



# Digitaltechnik EITF65

## Lecture 9: State Minimisation

# Equivalent graphs

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## Definition

**Equivalent graphs** Two graphs are said to be equivalent if they describe the same behaviour.

Build two realisations:



If the graphs are equivalent it is impossible to distinguish them by manipulating the input sequences and observing the output sequences.

# Equivalence relation

## Definition (Ch 3)

An **equivalence relation** is a binary relation,  $\equiv$ , over a set  $\mathcal{S}$  such that it is

- ▶ reflexive,

$$s \equiv s$$

- ▶ symmetric,

$$s_1 \equiv s_2 \Rightarrow s_2 \equiv s_1$$

- ▶ transitive,

$$s_1 \equiv s_2 \text{ and } s_2 \equiv s_3 \Rightarrow s_1 \equiv s_3$$

for all  $s, s_1, s_2, s_3 \in \mathcal{S}$ .

An equivalence relation over  $\mathcal{S}$  partitions  $\mathcal{S}$  into disjoint subsets, **equivalence classes**.

## Equivalence relation (Ex)

### Example

Consider the set of all integers,  $\mathbb{Z}$ . and let  $\equiv$  mean **congruent modulo 5**, i.e.,

$$a \equiv b \Leftrightarrow R_m(a) = R_m(b)$$

Then  $\equiv$  is an equivalence relation over  $\mathbb{Z}$ . There are 5 equivalence classes, and the  $i$ th contains all integers  $a$  such that  $R_5(a) = i$ ;

$$[0]_5 = \{\dots, -10, -5, 0, 5, 10, \dots\}$$

$$[1]_5 = \{\dots, -9, -4, 1, 6, 11, \dots\}$$

$$[2]_5 = \{\dots, -8, -3, 2, 7, 12, \dots\}$$

$$[3]_5 = \{\dots, -7, -2, 3, 8, 13, \dots\}$$

$$[4]_5 = \{\dots, -6, -1, 4, 9, 14, \dots\}$$

If we represent the classes with its least positive integer we are back to modulo calculation.

# State equivalence

## Definition

Two states  $s_1$  and  $s_2$  are **equivalent**,

$$s_1 \equiv s_2$$

if for all input sequences the corresponding output sequences are identical, regardless if we start in  $s_1$  or  $s_2$ .

State-equivalence is an equivalence relation, and the equivalence classes correspond to non-equivalent states.

## Definition (6.2)

If there are no equivalent states in a finite state machine, we say that the machine is on **reduced form**.

## RF-algorithm

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To find a reduced form of a graph we can use the following algorithm.

**Step 1** Form  $P_1$  by grouping states with the same output function in blocks.

**Step  $\ell$**  Let

$$P_{\ell-1} = \{\mathcal{B}_1, \dots, \mathcal{B}_m\}$$

Form

$$P_\ell = \{\tilde{\mathcal{B}}_1, \dots, \tilde{\mathcal{B}}_n\}$$

by grouping the states in each block  $\mathcal{B}_j$  by the groups of their next states.

If  $P_\ell = P_{\ell-1}$  then

$$\mathcal{S}_{\text{red}} = P_\ell$$

STOP

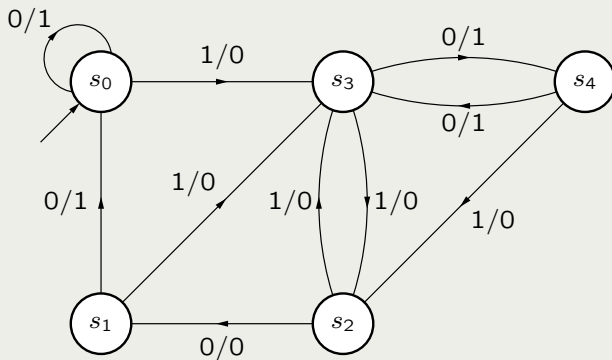
Otherwise

GOTO Step  $\ell + 1$ .

# RF-algorithm, example

## Example

Consider the graph



## Example (cont'd)

Tabular:

s	0	1
$s_0$	$s_0/1$	$s_3/0$
$s_1$	$s_0/1$	$s_3/0$
$s_2$	$s_1/0$	$s_3/0$
$s_3$	$s_4/1$	$s_2/0$
$s_4$	$s_3/1$	$s_2/0$

RF-algorithm:

$$P_1 = \{s_0 s_1 s_3 s_4\} \{s_2\}$$

$$P_2 = \{s_0 s_1\} \{s_3 s_4\} \{s_2\}$$

$$P_3 = \{s_0 s_1\} \{s_3 s_4\} \{s_2\}$$

$s_{01} \quad s_{34}$

Reduced form graph:

S	0	1
$s_{01}$	$s_{01}/1$	$s_{34}/0$
$s_2$	$s_{01}/0$	$s_{34}/0$
$s_{34}$	$s_{34}/1$	$s_2/0$



## Definition (6.3)

If it exists at least one path through the graph from state  $s_0$  to state  $s$ , then  $s$  is **reachable** from  $s_0$ .

A state transition graph with starting state  $s_0$  is **reachable** if all states are reachable from  $s_0$ .

A graph with starting state  $s_0$  can be realized by the reachable part of the graph.

## Definition (6.5)

A **minimal graph** is a reduced form of a reachable graph.

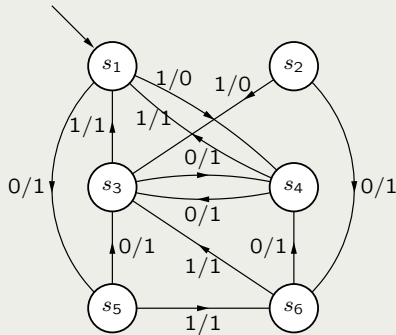
To find a minimal form of a graph

- ▶ Remove the states from  $\mathcal{S}$  that are not reachable from  $s_0$ , i.e., find the reachable part.
- ▶ Use the **RF-algorithm** to find a reduced form of this graph.

# An example

## Example

Minimise the following graph:



## Example (solution)

Draw the state transition table:

$s$	0	1
$s_1$	$s_5/1$	$s_4/0$
$s_2$	$s_6/1$	$s_3/0$
$s_3$	$s_4/1$	$s_1/1$
$s_4$	$s_3/1$	$s_1/1$
$s_5$	$s_3/1$	$s_6/1$
$s_6$	$s_4/1$	$s_3/1$

We see that state  $s_2$  is unreachable from the starting state  $s_1$ . Hence, the reachable graph is

$s$	0	1
$s_1$	$s_5/1$	$s_4/0$
$s_3$	$s_4/1$	$s_1/1$
$s_4$	$s_3/1$	$s_1/1$
$s_5$	$s_3/1$	$s_6/1$
$s_6$	$s_4/1$	$s_3/1$

## Example (solution)

RF-algorithm:

$$P_1 = \{s_1\}\{s_3s_4s_5s_6\}$$

$$P_2 = \{s_1\}\{s_3s_4\}\{s_5s_6\}$$

$$P_3 = \{s_1\}\{s_3s_4\}\{s_5\}\{s_6\}$$

$$P_4 = P_3$$

A minimal graph is given by

$s$	0	1
$s_1$	$s_5/1$	$s_{34}/0$
$s_{34}$	$s_{34}/1$	$s_1/1$
$s_5$	$s_{34}/1$	$s_6/1$
$s_6$	$s_{34}/1$	$s_{34}/1$

Or, in graphical form:

