Digitalteknik EITF65

Lecture 9: State Minimisation

August 27, 2020

Equivalent graphs

Definition

Equivalent graphs Two graphs are said to be equivalent if they describe the same behaviour.

Build two realisations:

If the graphs are equivalent it is impossible to distinguish them by manipulating the input sequences and observing the output sequences.

Equivalence relation

Definition (Ch 3)

An equivalence relation is a binary relation, $\equiv$, over a set $\mathcal{S}$ such that it is

- reflexive,

$$s \equiv s$$

- symmetric,

$$s_1 \equiv s_2 \;\Rightarrow\; s_2 \equiv s_1$$

- transitive,

$$s_1 \equiv s_2 \text{ and } s_2 \equiv s_3 \;\Rightarrow\; s_1 \equiv s_3$$

for all $s, s_1, s_2, s_3 \in \mathcal{S}$.

An equivalence relation over $\mathcal{S}$ partitions $\mathcal{S}$ into disjoint subsets, equivalence classes.

Equivalence relation (Ex)

Example

Consider the set of all integers, $\mathbb{Z}$. and let $\equiv$ mean congruent modulo 5, i.e.,

$$a \equiv b \;\Leftrightarrow\; R_m(a) = R_m(b)$$

Then $\equiv$ is an equivalence relation over $\mathbb{Z}$. There are 5 equivalence classes, and the ith contains all integers a such that $R_5(a) = i$;

$$\begin{aligned}
[0]_5 &= \{\ldots, -10, -5, 0, 5, 10, \ldots\} \\
[1]_5 &= \{\ldots, -9, -4, 1, 6, 11, \ldots\} \\
[2]_5 &= \{\ldots, -8, -3, 2, 7, 12, \ldots\} \\
[3]_5 &= \{\ldots, -7, -2, 3, 8, 13, \ldots\} \\
[4]_5 &= \{\ldots, -6, -1, 4, 9, 14, \ldots\}
\end{aligned}$$

If we represent the classes with its least positive integer we are back to modulo calculation.

State equivalence

Definition

Two states s_1 and s_2 are equivalent,

$$s_1 \equiv s_2$$

if for all input sequences the corresponding output sequences are identical, regardless if we start in s_1 or s_2.

State-equivalence is an equivalence relation, and the equivalence classes correspond to non-equivalent states.

Definition (6.2)

If there are no equivalent states in a finite state machine, we say that the machine is on reduced form.

RF-algorithm

To find a reduced form of a graph we can use the following algorithm.

Step 1 Form P_1 by grouping states with the same output function in blocks.

Step ℓ Let

$$P_{\ell-1} = \{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$$

Form

$$P_\ell = \{\tilde{\mathcal{B}}_1, \ldots, \tilde{\mathcal{B}}_n\}$$

by grouping the states in each block $\mathcal{B}_j$ by the groups of their next states.

If $P_\ell = P_{\ell-1}$ then

$\mathcal{S}_{\mathrm{red}} = P_\ell$

STOP

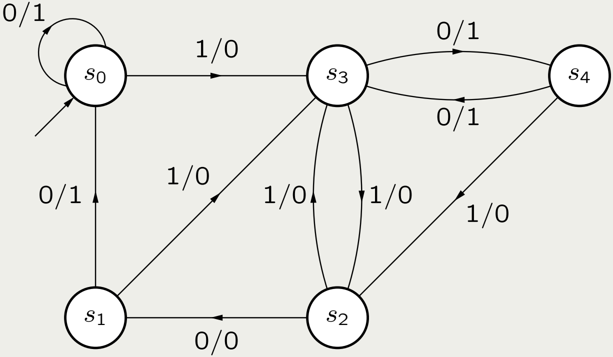
Otherwise

`GOTO` Step $\ell + 1$.

RF-algorithm, example

Example

Consider the graph



Example (cont'd)

Tabular:

s	0	1
s_0	$s_0/1$	$s_3/0$
s_1	$s_0/1$	$s_3/0$
s_2	$s_1/0$	$s_3/0$
s_3	$s_4/1$	$s_2/0$
s_4	$s_3/1$	$s_2/0$

RF-algorithm:

$$P_1 = \{s_0 s_1 s_3 s_4\}\{s_2\}$$

$$P_2 = \{s_0 s_1\}\{s_3 s_4\}\{s_2\}$$

$$P_3 = \underbrace{\{s_0 s_1\}}_{s_{01}}\underbrace{\{s_3 s_4\}}_{s_{34}}\{s_2\}$$

Reduced form graph:

S	0	1
s_{01}	$s_{01}/1$	$s_{34}/0$
s_2	$s_{01}/0$	$s_{34}/0$
s_{34}	$s_{34}/1$	$s_2/0$

Initialized graphs

Definition (6.3)

If it exists at least one path through the graph from state s_0 to state s, then s is reachable from s_0.
A state transition graph with starting state s_0 is reachable if all states are reachable from s_0.

A graph with starting state s_0 can be realized by the reachable part of the graph.

Minimal form

Definition (6.5)

A minimal graph is a reduced form of a reachable graph.

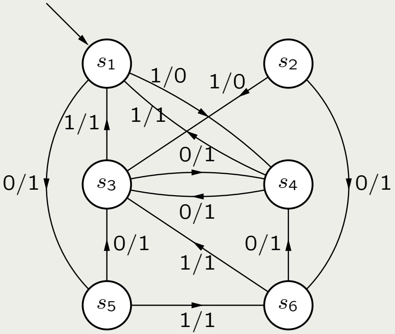
To find a minimal form of a graph

- Remove the states from $\mathcal{S}$ that are not reachable from s_0, i.e., find the reachable part.
- Use the RF-algorithm to find a reduced form of this graph.

An example

Example

Minimise the following graph:



Example (solution)

Draw the state transition table:

s	0	1
s_1	$s_5/1$	$s_4/0$
s_2	$s_6/1$	$s_3/0$
s_3	$s_4/1$	$s_1/1$
s_4	$s_3/1$	$s_1/1$
s_5	$s_3/1$	$s_6/1$
s_6	$s_4/1$	$s_3/1$

We see that state s_2 is unreachable from the starting state s_1. Hence, the reachable graph is

s	0	1
s_1	$s_5/1$	$s_4/0$
s_3	$s_4/1$	$s_1/1$
s_4	$s_3/1$	$s_1/1$
s_5	$s_3/1$	$s_6/1$
s_6	$s_4/1$	$s_3/1$

Example (solution)

RF-algorithm:

$$P_1 = \{s_1\}\{s_3 s_4 s_5 s_6\}$$
$$P_2 = \{s_1\}\{s_3 s_4\}\{s_5 s_6\}$$
$$P_3 = \{s_1\}\{s_3 s_4\}\{s_5\}\{s_6\}$$
$$P_4 = P_3$$

A minimal graph is given by

s	0	1
s_1	$s_5/1$	$s_{34}/0$
s_{34}	$s_{34}/1$	$s_1/1$
s_5	$s_{34}/1$	$s_6/1$
s_6	$s_{34}/1$	$s_{34}/1$

Or, in graphical form: