# Digitalteknik EITF65

Lecture 8: Karnaugh Maps

September 19, 2018

# Note regarding notation

Some notation in the Compendium has changed from older versions.

## Normal forms for Boolean functions

- DNF (Disjunctive normal form)
- CNF (Conjunctive normal form)
- Reed-Muller Canonical Form (RMF) = Ring-Sum Expansion (RSE) = Algebraic Normal Form (ANF)

# Idea of minimisation

**Theorem**

*There exists a prime cover that is a minimal cover.*

**Minimal prime cover**

- Find all prime implicants.
- Find the essential prime implicants. These must be part of a minimal prime cover a part of on-set.
- Cover the rest of on-set with a minmal number of prime implicants.

# Gray code

In a Gray code only one variable is changed for each (cyclic) step.

## Example

Example of NBCD and Gray code for $N = 4$ and $N = 8$.

$N = 4$

| n | NBCD | Gray |
|---|---|---|
| 0 | 00 | 00 |
| 1 | 01 | 01 |
| 2 | 10 | 11 |
| 3 | 11 | 10 |

$N = 8$

| n | NBCD | Gray |
|---|---|---|
| 0 | 000 | 000 |
| 1 | 001 | 001 |
| 2 | 010 | 011 |
| 3 | 011 | 010 |
| 4 | 100 | 110 |
| 5 | 101 | 111 |
| 6 | 110 | 101 |
| 7 | 111 | 100 |

(NBCD = Natural Binary Coded Decimal.)

# Karnaugh maps

A Karnaugh map is a function table in matrix form with

- at most two variables per dimension.
- the input combinations are listed as Gray code.

Then, between two consecutive positions (horisontal or vertical) there is only a change in one variable.

The Karnaugh map can be seen as a graphicaly interpretation of $B^n$. The maps useful on paper (2-dimensions) are for functions with 2, 3, or 4 variables.

# Karnaugh maps (cont'd)

$f(x_1 x_2)$:

| $f$ | $x_2$ = 0 | 1 |
|---|---|---|
| $x_1$ = 0 | $f(0)$ | $f(1)$ |
| 1 | $f(2)$ | $f(3)$ |

$f(x_1 x_2 x_3)$:

| $f$ | $x_2 x_3$ = 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| $x_1$ = 0 | $f(0)$ | $f(1)$ | $f(3)$ | $f(2)$ |
| 1 | $f(4)$ | $f(5)$ | $f(7)$ | $f(6)$ |

| $f$ | $x_3$ = 0 | 1 |
|---|---|---|
| $x_1 x_2$ = 00 | $f(0)$ | $f(1)$ |
| 01 | $f(2)$ | $f(3)$ |
| 11 | $f(6)$ | $f(7)$ |
| 10 | $f(4)$ | $f(5)$ |

$f(x_1 x_2 x_3 x_4)$:

| $f$ | $x_3 x_4$ = 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| $x_1 x_2$ = 00 | $f(0)$ | $f(1)$ | $f(3)$ | $f(2)$ |
| 01 | $f(4)$ | $f(5)$ | $f(7)$ | $f(6)$ |
| 11 | $f(12)$ | $f(13)$ | $f(15)$ | $f(14)$ |
| 10 | $f(8)$ | $f(9)$ | $f(11)$ | $f(10)$ |

# Minimisation

## Cubes

In a Karnaugh map a cube is a (cyclic) rectangular block covering $2^k$ positions.

## Minimisation by Karnaugh maps

- Find as big rectangular blocks as possible with $2^k$ 1s and $-$. These represent the prime implicants.
- Find the essential implicants. These must be part of the function.
- Chose a minimal number of the rest of the implicants such that the function is covered.

# Karnaugh minimisation (Ex)

## Example

Consider the function $f$ with

$$f^{-1}(1) = \{0, 2, 3, 7\}, \quad f^{-1}(-) = \{5, 6\}$$

The Karnaugh map is

| $f$ ($x_1x_2$ \ $x_3$) | 0 | 1 |
|---|---|---|
| 00 | 1 | 0 |
| 01 | 1 | 1 |
| 11 | - | 1 |
| 10 | 0 | - |

Prime implicants:

$$A = c^{(0B0)}(\mathbf{x}) = x_1'x_3'$$

$$B = c^{(B1B)}(\mathbf{x}) = x_2$$

$$C = c^{(1B1)}(\mathbf{x}) = x_1x_3$$

$A$ and $B$ are essential, and covers the function, i.e.,

$$f_{\text{min}} = A \vee B = x_1'x_3' \vee x_2$$

is a minimal function (disjunctive form).

