# Digitalteknik EITF65

Lecture 7: Minimal Functions

# Implicants and function cover

## Definition

Let $\mathcal{F}$ be a set of cubes and $V(\mathcal{F})$ the corresponding vertices. Then the cubes in $\mathcal{F}$ cover the function $f$ if and only if

$$f^{-1}(1) \subseteq V(\mathcal{F}) \subseteq f^{-1}(1) \cup f^{-1}(-)$$

A cube function $c^{\mathcal{C}}(\mathsf{x})$ for a cube in a cover of $f$, i.e.

$$c^{\mathcal{C}}(\mathsf{x}) \quad \text{s.t.} \quad V(c) \subseteq f^{-1}(1) \cup f^{-1}(-)$$

is called an implicant of $f$.

If $\mathcal{F}$ cover the function $f$, then $f$ can be written

$$f(\mathsf{x}) = \bigvee_{\mathcal{C} \in \mathcal{F}} c^{\mathcal{C}}(\mathsf{x}).$$

# Minimal cover

**Definition (5.1)**

Let $\mathcal{I}$ be a set of implicants that covers the function $f$. If the number of implicants in the cover is minimal over all possible covers of $f$, it is a minimal cover.

Our goal is to find a minimal cover of $f$.

# Prime implicants

**Definition**

A prime implicant is an implicant that is not covered by any other implicant. If a cover consists of only prime implicants it is a prime cover.
An essential prime implicant covers a minterm (vertex) that is not covered by any other prime implicant.

A prime cover must contain the essential primes.

# Prime cover

**Theorem (5.2)**

*A minimal cover of the incompletely specified function $f$ can be obtained by a minimal cover of $f^{-1}(1)$ with prime implicants of $f^{-1}(1) \cup f^{-1}(-)$.*

- Find all prime implicants of $f^{-1}(1) \cup f^{-1}(-)$
- Use them to find a minimal cover of $f^{-1}(1)$

# Iterative consensus

One way to derive all prime implicants for a Boolean expressions is to, for each term, iteratively

- expand with all consensus terms,

$$ab \vee a'c = ab \vee a'c \vee bc$$

- simplify with absorption,

$$a \vee ab = a$$

# Example of iterative consensus

## Example

Simplify $f = x_1'x_2' \vee x_1x_2'x_3 \vee x_1x_2x_3$

$$
\begin{aligned}
f &= \underbrace{x_1'x_2'}_{A} \vee \underbrace{x_1x_2'x_3}_{B} \vee \underbrace{x_1x_2x_3}_{C} \vee \underbrace{x_2'x_3}_{D=C(A,B)} && \text{(Add consensus for A)} \\
&= \underbrace{x_1'x_2'}_{A} \vee \underbrace{x_1x_2'x_3}_{B=x_1 D} \vee \underbrace{x_1x_2x_3}_{C} \vee \underbrace{x_2'x_3}_{D} && \text{(Remove with absoption)} \\
&= \underbrace{x_1'x_2'}_{A} \vee \underbrace{x_1x_2x_3}_{C} \vee \underbrace{x_2'x_3}_{D} \vee \underbrace{x_1x_3}_{E=C(C,D)} && \text{(Add consensus for C)} \\
&= \underbrace{x_1'x_2'}_{A} \vee \underbrace{x_1x_2x_3}_{C=x_2 E} \vee \underbrace{x_2'x_3}_{D} \vee \underbrace{x_1x_3}_{E} && \text{(Remove with absoption)} \\
&= \underbrace{x_1'x_2'}_{A} \vee \underbrace{x_2'x_3}_{D} \vee \underbrace{x_1x_3}_{E} && \text{(All primes)}
\end{aligned}
$$

Since $x_2'x_3 = C(A,E)$ this can be removed to get the minimal solution

$$f = x_1'x_2' \vee x_1x_3$$

# Gray code

In a Gray code only one variable is changed for each (cyclic) step.

## Example

Example of NBCD and Gray code for $N = 4$ and $N = 8$.

$N = 4$

| n | NBCD | Gray |
|---|---|---|
| 0 | 00 | 00 |
| 1 | 01 | 01 |
| 2 | 10 | 11 |
| 3 | 11 | 10 |

$N = 8$

| n | NBCD | Gray |
|---|---|---|
| 0 | 000 | 000 |
| 1 | 001 | 001 |
| 2 | 010 | 011 |
| 3 | 011 | 010 |
| 4 | 100 | 110 |
| 5 | 101 | 111 |
| 6 | 110 | 101 |
| 7 | 111 | 100 |

(NBCD = Natural Binary Coded Decimal.)

