Digitalteknik EITF65

Lecture 7: Minimal Functions

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Implicants and function cover

Definition

Let $\mathcal{F}$ be a set of cubes and $V(\mathcal{F})$ the corresponding vertices. Then the cubes in $\mathcal{F}$ cover the function f if and only if

$$f^{-1}(1) \subseteq V(\mathcal{F}) \subseteq f^{-1}(1) \cup f^{-1}(-)$$

A cube function $c^{\mathcal{C}}(\mathsf{x})$ for a cube in a cover of f, i.e.

$$c^{\mathcal{C}}(\mathsf{x}) \quad \text{s.t.} \quad V(\mathcal{c}) \subseteq f^{-1}(1) \cup f^{-1}(-)$$

is called an implicant of f.

If $\mathcal{F}$ cover the function f, then f can be written

$$f(\mathsf{x}) = \bigvee_{\mathcal{C} \in \mathcal{F}} c^{\mathcal{C}}(\mathsf{x}).$$

Minimal cover

Definition (5.1)

Let $\mathcal{I}$ be a set of implicants that covers the function f. If the number of implicants in the cover is minimal over all possible covers of f, it is a minimal cover.

Our goal is to find a minimal cover of f.

Prime implicants

Definition

A prime implicant is an implicant that is not covered by any other implicant. If a cover consists of only prime implicants it is a prime cover.
An essential prime implicant covers a minterm (vertex) that is not covered by any other prime implicant.

A prime cover must contain the essential primes.

Prime cover

Theorem (5.2)

A minimal cover of the incompletely specified function f can be obtained by a minimal cover of $f^{-1}(1)$ with prime implicants of $f^{-1}(1) \cup f^{-1}(-)$.

- Find all prime implicants of $f^{-1}(1) \cup f^{-1}(-)$
- Use them to find a minimal cover of $f^{-1}(1)$

Iterative consensus

One way to derive all prime implicants for a Boolean expressions is to, for each term, iteratively

- expand with all consensus terms,

$$ab \vee a'c = ab \vee a'c \vee bc$$

- simplify with absorption,

$$a \vee ab = a$$

Example of iterative consensus

Example

Simplify $f = x_1'x_2' \vee x_1x_2'x_3 \vee x_1x_2x_3$

$$\begin{aligned}
f &= \underbrace{x_1'x_2'}_{A} \vee \underbrace{x_1x_2'x_3}_{B} \vee \underbrace{x_1x_2x_3}_{C} \vee \underbrace{x_2'x_3}_{D=C(A,B)} && \text{(Add consensus for A)}\\
&= \underbrace{x_1'x_2'}_{A} \vee \underbrace{x_1x_2'x_3}_{B=x_1D} \vee \underbrace{x_1x_2x_3}_{C} \vee \underbrace{x_2'x_3}_{D} && \text{(Remove with absoption)}\\
&= \underbrace{x_1'x_2'}_{A} \vee \underbrace{x_1x_2x_3}_{C} \vee \underbrace{x_2'x_3}_{D} \vee \underbrace{x_1x_3}_{E=C(C,D)} && \text{(Add consensus for C)}\\
&= \underbrace{x_1'x_2'}_{A} \vee \underbrace{x_1x_2x_3}_{C=x_2E} \vee \underbrace{x_2'x_3}_{D} \vee \underbrace{x_1x_3}_{E} && \text{(Remove with absoption)}\\
&= \underbrace{x_1'x_2'}_{A} \vee \underbrace{x_2'x_3}_{D} \vee \underbrace{x_1x_3}_{E} && \text{(All primes)}
\end{aligned}$$

Since $x_2'x_3 = C(A,E)$ this can be removed to get the minimal solution

$$f = x_1'x_2' \vee x_1x_3$$

Gray code

In a Gray code only one variable is changed for each (cyclic) step.

Example

Example of NBCD and Gray code for $N = 4$ and $N = 8$.

$N = 4$

n	NBCD	Gray
0	00	00
1	01	01
2	10	11
3	11	10

$N = 8$

n	NBCD	Gray
0	000	000
1	001	001
2	010	011
3	011	010
4	100	110
5	101	111
6	110	101
7	111	100

(NBCD = Natural Binary Coded Decimal.)

Karnaugh maps

A Karnaugh map is a function table in matrix form with

- at most two variables per dimension.
- the input combinations are listed as Gray code.

Then, between two consecutive positions (horisontal or vertical) there is only a change in one variable.

The Karnaugh map can be seen as a graphicaly interpretation of B^n. The maps useful on paper (2-dimensions) are for functions with 2, 3, or 4 variables.

Karnaugh maps (cont'd)

$f(x_1x_2)$:

f	x_2 = 0	x_2 = 1
x_1 = 0	$f(0)$	$f(1)$
x_1 = 1	$f(2)$	$f(3)$

$f(x_1x_2x_3)$:

f	x_2x_3 = 00	01	11	10
x_1 = 0	$f(0)$	$f(1)$	$f(3)$	$f(2)$
x_1 = 1	$f(4)$	$f(5)$	$f(7)$	$f(6)$

f	x_3 = 0	x_3 = 1
x_1x_2 = 00	$f(0)$	$f(1)$
01	$f(2)$	$f(3)$
11	$f(6)$	$f(7)$
10	$f(4)$	$f(5)$

$f(x_1x_2x_3x_4)$:

f	x_3x_4 = 00	01	11	10
x_1x_2 = 00	$f(0)$	$f(1)$	$f(3)$	$f(2)$
01	$f(4)$	$f(5)$	$f(7)$	$f(6)$
11	$f(12)$	$f(13)$	$f(15)$	$f(14)$
10	$f(8)$	$f(9)$	$f(11)$	$f(10)$

Minimisation

In a Karnaugh map an implicant is a (cyclic) rectangular block covering 2^k 1s.

Minimisation by Karnaugh maps:

- Find as big rectangular blocks as possible with 2^k 1s and $-$. These represent the prime implicants.
- Find the essential implicants. These must be part of the function.
- Chose a minimal number of the rest of the implicants such that the function is covered.

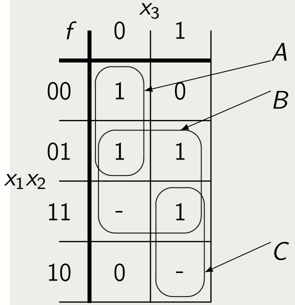
Karnaugh minimisation (Ex)

Example

Consider the function f with

$$f^{-1}(1) = \{0, 2, 3, 7\}, \quad f^{-1}(-) = \{5, 6\}$$

The Karnaugh map is



Prime implicants:

$$A = c^{(0B0)}(\mathbf{x}) = x_1'x_3'$$

$$B = c^{(B1B)}(\mathbf{x}) = x_2$$

$$C = c^{(1B1)}(\mathbf{x}) = x_1x_3$$

A and B are essential, and covers the function, i.e.,

$$f_{\text{min}} = A \vee B = x_1'x_3' \vee x_2$$

is a minimal function (disjunctive form).

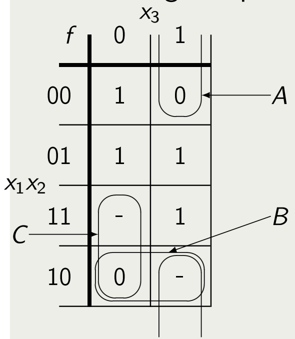
Karnaugh minimisation (Conjunctive form)

Example

Consider the function f with

$$f^{-1}(1) = \{0, 2, 3, 7\}, \quad f^{-1}(-) = \{5, 6\}$$

The Karnaugh map is



(Anti) prime implicants:

$$A = \left(c^{(B01)}(\mathbf{x})\right)' = (x_2'x_3)' = x_2 \vee x_3'$$

$$B = \left(c^{(10B)}(\mathbf{x})\right)' = (x_1x_2')' = x_1' \vee x_2$$

$$C = \left(c^{(1B0)}(\mathbf{x})\right)' = (x_1x_3')' = x_1' \vee x_3$$

A is essential. Minimal conjunctive form:

$$f_{\text{min}} = A \wedge B = (x_2 \vee x_3')(x_1' \vee x_2)$$

$$f_{\text{min}} = A \wedge C = (x_2 \vee x_3')(x_1' \vee x_3)$$