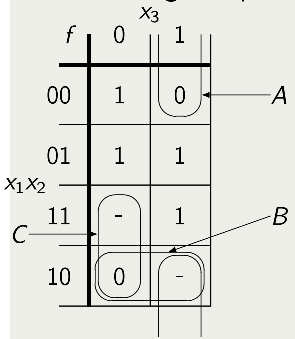
# Karnaugh minimisation (Conjunctive form)

## Example

Consider the function $f$ with

$$f^{-1}(1) = \{0, 2, 3, 7\}, \quad f^{-1}(-) = \{5, 6\}$$

The Karnaugh map is




(Anti) prime implicants:

$$A = \left(c^{(B01)}(\mathbf{x})\right)' = (x_2'x_3)' = x_2 \vee x_3'$$

$$B = \left(c^{(10B)}(\mathbf{x})\right)' = (x_1x_2')' = x_1' \vee x_2$$

$$C = \left(c^{(1B0)}(\mathbf{x})\right)' = (x_1x_3')' = x_1' \vee x_3$$

$A$ is essential. Minimal conjunctive form:

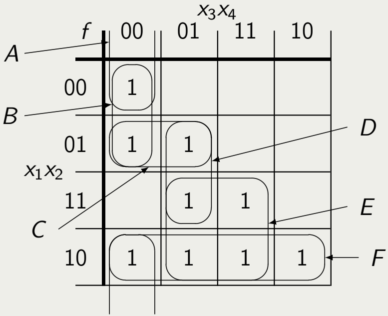
$$f_{\min} = A \wedge B = (x_2 \vee x_3')(x_1' \vee x_2)$$

$$f_{\min} = A \wedge C = (x_2 \vee x_3')(x_1' \vee x_3)$$

# Minimal disjunctive form

## Example 5.7 (disjunctive form)

The function $f(x_1x_2x_3x_4)$ has $f^{-1}(1) = \{0, 4, 5, 8, 9, 10, 11, 13, 15\}$




Prime implicants:

$$A = x_2'x_3'x_4' \quad B = x_1'x_3'x_4' \quad C = x_1'x_2x_3'$$
$$D = x_2x_3'x_4 \quad E = x_1x_4 \quad F = x_1x_2'$$

# Prime table

A prime table (P-table) shows which minterms the prime implicants cover.
Rows correspond to prime implicants.
Columns correspond to minterms.
A cross $\times$ shows when a minterm is covered by the implicant.
Then

- an essential row contains a $\times$ that is not represented in any other rows. Essential rows must be in the cover.
- row $r$ dominates row $s$, $r \supset s$, if all $\times$ in $s$ also exist in $r$. The dominated row, $s$, can be deleted.
- column $c$ dominates column $d$, $c \supset d$, if all $\times$ in $d$ also exist in $c$.
  The dominating column, $c$, can be deleted.

# Prime table (Ex 5.7)

| | 0 | 4 | 5 | 8 | 9 | 10 | 11 | 13 | 15 |
|---|---|---|---|---|---|---|---|---|---|
| $A$ | x | | | x | | | | | |
| $B$ | x | x | | | | | | | |
| $C$ | | x | x | | | | | | |
| $D$ | | | x | | | | | x | |
| $E$ | | | | | x | | x | x | x |
| $F$ | | | | x | x | x | x | | |

Essential: $E$ from 15
$F$ from 10

All columns that have $\times$ in row $E$ and $F$ are deleted from the table, they are covered by $E$ and $F$.

# Prime table (Ex 5.7)

| | 0 | 4 | 5 |
|---|---|---|---|
| $B \supset A$ | x | | |
| $B$ | x | x | |
| $C$ | | x | x |
| $C \supset D$ | | | x |

Here, $B$ dominates $A$ and $C$ dominates $D$.

| | 0 | 4 | 5 |
|---|---|---|---|
| $B$ | x | x | |
| $C$ | | x | x |

Now we see that 4 dominates both 0 and 5. Henc, the column 4 is deleted. Then, both $B$ and $C$ are (secondary) essential.

$$f = E \vee F \vee B \vee C$$

Remark: From the Karnaugh map we can find two other minimal covers: $f = E \vee F \vee B \vee D$ and $f = E \vee F \vee A \vee C$

# Cyclic prime table

## Example

The function $f$ i specified by

$$f^{-1}(1) = \{5, 6, 15\}$$
$$f^{-1}(0) = \{0, 2, 8, 10, 12\}$$

| | $f$ ($x_1x_2$ \ $x_3x_4$) | 00 | 01 | 11 | 10 | |
|---|---|---|---|---|---|---|
| $A$ | 00 | 0 | - | - | 0 | $B$ |
| | 01 | - | 1 | - | 1 | |
| | 11 | 0 | - | 1 | - | $C$ |
| | 10 | 0 | - | - | 0 | |

# Cyclic prime table

## Example (cont'd)

| | 5 | 6 | 15 |
|---|---|---|---|
| $A$ | x | x | |
| $B$ | x | | x |
| $C$ | | x | x |

The table is cyclic (No essential rows, dominating rows or dominating columns).

$$\begin{cases} f = A \vee B = x_1' x_2 \vee x_4 \\ f = A \vee C = x_1' x_2 \vee x_2 x_3 \\ f = B \vee C = x_4 \vee x_2 x_3 \end{cases}$$

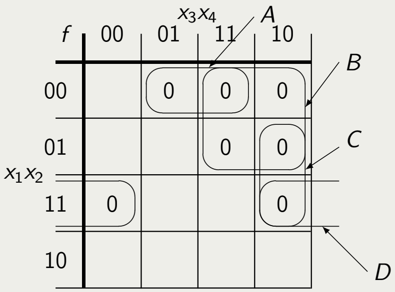
# Minimal conjunctive form

## Example 5.7 (conjunctive form)

The function $f$ has

$$f^{-1}(0) = \{1, 2, 3, 6, 7, 12, 14\}$$

Karnaugh map:



# Minimal conjunctive form

## Example (cont'd)

(Dual of) implicants:

$$\begin{aligned} A &= (x_1'x_2'x_4)' = x_1 \vee x_2 \vee x_4' \\ B &= (x_1'x_3)' = x_1 \vee x_3' \\ C &= (x_2x_3x_4')' = x_2' \vee x_3' \vee x_4 \\ D &= (x_1x_2x_4')' = x_1' \vee x_2' \vee x_4 \end{aligned}$$

Essential: $A$, $B$, and $D$

The essentials cover $f$. $\Rightarrow$

$$\begin{aligned} f &= A \wedge B \wedge D \\ &= (x_1 \vee x_2 \vee x_4')(x_1 \vee x_3')(x_1' \vee x_2' \vee x_4) \end{aligned}$$

# 5 variables

## Example 5.9

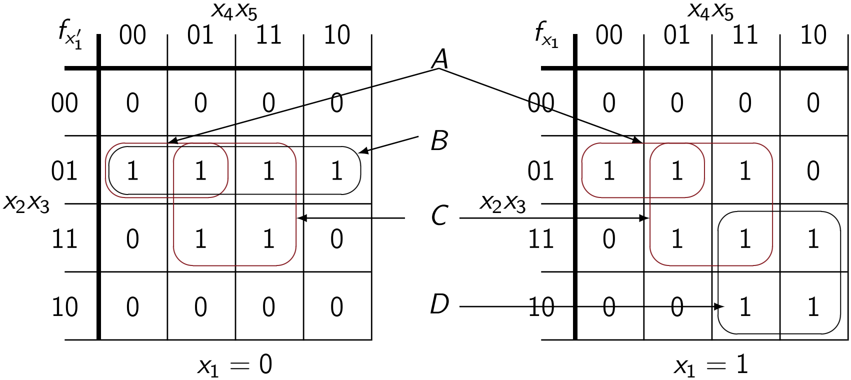
Minimise $f(x_1x_2x_3x_4x_5)$ where

$$f^{-1}(1) = \{4, 5, 6, 7, 13, 15, 20, 21, 23, 26, 27, 29, 30, 31\}$$

To solve this we need three dimensions. Use two separate maps and view them as the third dimension (on top of each other).