# Karnaugh maps

A Karnaugh map is a function table in matrix form with

- at most two variables per dimension.
- the input combinations are listed as Gray code.

Then, between two consecutive positions (horisontal or vertical) there is only a change in one variable.

The Karnaugh map can be seen as a graphicaly interpretation of $B^n$. The maps useful on paper (2-dimensions) are for functions with 2, 3, or 4 variables.

# Karnaugh maps (cont'd)

$f(x_1x_2)$:

| $f$ | $x_1$ \ $x_2$ | 0 | 1 |
|---|---|---|---|
| $x_1$ | 0 | $f(0)$ | $f(1)$ |
| | 1 | $f(2)$ | $f(3)$ |

$f(x_1x_2x_3)$:

| $f$ | $x_1$ \ $x_2x_3$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|---|
| $x_1$ | 0 | $f(0)$ | $f(1)$ | $f(3)$ | $f(2)$ |
| | 1 | $f(4)$ | $f(5)$ | $f(7)$ | $f(6)$ |

| $f$ | $x_1x_2$ \ $x_3$ | 0 | 1 |
|---|---|---|---|
| $x_1x_2$ | 00 | $f(0)$ | $f(1)$ |
| | 01 | $f(2)$ | $f(3)$ |
| | 11 | $f(6)$ | $f(7)$ |
| | 10 | $f(4)$ | $f(5)$ |

$f(x_1x_2x_3x_4)$:

| $f$ | $x_1x_2$ \ $x_3x_4$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|---|
| $x_1x_2$ | 00 | $f(0)$ | $f(1)$ | $f(3)$ | $f(2)$ |
| | 01 | $f(4)$ | $f(5)$ | $f(7)$ | $f(6)$ |
| | 11 | $f(12)$ | $f(13)$ | $f(15)$ | $f(14)$ |
| | 10 | $f(8)$ | $f(9)$ | $f(11)$ | $f(10)$ |

# Minimisation

In a Karnaugh map an implicant is a (cyclic) rectangular block covering $2^k$ 1s.

Minimisation by Karnaugh maps:

- Find as big rectangular blocks as possible with $2^k$ 1s and $-$. These represent the prime implicants.
- Find the essential implicants. These must be part of the function.
- Chose a minimal number of the rest of the implicants such that the function is covered.

## Example

Consider the function $f$ with

$$f^{-1}(1) = \{0, 2, 3, 7\}, \quad f^{-1}(-) = \{5, 6\}$$

The Karnaugh map is

| $f$ ($x_1x_2$ \ $x_3$) | 0 | 1 |
|---|---|---|
| 00 | 1 | 0 |
| 01 | 1 | 1 |
| 11 | - | 1 |
| 10 | 0 | - |

Prime implicants:

$$A = c^{(0B0)}(\mathbf{x}) = x_1'x_3'$$

$$B = c^{(B1B)}(\mathbf{x}) = x_2$$

$$C = c^{(1B1)}(\mathbf{x}) = x_1x_3$$

$A$ and $B$ are essential, and covers the function, i.e.,

$$f_{\text{min}} = A \vee B = x_1'x_3' \vee x_2$$

is a minimal function (disjunctive form).

# Karnaugh minimisation (Conjunctive form)

## Example

Consider the function $f$ with

$$f^{-1}(1) = \{0, 2, 3, 7\}, \quad f^{-1}(-) = \{5, 6\}$$

The Karnaugh map is

| $f$ ($x_1x_2$ \ $x_3$) | 0 | 1 |
|---|---|---|
| 00 | 1 | 0 |
| 01 | 1 | 1 |
| 11 | - | 1 |
| 10 | 0 | - |

(Anti) prime implicants:

$$A = \left(c^{(B01)}(\mathbf{x})\right)' = (x_2'x_3)' = x_2 \vee x_3'$$

$$B = \left(c^{(10B)}(\mathbf{x})\right)' = (x_1x_2')' = x_1' \vee x_2$$

$$C = \left(c^{(1B0)}(\mathbf{x})\right)' = (x_1x_3')' = x_1' \vee x_3$$

$A$ is essential. Minimal conjunctive form:

$$f_{\min} = A \wedge B = (x_2 \vee x_3')(x_1' \vee x_2)$$

$$f_{\min} = A \wedge C = (x_2 \vee x_3')(x_1' \vee x_3)$$