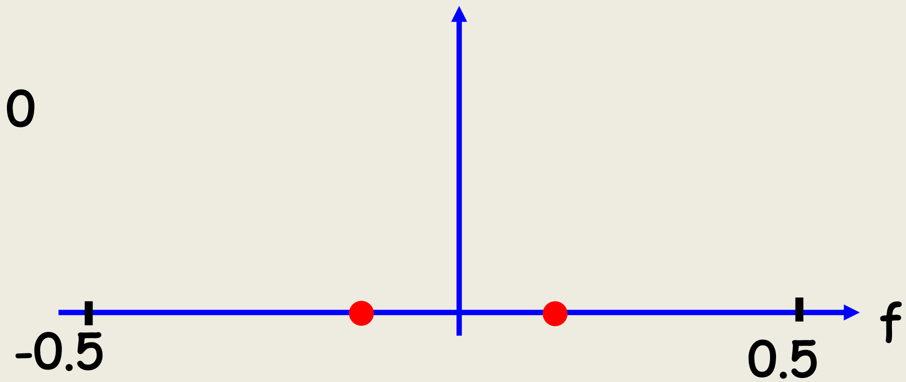
EITF75, DTFT

Pole-zero plot



DTFT is H(z) at unit circle

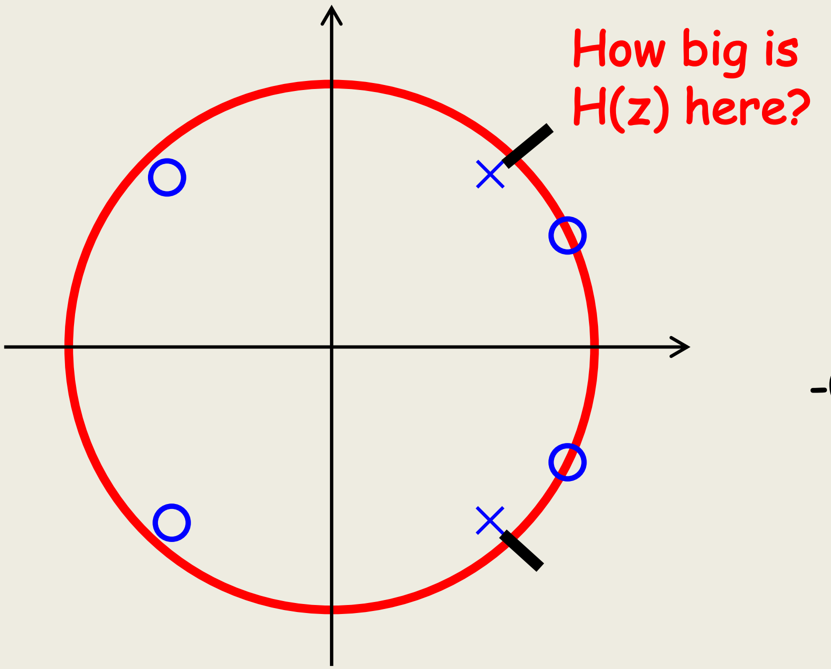


We are at a zero

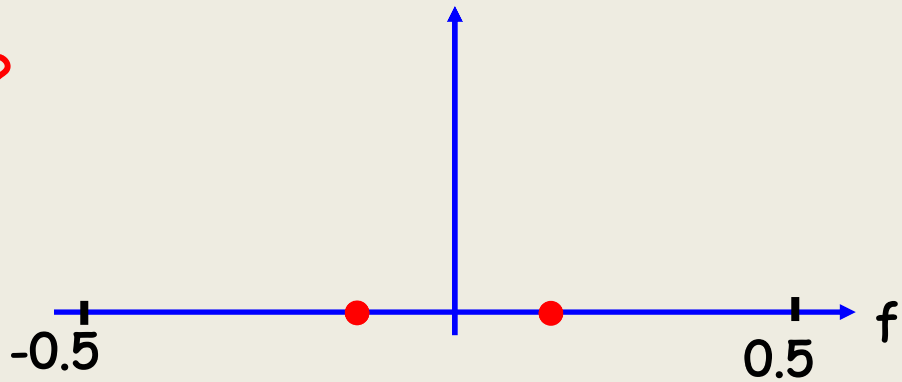
$$H(z) = \frac{\boxed{(z-z_1)}\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)}$$

EITF75, DTFT

Pole-zero plot



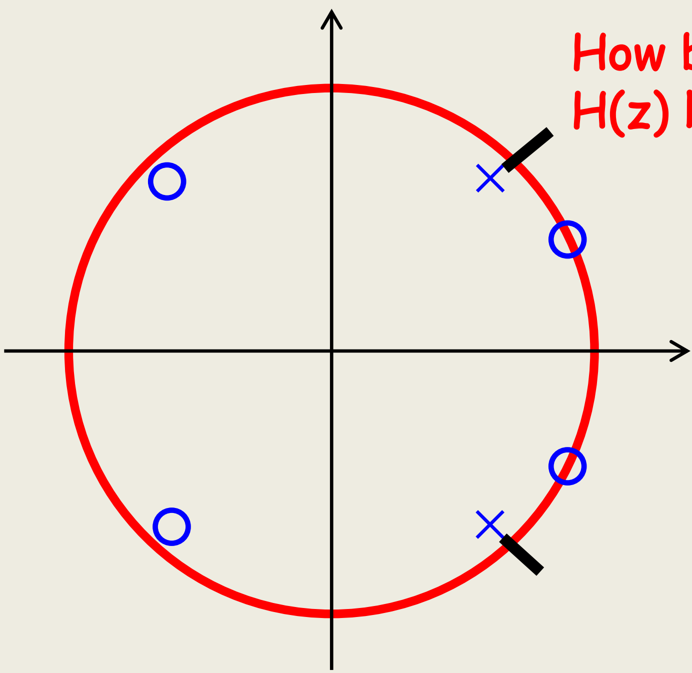
DTFT is H(z) at unit circle



$$H(z) = \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}$$

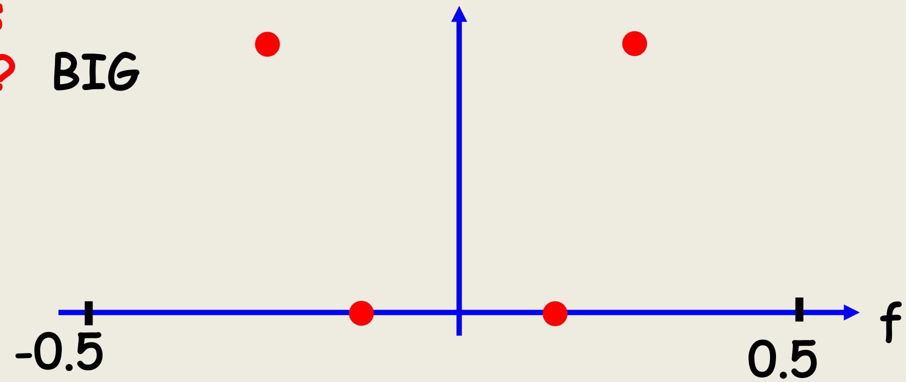
EITF75, DTFT

Pole-zero plot



How big is H(z) here? BIG

DTFT is H(z) at unit circle



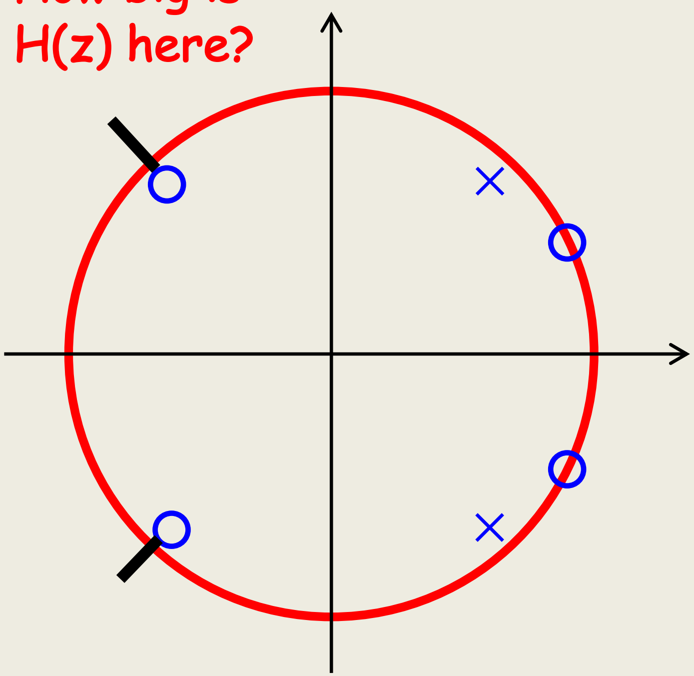
We are close to a pole

$$\boxed{H(z) = \frac{(z-z_1)\cdots(z-z_M)}{\boxed{(z-p_1)}\cdots(z-p_N)}}$$

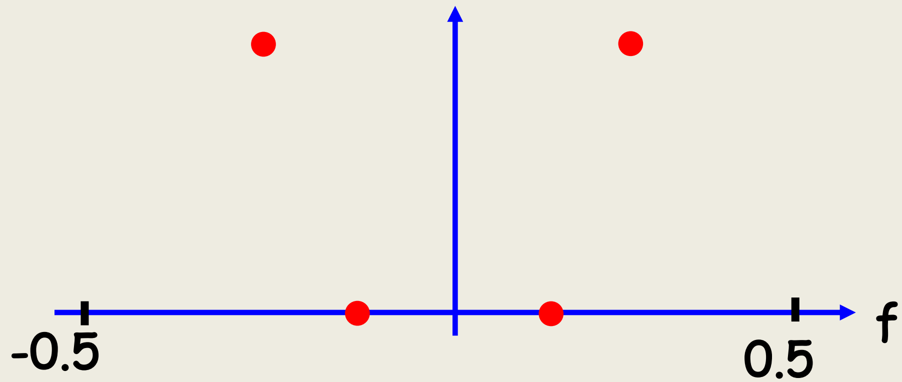
EITF75, DTFT

Pole-zero plot

How big is H(z) here?