# Ex 5.9 (solution)

Split the function in two Karnaugh maps, one for $x_1 = 0$ and one for $x_1 = 1$:




Prime implicants: $A = x_2'x_3x_4'$ $\quad C = x_3x_5$

$B = x_1'x_2'x_3$ $\quad D = x_1x_2x_4$

All prime implicants are essential, $f = A \vee B \vee C \vee D$

# More than one function

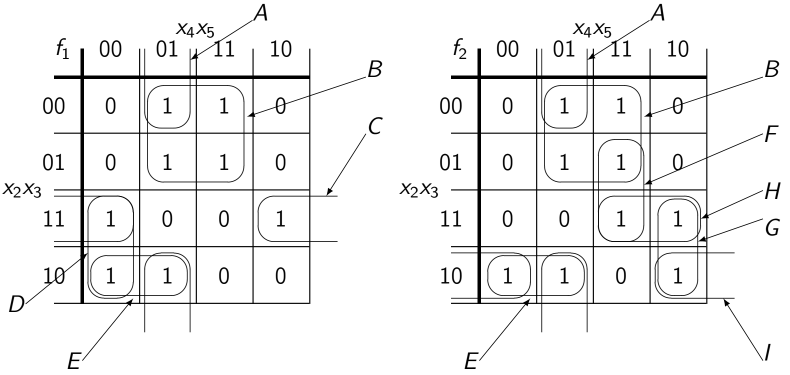
**Example 5.10**

Minimise a minimal circuit that realises the functions

$$f_1^{-1}(1) = \{1, 3, 5, 7, 8, 9, 12, 14\}$$
$$f_2^{-1}(1) = \{1, 3, 5, 7, 8, 9, 10, 14, 15\}$$

# Ex 5.10 (solution)

Karnaugh maps for $f_1$ and $f_2$ to get implicants for the separeted functions:



# Ex 5.10 (solution)

Implicants that can be used in both $f_1$ and $f_2$:

| $f_1 \wedge f_2$ ($x_1x_2$ \ $x_3x_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 1 | 0 |
| 01 | 0 | 1 | 1 | 0 |
| 11 | 0 | 0 | 0 | 1 |
| 10 | 1 | 1 | 0 | 0 |

$A$ $B$ $J$ $E$

# P-table (Ex 5.10)

| | $f_1$ | | | | | | | | $f_2$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 3 | 5 | 7 | 8 | 9 | 12 | 14 | 1 | 3 | 5 | 7 | 8 | 9 | 10 | 14 | 15 |
| $A$ | x | | | | | x | | | x | | | | | x | | | |
| $B$ | x | x | x | x | | | | | x | x | x | x | | | | | |
| $C$ | | | | | | | x | x | | | | | | | | | |
| $D$ | | | | | x | | x | | | | | | | | | | |
| $E$ | | | | | x | x | | | | | | | x | x | | | |
| $F$ | | | | | | | | | | | | x | | | | | x |
| $G$ | | | | | | | | | | | | | | | x | x | |
| $H$ | | | | | | | | | | | | | | | | x | x |
| $I$ | | | | | | | | | | | | | x | | x | | |
| $J$ | | | | | | | | x | | | | | | | | x | |

# 1st reduction

| | $f_1$ | | | | $f_2$ | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 8 | 9 | 12 | 14 | 8 | 9 | 10 | 14 | 15 |
| $A$ | | x | | | | x | | | |
| $C$ | | | x | x | | | | | |
| $D$ | x | | x | | | | | | |
| $E$ | x | x | | | x | x | | | |
| $F$ | | | | | | | | | x |
| $G$ | | | | | | | x | x | |
| $H$ | | | | | | | | x | x |
| $I$ | | | | | x | | x | | |
| $J$ | | | | x | | | | x | |

# 2nd reduction

$A \subset E \Rightarrow$ delete $A$. Then $E$ is (sec.) essential in both $f_1$ and $f_2$.
$D \subset C \Rightarrow$ delete $D$
$I \subset G \Rightarrow$ delete $I$
$F \subset H \Rightarrow$ delete $F$

| | $f_1$ | | $f_2$ | | |
|---|---|---|---|---|---|
| | 12 | 14 | 10 | 14 | 15 |
| $C$ | x | x | | | |
| $G$ | | | x | x | |
| $H$ | | | | x | x |
| $J$ | | x | | x | |

Then $C$ is (sec.) essential in $f_1$ and $G$ and $H$ are (sec.) essential in $f_2$.

$$f_1 = B \vee E \vee C$$

$$f_2 = B \vee E \vee G \vee H$$