



Digitalteknik EITF65

Lecture 8: Karnaugh Maps

Gray code

In a Gray code only one variable is changed for each (cyclic) step.

Example

Example of NBCD and Gray code for $N = 4$ and $N = 8$.

$N = 4$		
n	NBCD	Gray
0	00	00
1	01	01
2	10	11
3	11	10

$N = 8$		
n	NBCD	Gray
0	000	000
1	001	001
2	010	011
3	011	010
4	100	110
5	101	111
6	110	101
7	111	100

(NBCD = Natural Binary Coded Decimal.)

Karnaugh maps

A **Karnaugh map** is a function table in matrix form with

- ▶ at most two variables per dimension.
- ▶ the input combinations are listed as Gray code.

Then, between two consecutive positions (horizontal or vertical) there is only a change in one variable.

The Karnaugh map can be seen as a graphical interpretation of B^n . The maps useful on paper (2-dimensions) are for functions with **2, 3, or 4 variables**.

Karnaugh maps (cont'd)

$f(x_1x_2)$:

		x_2	
		0	1
x_1	0	$f(0)$	$f(1)$
	1	$f(2)$	$f(3)$

$f(x_1x_2x_3)$:

		x_2x_3			
		00	01	11	10
x_1	0	$f(0)$	$f(1)$	$f(3)$	$f(2)$
	1	$f(4)$	$f(5)$	$f(7)$	$f(6)$

		x_3	
		0	1
x_1x_2	00	$f(0)$	$f(1)$
	01	$f(2)$	$f(3)$
	11	$f(6)$	$f(7)$
	10	$f(4)$	$f(5)$

$f(x_1x_2x_3x_4)$:

		x_3x_4			
		00	01	11	10
x_1x_2	00	$f(0)$	$f(1)$	$f(3)$	$f(2)$
	01	$f(4)$	$f(5)$	$f(7)$	$f(6)$
	11	$f(12)$	$f(13)$	$f(15)$	$f(14)$
	10	$f(8)$	$f(9)$	$f(11)$	$f(10)$

Minimisation

Cubes

In a Karnaugh map a cube is a (cyclic) rectangular block covering 2^k positions.

Minimisation by Karnaugh maps

- ▶ Find as big rectangular blocks as possible with 2^k 1s and —. These represent the prime implicants.
- ▶ Find the essential implicants. These must be part of the function.
- ▶ Chose a minimal number of the rest of the implicants such that the function is covered.

Karnaugh minimisation (Ex)

Example

Consider the function f with

$$f^{-1}(1) = \{0, 2, 3, 7\}, \quad f^{-1}(0) = \{5, 6\}$$

The Karnaugh map is

		x_3	
		0	1
x_1x_2	00	1	0
	01	1	1
	11	-	1
	10	0	-

Diagram illustrating the Karnaugh map for function f . The map is a 4x2 grid with columns labeled x_3 (0 and 1) and rows labeled x_1x_2 (00, 01, 11, 10). The function values are: (00,0)=1, (00,1)=0, (01,0)=1, (01,1)=1, (11,0)=-, (11,1)=1, (10,0)=0, (10,1)=-. Three prime implicants are circled: A (top row), B (middle two rows), and C (right column).

Prime implicants:

$$A = c^{(0B0)}(\mathbf{x}) = x'_1x'_3$$

$$B = c^{(B1B)}(\mathbf{x}) = x_2$$

$$C = c^{(1B1)}(\mathbf{x}) = x_1x_3$$

A and B are essential, and covers the function, i.e.,

$$f_{\min} = A \vee B = x'_1x'_3 \vee x_2$$

is a minimal function (disjunctive form).

Karnaugh minimisation (Conjunctive form)

Example

Consider the function f with

$$f^{-1}(1) = \{0, 2, 3, 7\}, \quad f^{-1}(-) = \{5, 6\}$$

The Karnaugh map is

		x_3	
		0	1
x_1x_2	00	1	0
	01	1	1
	11	-	1
	10	0	-

Annotations: A circle around the '0' in cell (00,1) is labeled 'A'. A circle around the '1' in cell (11,1) is labeled 'B'. A circle around the '0' in cell (10,0) is labeled 'C'.

(Anti) prime implicants:

$$A = \left(c^{(B01)}(\mathbf{x}) \right)' = (x_2'x_3)' = x_2 \vee x_3'$$

$$B = \left(c^{(10B)}(\mathbf{x}) \right)' = (x_1x_2')' = x_1' \vee x_2$$

$$C = \left(c^{(1B0)}(\mathbf{x}) \right)' = (x_1x_3')' = x_1' \vee x_3$$

A is essential. Minimal conjunctive form:

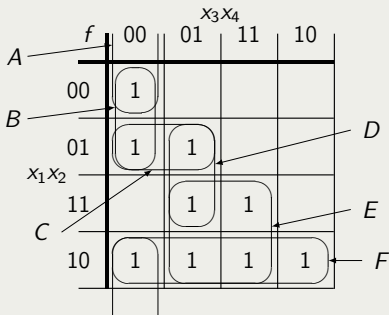
$$f_{\min} = A \wedge B = (x_2 \vee x_3')(x_1' \vee x_2)$$

$$f_{\min} = A \wedge C = (x_2 \vee x_3')(x_1' \vee x_3)$$

Minimal disjunctive form

Example 5.7 (disjunctive form)

The function $f(x_1x_2x_3x_4)$ has $f^{-1}(1) = \{0, 4, 5, 8, 9, 10, 11, 13, 15\}$



Prime implicants:

$$\begin{aligned} A &= x_2'x_3'x_4' & B &= x_1'x_3'x_4' & C &= x_1'x_2x_3' \\ D &= x_2x_3'x_4 & E &= x_1x_4 & F &= x_1x_2' \end{aligned}$$

Prime table

A **prime table** (P-table) shows which minterms the prime implicants cover.

Rows correspond to prime implicants.

Columns correspond to minterms.

A cross \times shows when a minterm is covered by the implicant.

Then

- ▶ an **essential row** contains a \times that is not represented in any other rows. Essential rows must be in the cover.
- ▶ **row r dominates row s** , $r \supset s$, if all \times in s also exist in r . The dominated row, s , can be deleted.
- ▶ **column c dominates column d** , $c \supset d$, if all \times in d also exist in c . The dominating column, c , can be deleted.

Prime table (Ex 5.7)

	0	4	5	8	9	10	11	13	15
A	x			x					
B	x	x							
C		x	x						
D			x					x	
E					x		x	x	⊗
F				x	x	⊗	x		

Essential: E from 15

F from 10

All columns that have \times in row E and F are deleted from the table, they are covered by E and F .

Prime table (Ex 5.7)

	0	4	5
$B \supset A$	x		
B	x	x	
C		x	x
$C \supset D$			x

Here, B dominates A and C dominates D .

	0	4	5
B	⊗	x	
C		x	⊗

Now we see that 4 dominates both 0 and 5. Hence, the column 4 is deleted. Then, both B and C are (secondary) essential.

$$f = E \vee F \vee B \vee C$$

Remark: From the Karnaugh map we can find two other minimal covers: $f = E \vee F \vee B \vee D$ and $f = E \vee F \vee A \vee C$

Cyclic prime table

Example

The function f is specified by

$$f^{-1}(1) = \{5, 6, 15\}$$

$$f^{-1}(0) = \{0, 2, 8, 10, 12\}$$

		x_3x_4			
		f	00	01	11
x_1x_2	00	0	-	-	0
	01	-	1	-	1
	11	0	-	1	-
	10	0	-	-	0

Diagram annotations: A points to the cell (01, 00) containing '-'. B points to the cell (00, 10) containing '0'. C points to the cell (11, 10) containing '-'. There are also three rounded rectangular boxes: one around the row (01, -), one around the column (11, -), and one around the cell (11, 10).

Cyclic prime table

Example (cont'd)

	5	6	15
A	x	x	
B	x		x
C		x	x

The table is cyclic (No essential rows, dominating rows or dominating columns).

$$\begin{cases} f = A \vee B = x_1'x_2 \vee x_4 \\ f = A \vee C = x_1'x_2 \vee x_2x_3 \\ f = B \vee C = x_4 \vee x_2x_3 \end{cases}$$

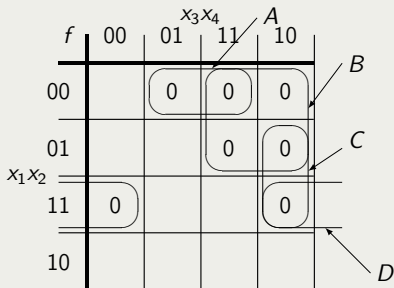
Minimal conjunctive form

Example 5.7 (conjunctive form)

The function f has

$$f^{-1}(0) = \{1, 2, 3, 6, 7, 12, 14\}$$

Karnaugh map:



Minimal conjunctive form

Example (cont'd)

(Dual of) implicants:

$$A = (x_1'x_2'x_4)' = x_1 \vee x_2 \vee x_4'$$

$$B = (x_1'x_3)' = x_1 \vee x_3'$$

$$C = (x_2x_3x_4')' = x_2' \vee x_3' \vee x_4$$

$$D = (x_1x_2x_4')' = x_1' \vee x_2' \vee x_4$$

Essential: A , B , and D

The essentials cover f . \Rightarrow

$$\begin{aligned} f &= A \wedge B \wedge D \\ &= (x_1 \vee x_2 \vee x_4')(x_1 \vee x_3')(x_1' \vee x_2' \vee x_4) \end{aligned}$$

5 variables

Example 5.9

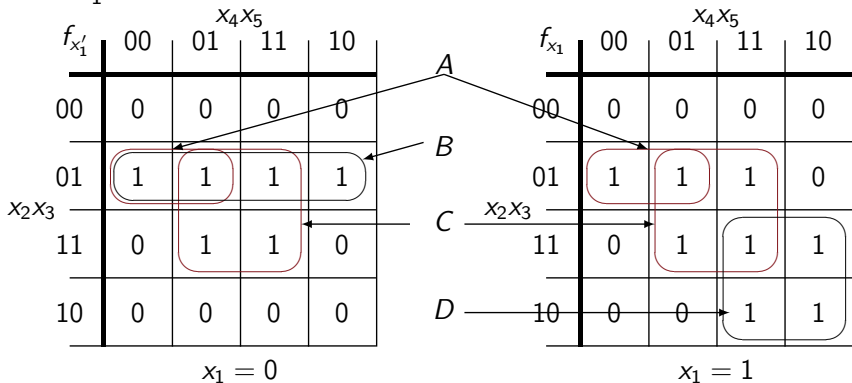
Minimise $f(x_1x_2x_3x_4x_5)$ where

$$f^{-1}(1) = \{4, 5, 6, 7, 13, 15, 20, 21, 23, 26, 27, 29, 30, 31\}$$

To solve this we need three dimensions. Use two separate maps and view them as the third dimension (on top of each other).

Ex 5.9 (solution)

Split the function in two Karnaugh maps, one for $x_1 = 0$ and one for $x_1 = 1$:



Prime implicants: $A = x_2'x_3x_4'$ $C = x_3x_5$
 $B = x_1'x_2'x_3$ $D = x_1x_2x_4$

All prime implicants are essential, $f = A \vee B \vee C \vee D$

More than one function

Example 5.10

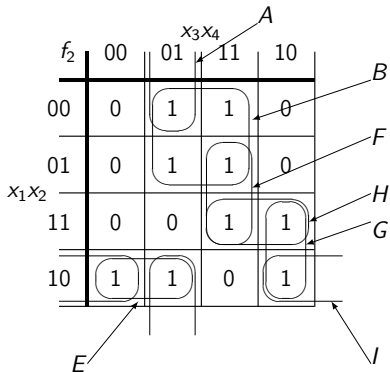
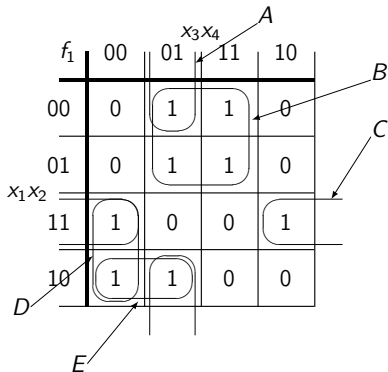
Minimise a minimal circuit that realises the functions

$$f_1^{-1}(1) = \{1, 3, 5, 7, 8, 9, 12, 14\}$$

$$f_2^{-1}(1) = \{1, 3, 5, 7, 8, 9, 10, 14, 15\}$$

Ex 5.10 (solution)

Karnaugh maps for f_1 and f_2 to get implicants for the separated functions:



Ex 5.10 (solution)

Implicants that can be used in both f_1 and f_2 :

$f_1 \wedge f_2$	00	01	11	10
00	0	1	1	0
01	0	1	1	0
11	0	0	0	1
10	1	1	0	0

Diagram illustrating the Karnaugh map for the function $f_1 \wedge f_2$. The map is a 4x4 grid with columns labeled x_3x_4 (00, 01, 11, 10) and rows labeled x_1x_2 (00, 01, 11, 10). The values in the cells are: (00,00)=0, (01,00)=1, (11,00)=1, (10,00)=0; (00,01)=0, (01,01)=1, (11,01)=1, (10,01)=0; (00,11)=0, (01,11)=0, (11,11)=0, (10,11)=1; (00,10)=1, (01,10)=1, (11,10)=0, (10,10)=0. Circled 1s indicate prime implicants: a vertical group of two 1s in the 01 column (labeled A), a horizontal group of two 1s in the 00 row (labeled B), a horizontal group of two 1s in the 10 row (labeled E), a vertical group of two 1s in the 01 column (labeled J), and a horizontal group of two 1s in the 10 row (labeled J).

P-table (Ex 5.10)

	f_1								f_2								
	1	3	5	7	8	9	12	14	1	3	5	7	8	9	10	14	15
A	x					x			x					x			
B	x	⊗	⊗	⊗					x	⊗	⊗	x					
C							x	x									
D					x		x										
E					x	x							x	x			
F												x					x
G															x	x	
H																x	x
I													x	x			
J								x								x	

1st reduction

	f_1				f_2				
	8	9	12	14	8	9	10	14	15
<i>A</i>		x				x			
<i>C</i>			x	x					
<i>D</i>	x		x						
<i>E</i>	x	x			x	x			
<i>F</i>									x
<i>G</i>							x	x	
<i>H</i>								x	x
<i>I</i>					x		x		
<i>J</i>				x				x	

2nd reduction

$A \subset E \Rightarrow$ delete A . Then E is (sec.) essential in both f_1 and f_2 .

$D \subset C \Rightarrow$ delete D

$I \subset G \Rightarrow$ delete I

$F \subset H \Rightarrow$ delete F

	f_1		f_2		
	12	14	10	14	15
C	⊗	x			
G			⊗	x	
H				x	⊗
J		x		x	

Then C is (sec.) essential in f_1 and G and H are (sec.) essential in f_2 .

$$f_1 = B \vee E \vee C$$

$$f_2 = B \vee E \vee G \vee H$$