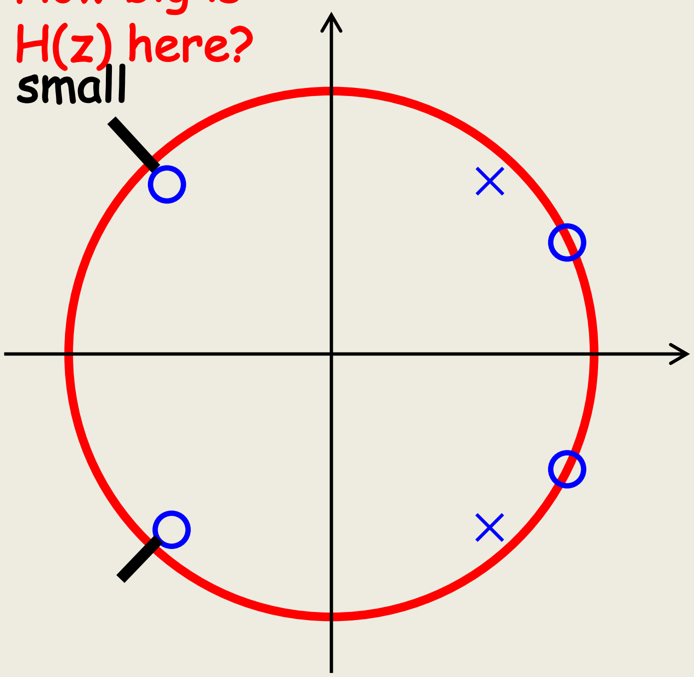


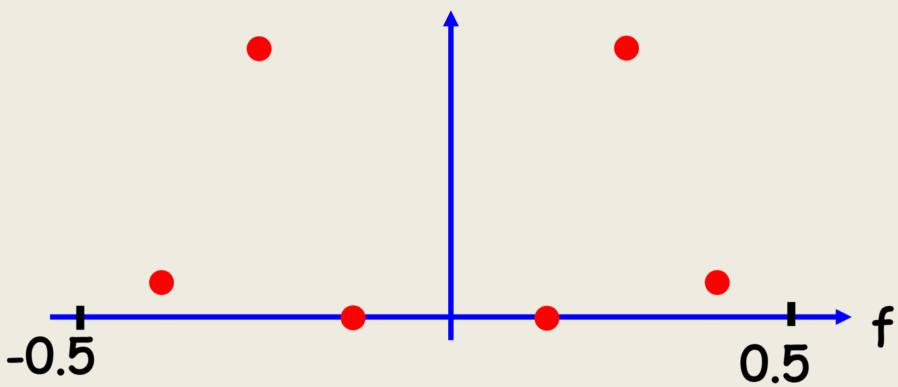
DTFT is H(z) at unit circle



$$H(z) = \frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)}$$

EITF75, DTFT

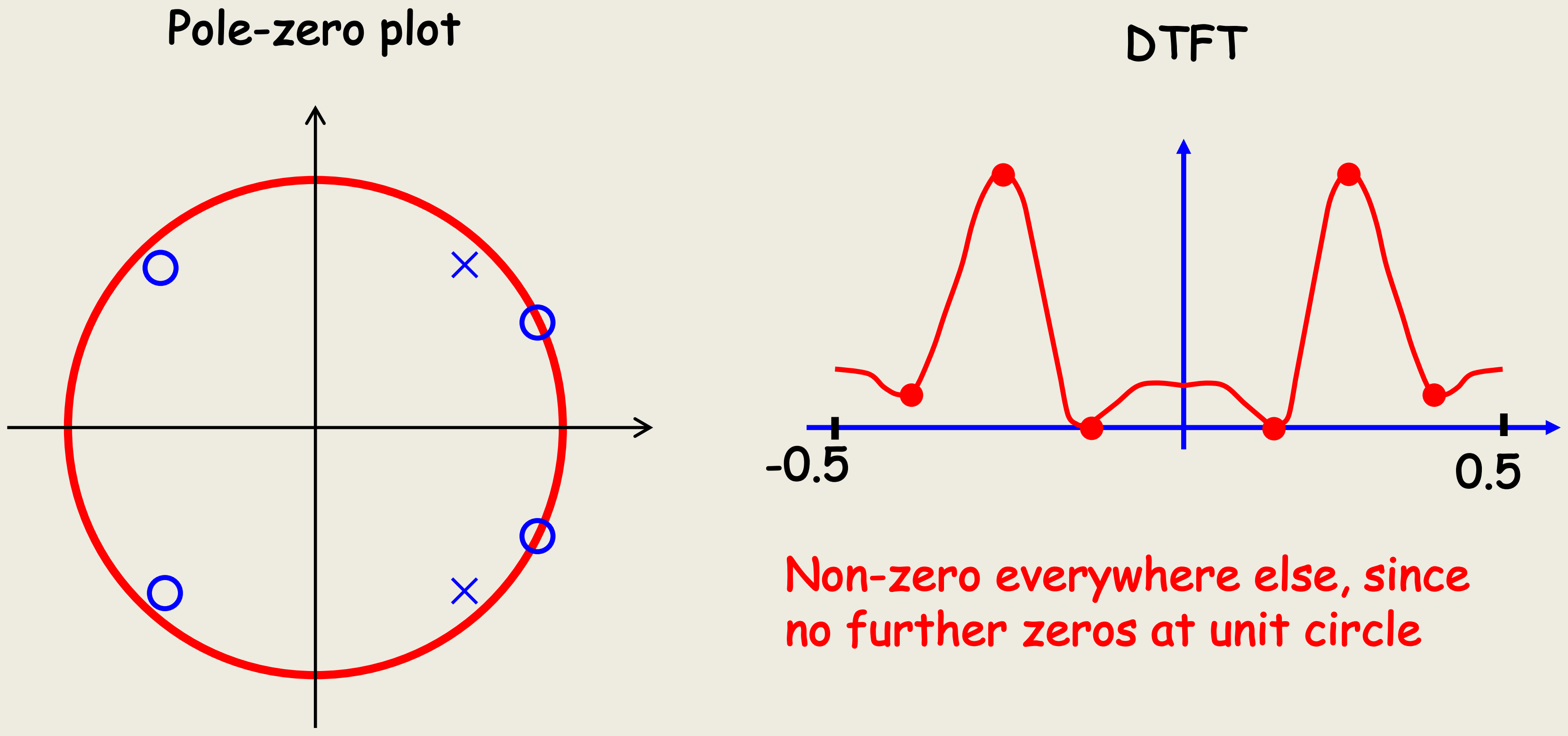




We are close to a zero

$$H(z) = \frac{(z - z_1)\cdots\boxed{(z - z_M)}}{(z - p_1)\cdots(z - p_N)}$$

EITF75, DTFT

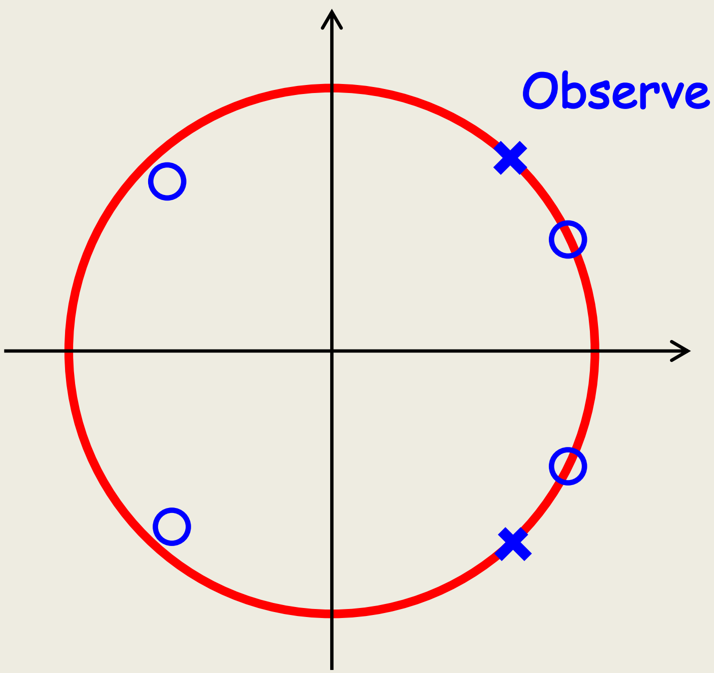


Non-zero everywhere else, since
no further zeros at unit circle

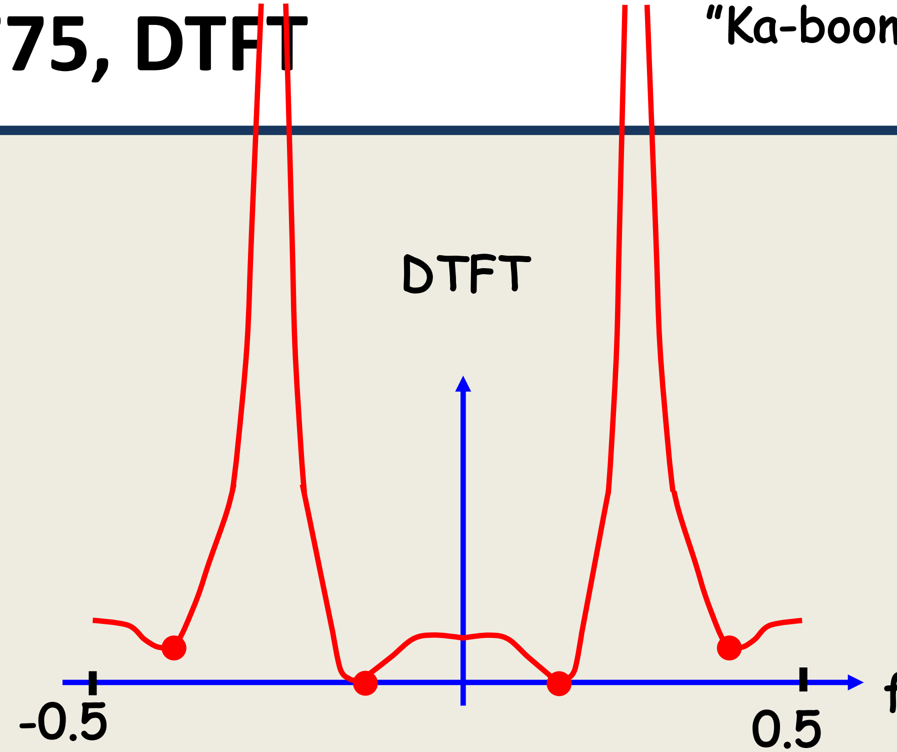
EITF75, DTFT

"Ka-boom"

Pole-zero plot

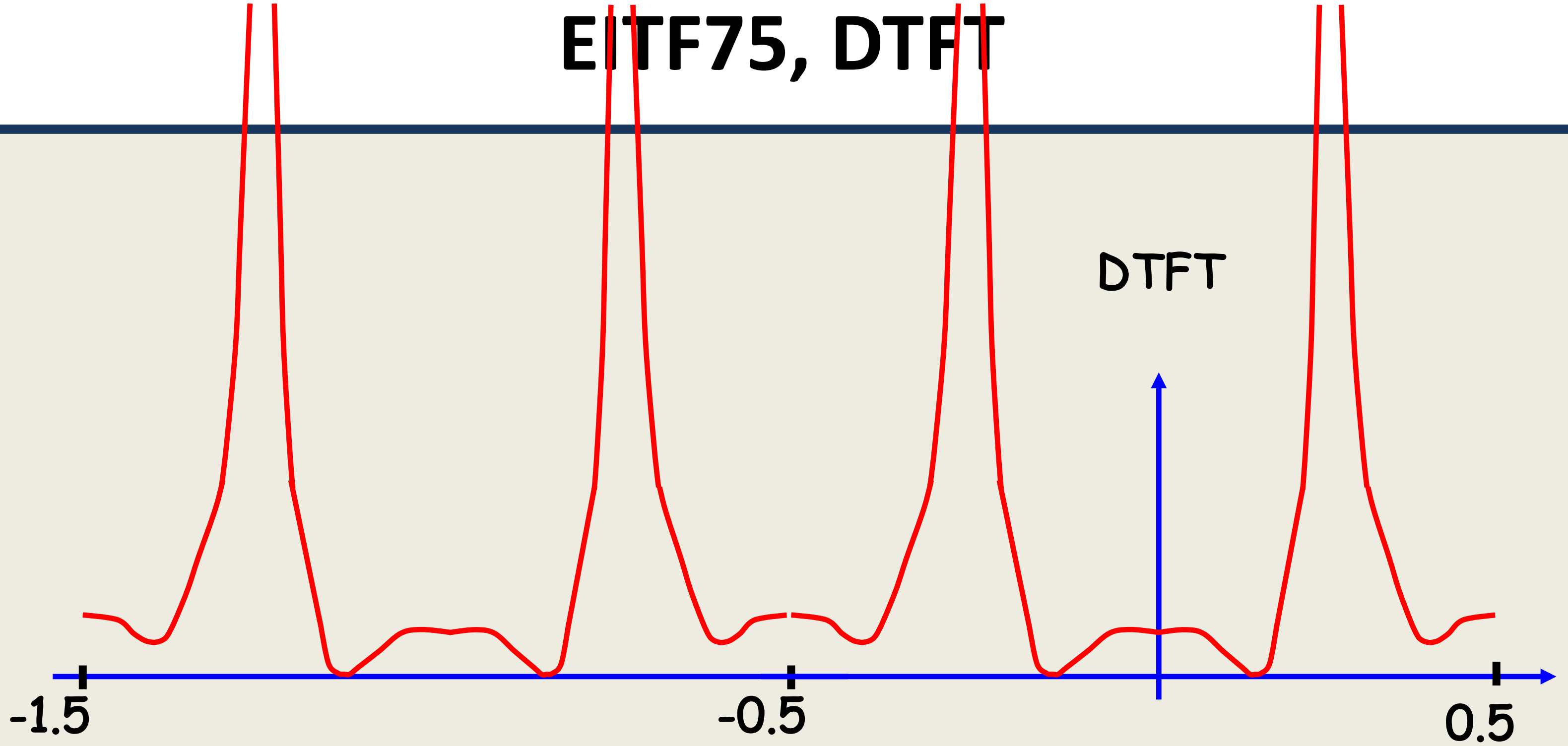


DTFT



Unstable

EITF75, DTFT



Final remark: X(f) is periodic

EITF75 Systems and Signals

DTFT Z-transform

For stable h(n) $H(f) = H\left(e^{i2\pi f}\right)$

For input $x(n) = \exp(i2\pi f_0 n)$

We get the output $y(n) = H(f_0)\exp(i2\pi f_0 n)$

Important: An LTI system cannot create frequencies not present in the input signal

EITF75 Systems and Signals

For cos/sin

Inputs

$$x(n) = \cos(2\pi f_0 n)$$

$$x(n) = \sin(2\pi f_0 n)$$

Outputs (LTI system)

$$y(n) = |H(f_0)| \cos(2\pi f_0 + \Theta(f_0))$$

$$y(n) = |H(f_0)| \sin(2\pi f_0 + \Theta(f_0))$$

ot

EITF75 Systems and Signals

Assume oscillating input, but turned on at n=0

$$x(n) = \cos(\omega_0 n) \cdot u(n) \quad \omega_0 = \frac{2\pi}{16}$$

Steady state solution (i.e., y(n) at big n) is the same as before. At small n, there is a transient behavior.

$$Y(z) = H(z)X(z) = \frac{B(z)}{A(z)}X(z)$$

$$= \frac{B(z)}{A(z)} \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

$$= \sum_{n=1}^{N} \frac{A_n}{1-p_n z^{-1}} + \frac{X_1 + X_2 z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$$

Transient (if all poles inside unit circle)

Steady state (same as for infinite cos)

EITF75 Systems and Signals

Parseval's formula

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \int_{-0.5}^{0.5} X(f)Y^*(f)\mathrm{d}f$$

Special case: $y(n) = x(n)$

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \int_{-0.5}^{0.5} |X(f)|^2\mathrm{d}f$$

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Some filter design

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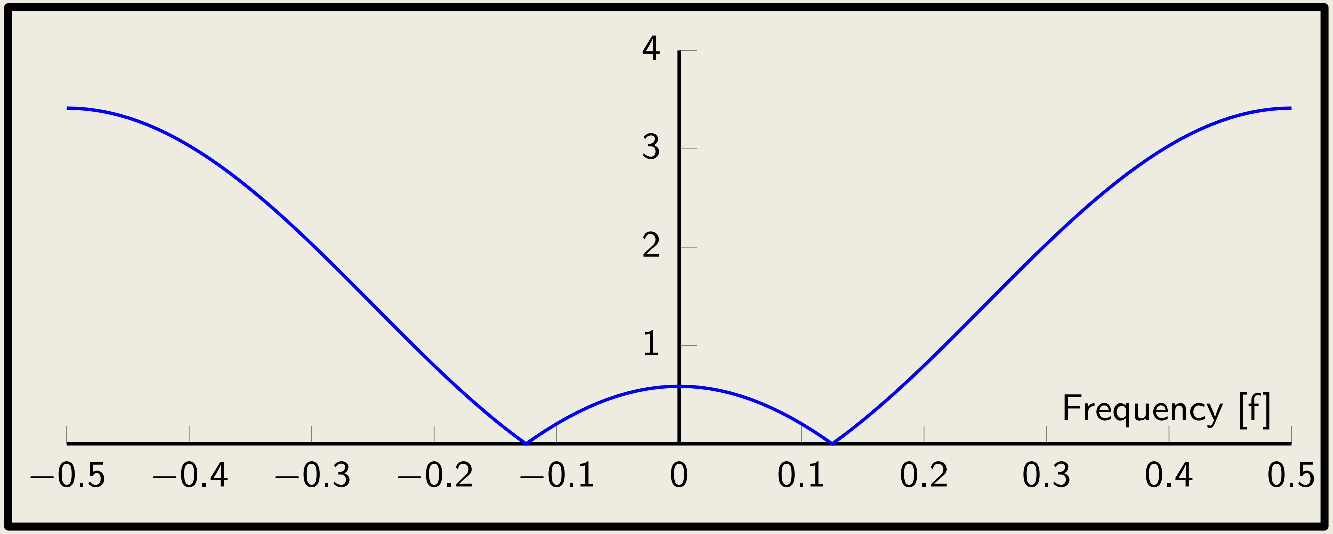
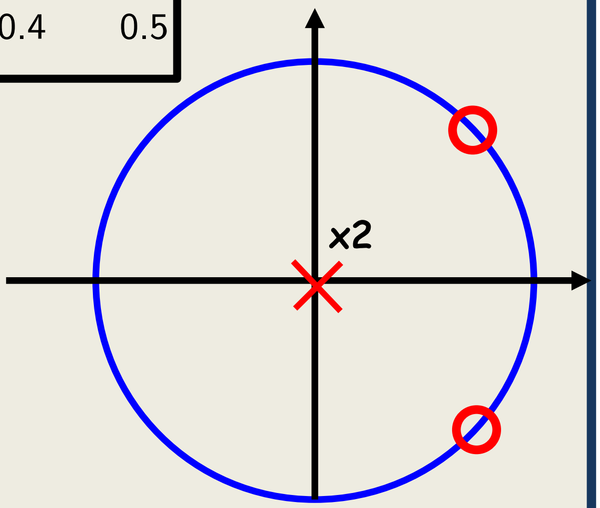


Magnitude response

FIASCO

An attempt to cancel f=0.125 by using two zeros



$h(n) = \{ \underline{1} \quad -2\cos(w_0) \quad 1 \}$

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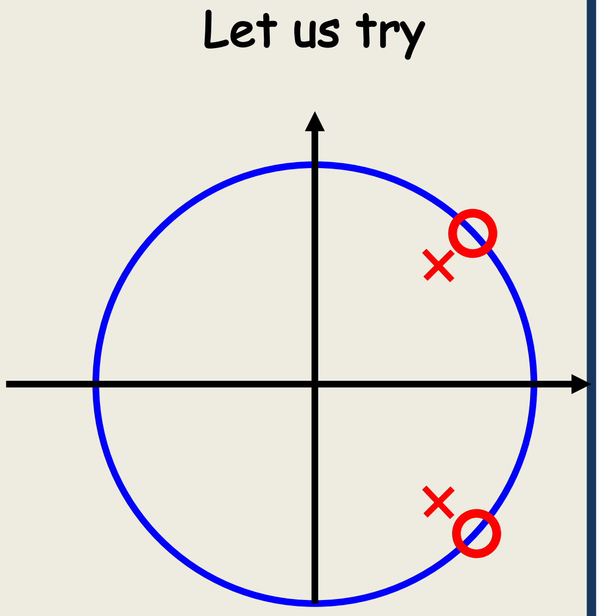

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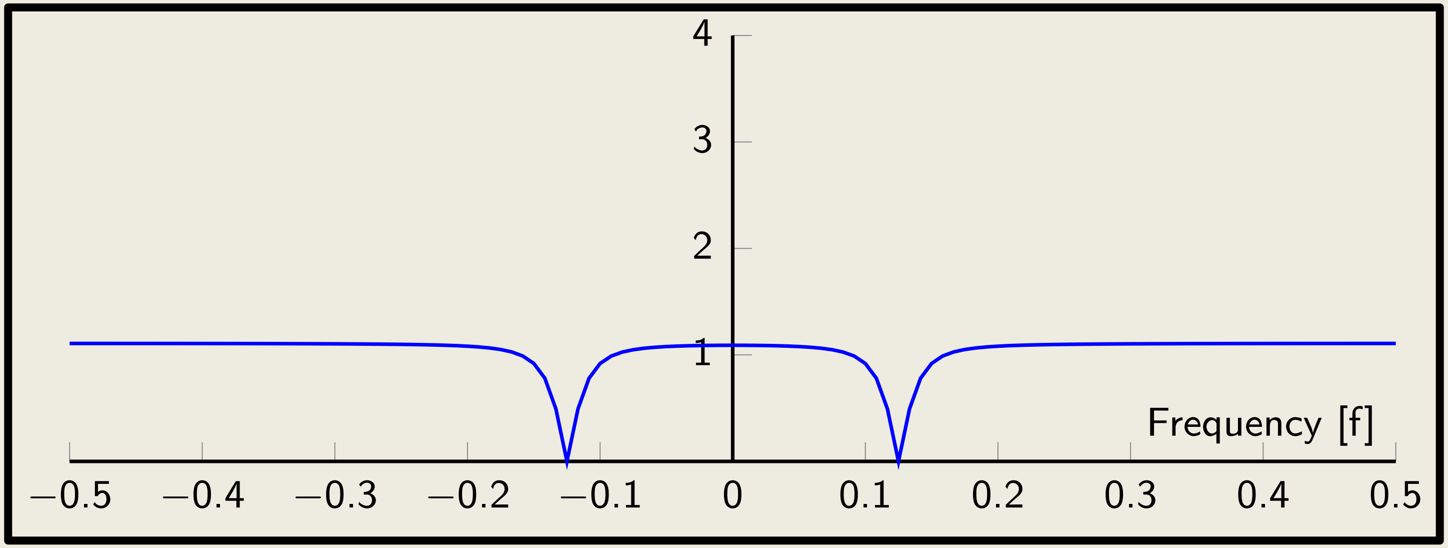
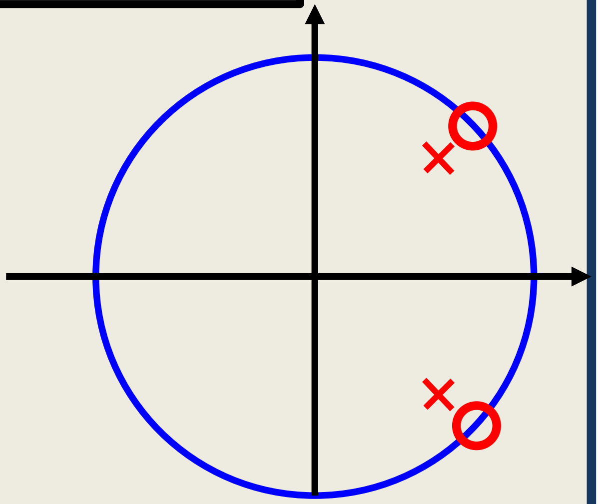


Magnitude response

Much better

NOTCH filter



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FIR filters with linear phase

Linear phase is desirable since it delays all frequencies equally much

Linear phase is defined as $$\Theta(\omega) = \kappa\omega + \pi\ell$$

Whenever there is a phase jump with π, this should be seen as a magnitude response that is negative

$h(n) = h(-n)$	Symmetry around n=0. Not causal
$h(n) = h(N-n)$	Symmetry around n=(N-1)/2.
$h(n) = -h(N-n)$	Anti-symmetry around n=(N-1)/2.

Three types of linear phase filters

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Example TYPE 1

$$h(n) = \left\{ \begin{matrix} 1 & 2 & \underline{3} & 2 & 1 \end{matrix} \right\} \qquad H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

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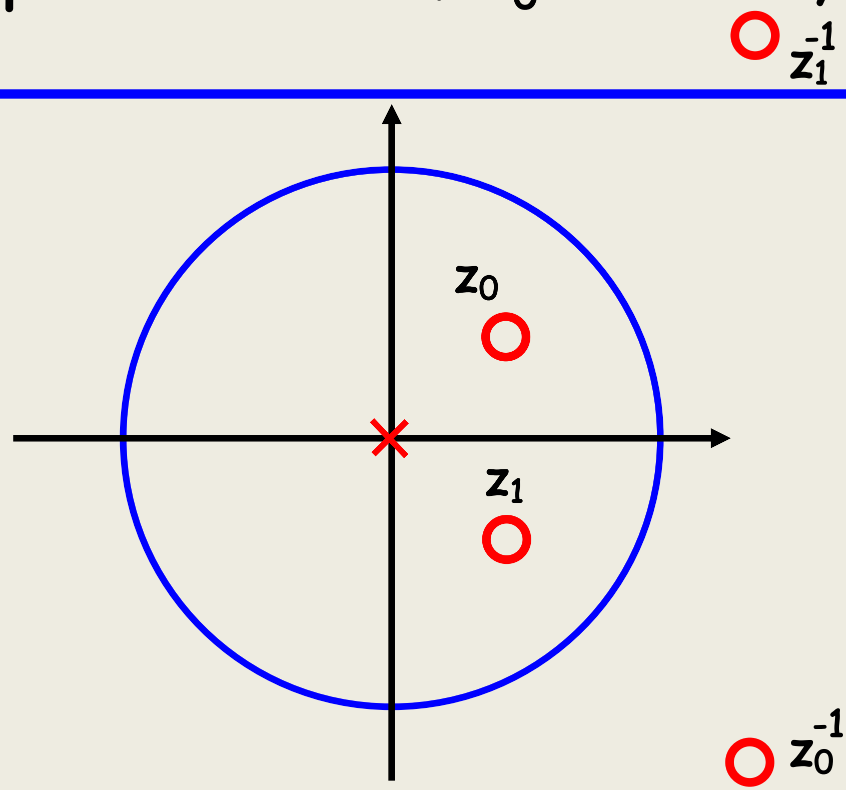
Pole-Zero diagram for linear phase FIR filters

Linear phase <-> If z_0 is a zero, so is z_0^{-1}



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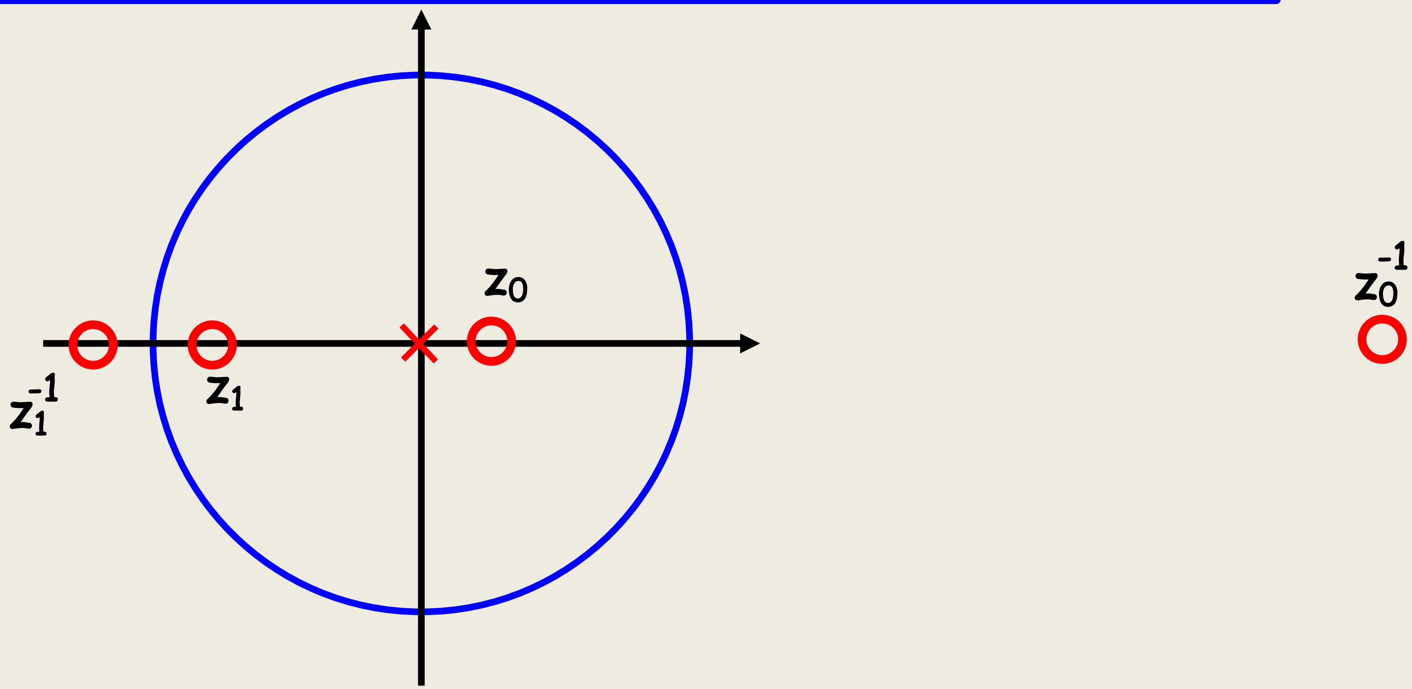
Pole-Zero diagram for linear phase FIR filters

Linear phase <-> If z_0 is a zero, so is z_0^{-1}



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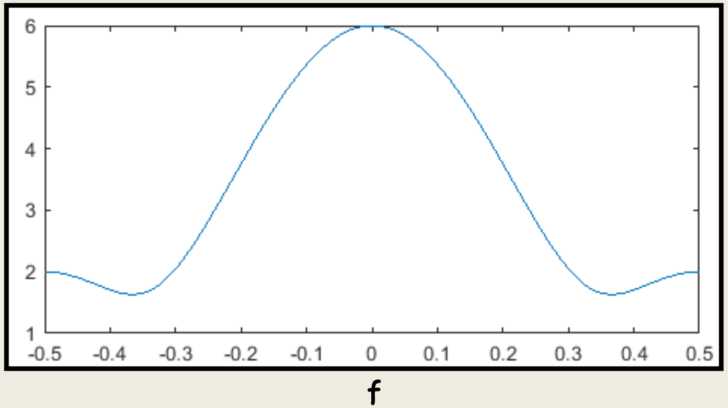
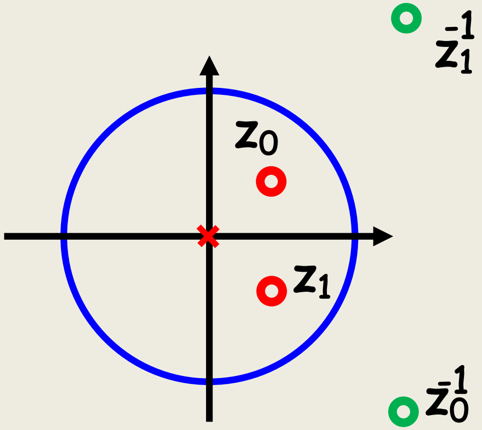
Minimum phase filters

$$|H(f)| = |H^*(f)|$$

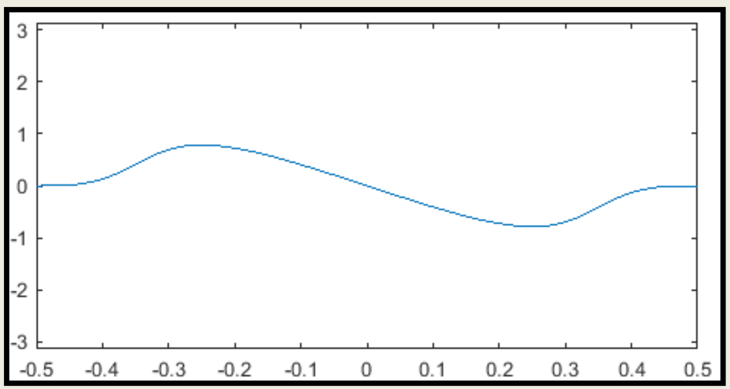


This is a general rule:
A filter with all zeros inside the unit circle has smaller phase.

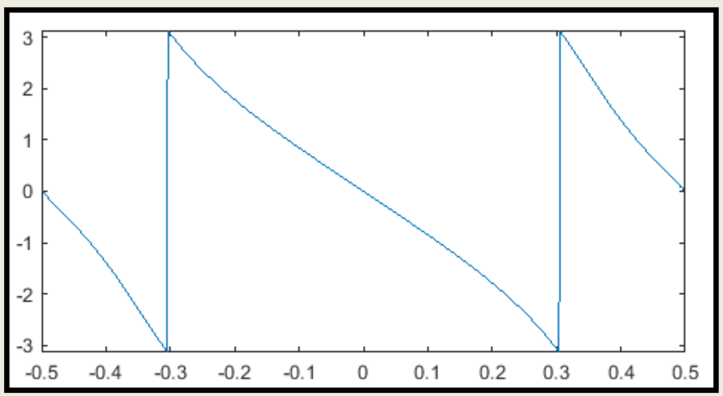
Minimum phase filter
Maximum phase filter



$\Theta(f)$



$\Theta(f)$



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The DFT

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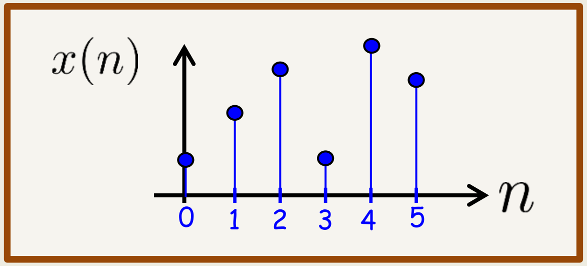
Background and motivation for yet another transform

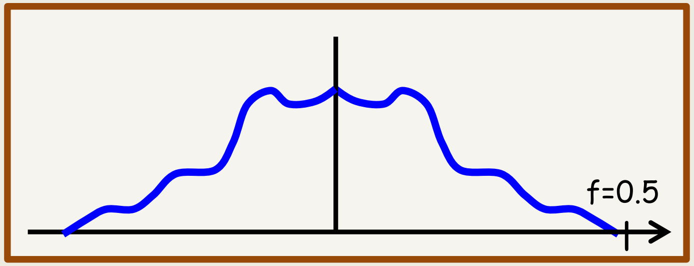
The discrete Fourier Transform (DFT) in one sentence:
A Fourier version of x(n) with 6 numbers

Besides, the DTFT is terribly inefficient



These 6 numbers, are in the frequency domain represened by a continuous curve !

It should be possible to Fourier represent x(n) by 6 numbers as well



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Formal definition

For a sequence $x(n)$ of arbitrary length, the N-point DFT is defined as

$$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } k = 0, 1, \ldots, N-1$$

and the inverse transform (IDFT) as

$$x_{\mathrm{IDFT}}(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) e^{j2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } n = 0, 1, \ldots, N-1$$

Result

if the length of $x(n)$ is N, then

$x_{\mathrm{IDFT}}(n) = x(n)$ and $X_{\mathrm{DFT}}(k) = X(f \mid f = k/N)$

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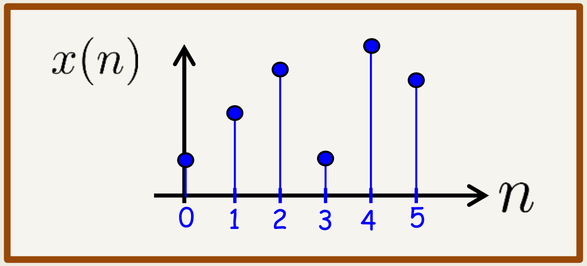
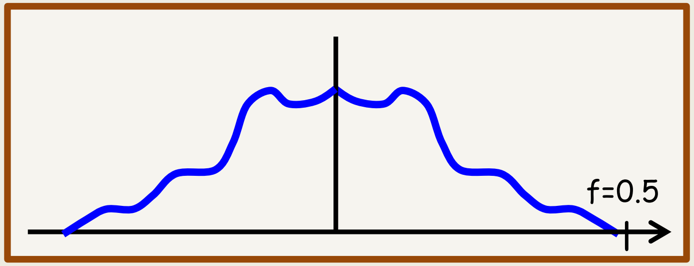
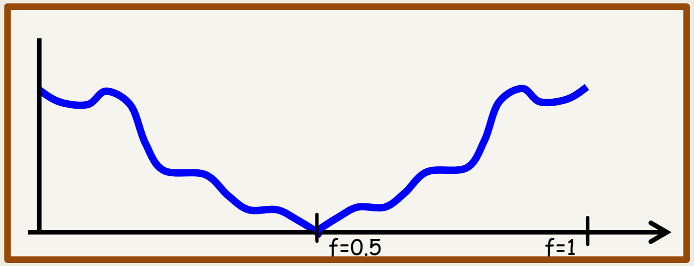




The DTFT is periodic



We can represent it like this

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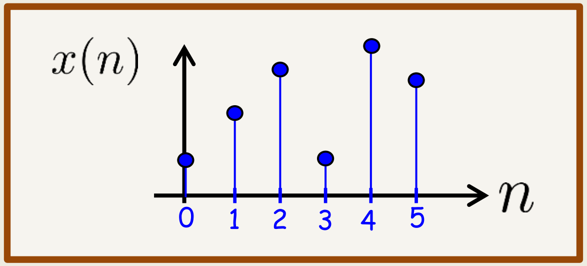

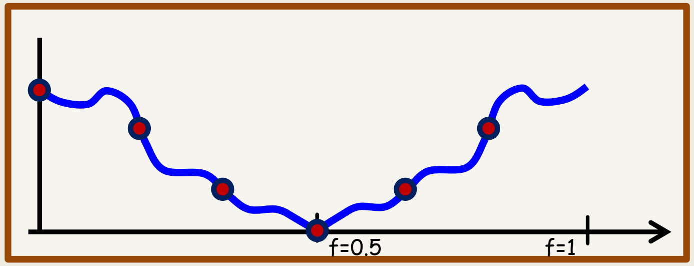

A 6-point DFT would compute the samples of the DTFT

This is sufficient to represent x(n)

Important: The DFT size must be at least as long as the signal, otherwise there is a loss (aliasing in time)

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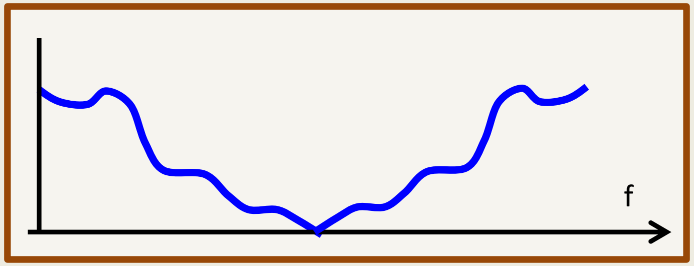

Assume a DTFT of a 6-tap signal

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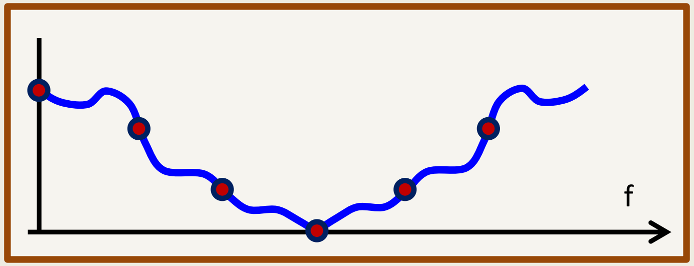

Sample it

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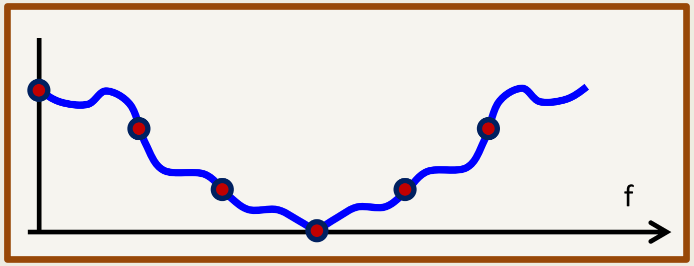
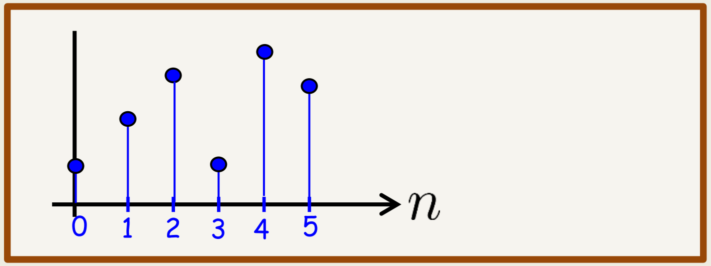


Sample it



Compute the "other domain" representation from samples. In this case, the time domain

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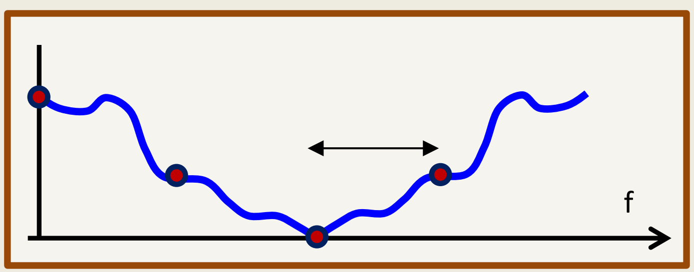
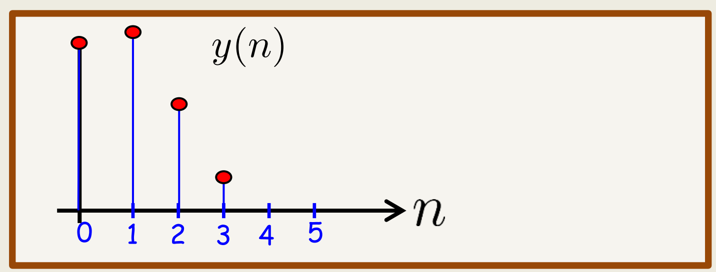
The time-aliasing only occurs if we are not careful with the DFT size. If it is equal or larger than the length of the signal, there is no time-aliasing



But if sample spacing is too small…



There is aliasing

Periodically extended

Aliasing $y(n) = \sum_{m=-\infty}^{\infty} x(n - mN)$

No aliasing $y(n) = x(n)$

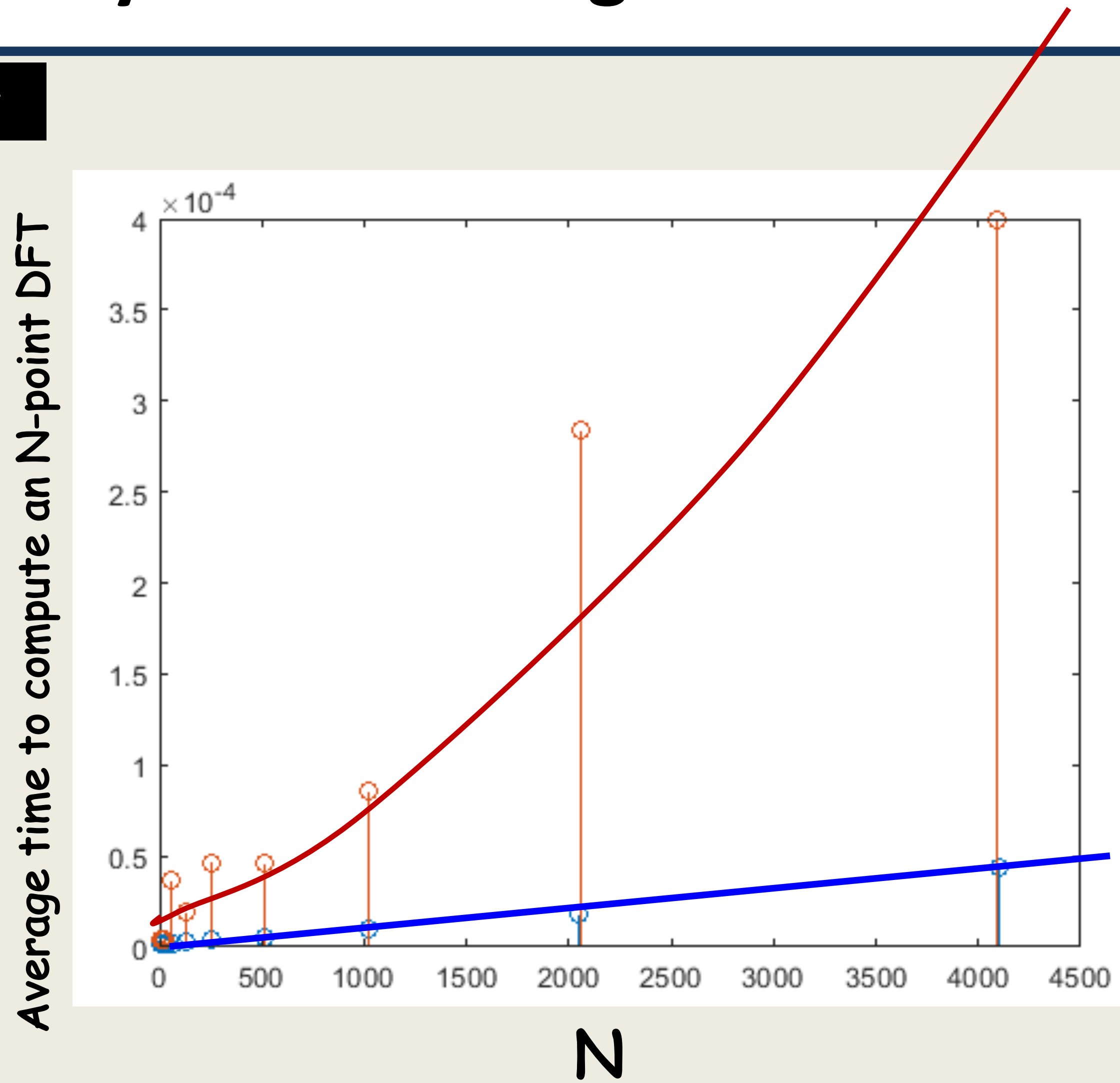
EITF75 Systems and Signals

Computational complexity

Test in Matlab

2048 is 2^{11}

Significant speed-up possible for $N=2^k$



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Computational complexity

FFT not included in course, but good to know about

Cooley and Tukey 1965

Method known to, and used by, Gauss in 1805

Fast Fourier transform (FFT)

If $N=2^k$, then $N \log_2(N)$ complexity to compute

$$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } k = 0, 1, \ldots, N-1$$

Made possible by some algebraic manipulations and tricks.

The importance of the FFT cannot be underestimated. WIFI and 4G, etc could not been implemented without the FFT

For a computer,

1. It can avoid the continuous DTFT
2. It can compute the DFT extremely fast

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Properties

For DTFTs, we have

$$x(n) \star y(n) \leftrightarrow X(f)Y(f)$$

$$x(n) \leftrightarrow X(f) \qquad x(n-n_0) \leftrightarrow X(f)\mathrm{e}^{-i2\pi f n_0}$$

For DFTs, we have

$$x_1(n) \otimes x_2(n) \leftrightarrow X(k)Y(k)$$

$$x(n-n_0 \bmod N) \leftrightarrow X(k)\mathrm{e}^{-i2\pi k n_0/N}$$

where $x_1(n) \otimes x_2(n) = \sum_{k=0}^{N-1} x_1(k)x_2(n-k \bmod N)$

Circular convolution

EITF75 Systems and Signals

Example

Linear convolution computed via DFTs

Given: Two length N sequences, x(n), y(n)

Task: Compute their linear convolution by using DFT and its inverse IDFT

```
>> x=[1 2 3 4];
>> y=[2 2 1 1];
>> yL=conv(x,y)
yL =
     2     6    11    17    13     7     4
```

This is the result, But not computed via DFT

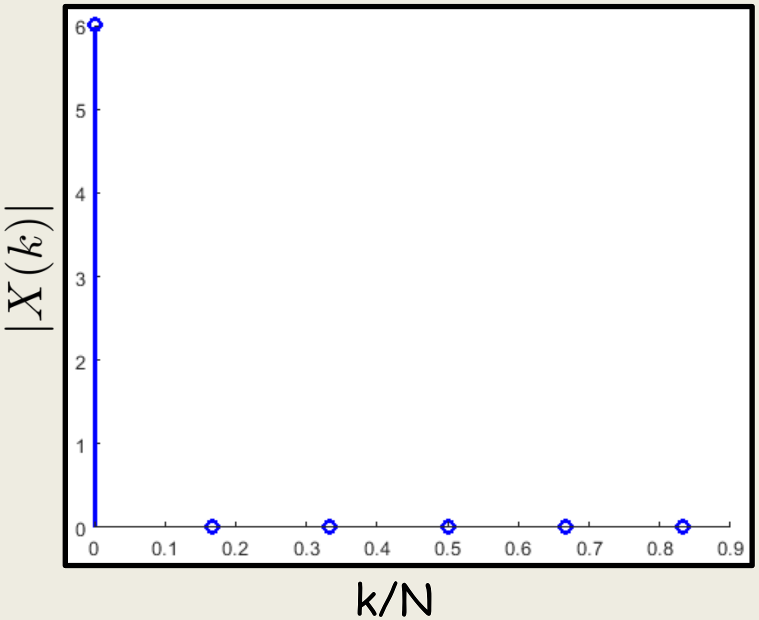
```
>> xp=[1 2 3 4 0 0 0 0];
>> yp=[2 2 1 1 0 0 0 0];
>> yL=ifft(fft(xp).*fft(yp))
yL =
    2.0000    6.0000   11.0000   17.0000   13.0000    7.0000    4.0000   -0.0000
```

Still a circular convolution carried out, but due to zero-padding, it behaves linear.

More examples: Resolution increase

$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\}$$

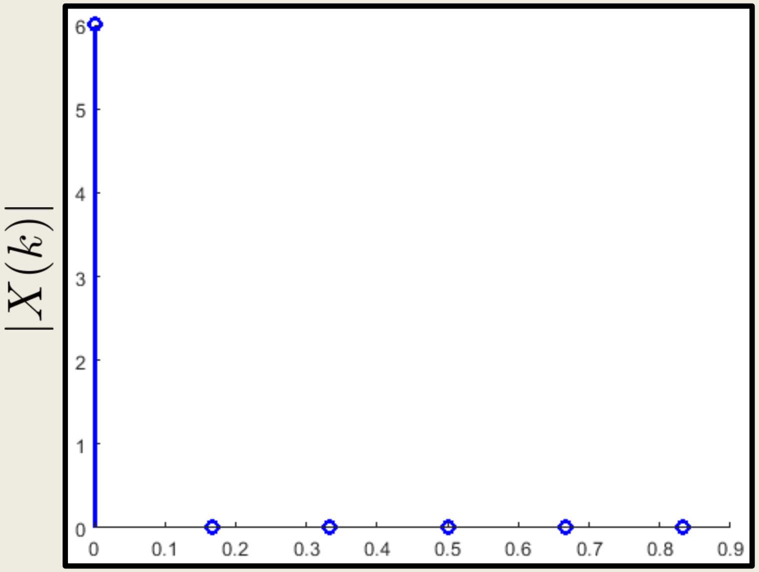
Compute DFT (N=6)



More examples: Resolution increase

$$x(n) = \{1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\}$$

Compute DFT (N=8)



k/N

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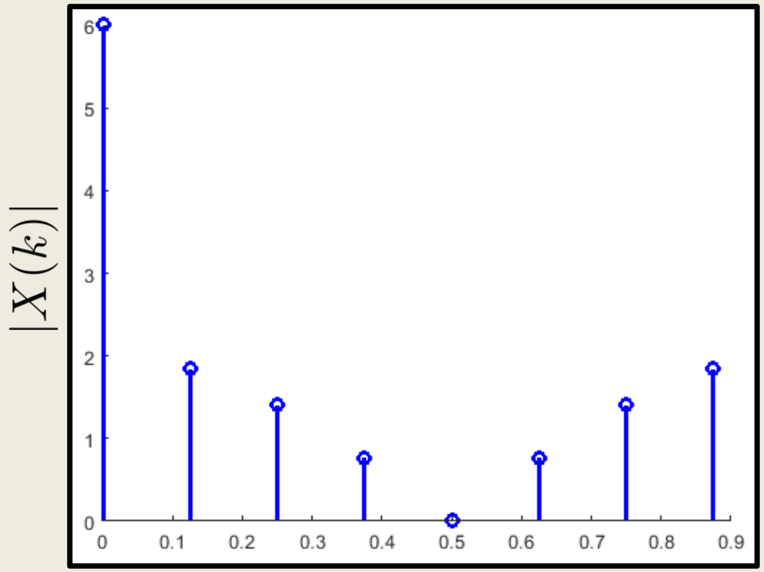
More examples: Resolution increase

$$x(n) = \{1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \;\; 0\}$$

Compute DFT (N=8)



k/N

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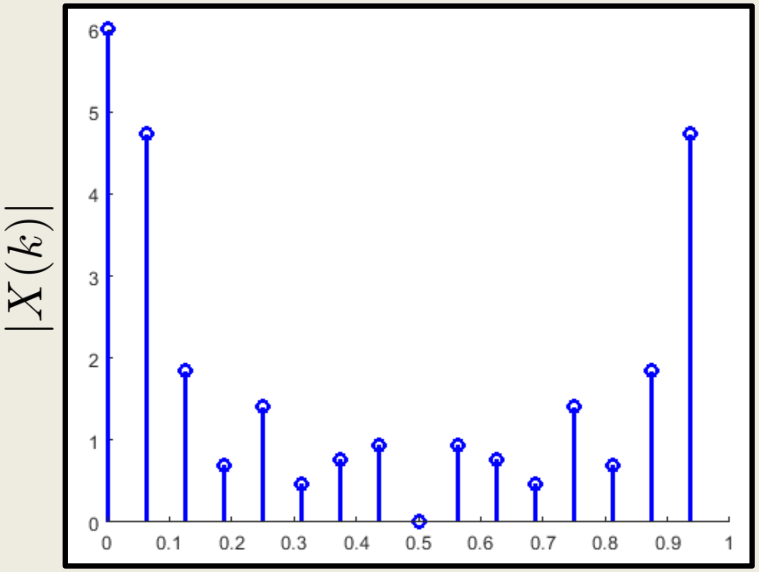
More examples: Resolution increase

$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \cdots\cdots\}$$

Compute DFT (N=16)



k/N

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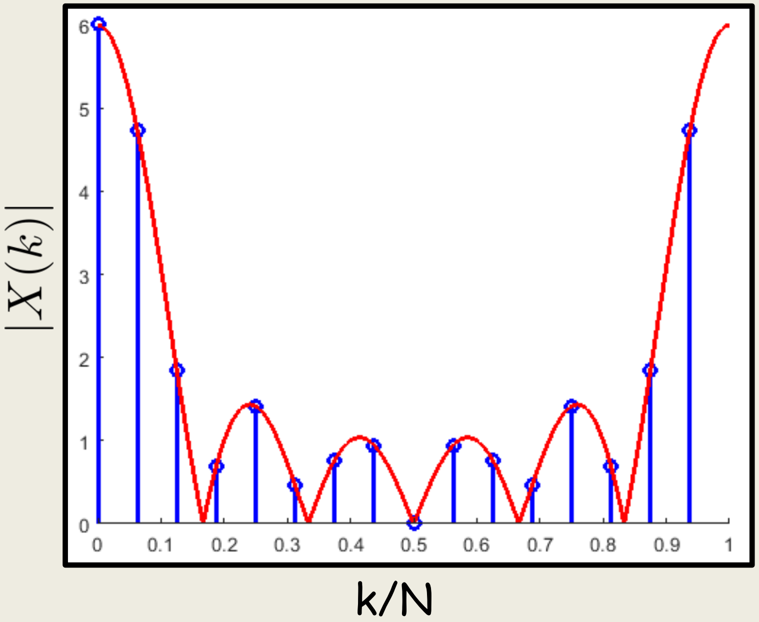
More examples: Resolution increase

$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \cdots\cdots\}$$

Compute DFT (N=16)



What is this line?

DFT size larger-or-equal to the length of x(n)

Therefore, DFT samples of DTFT