Digitalteknik EITF65

Lecture 6: Boolean Functions and Normal Forms

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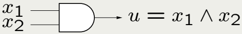
$(\mathbb{Z}_2, \wedge, \vee, ')$

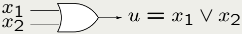
AND

$x_1 x_2$	$x_1 \wedge x_2$
0 0	0
0 1	0
1 0	0
1 1	1



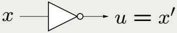
OR

$x_1 x_2$	$x_1 \vee x_2$
0 0	0
0 1	1
1 0	1
1 1	1



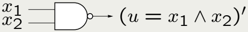
NOT

x	x'
0	1
1	0



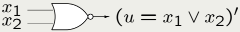
NAND

$x_1 x_2$	$(x_1 \wedge x_2)'$
0 0	1
0 1	1
1 0	1
1 1	0



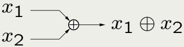
NOR

$x_1 x_2$	$(x_1 \vee x_2)'$
0 0	1
0 1	0
1 0	0
1 1	0



MOD 2 ADD

$x_1 x_2$	$x_1 \oplus x_2$
0 0	0
0 1	1
1 0	1
1 1	0



Boolean functions (Notation)

Definition

Let $B = \{0, 1\}$ denote the Boolean values. Then,

- $B^n = \{0, 1\}^n$ is an n-dimensional Boolean space (with values 0 and 1 in each dimension).
- B_n is the set of all functions from B^n to $\{0, 1\}$.
- B_n^* is the set of all functions from B^n to $\{0, 1, -\}$.

Inverse Boolean function

Definition (4.3)

The set of input combinations for which a Boolean function f gives the output

- 0 is called the off-set of the function, $f^{-1}(0)$.
- 1 is called the on-set of the function, $f^{-1}(1)$.
- $-$ is called the don't care-set of the function, $f^{-1}(-)$.

Inverse function (Example)

Example

Define $f \in B_3^*$ as

$x_1 x_2 x_3$	f
0 0 0	1
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	-
1 1 0	-
1 1 1	1

Then

$$\begin{aligned} f^{-1}(1) &= \{(000), (010), (011), (111)\} \\ &= \{0, 2, 3, 7\} \\ f^{-1}(0) &= \{(001), (100)\} = \{1, 4\} \\ f^{-1}(-) &= \{(101), (110)\} = \{5, 6\} \end{aligned}$$

Lattice exponent

Definition

Let $\mathcal{C} \subseteq B = \{0, 1\}$. Then the lattice exponent $x^{(\mathcal{C})}$ is defined as

$$x^{(\mathcal{C})} = \begin{cases} 1, & \text{if } x \in \mathcal{C} \\ 0, & \text{if } x \notin \mathcal{C} \end{cases}$$

In other words

$$\begin{aligned} x^{(1)} &= x & \qquad x^{(B)} &= 1 \\ x^{(0)} &= x' & \qquad x^{(\emptyset)} &= 0 \end{aligned}$$

Cubes and cube functions

Definition

The vector $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_n)$, $\mathcal{C}_i \in \{\emptyset, 0, 1, B\}$, describes a cube in the n-dimensional space B^n.
The corresponding cube function is formed by

$$c^{\mathcal{C}}(\mathbf{x}) = \bigwedge_{i=1}^{n} x_i^{(\mathcal{C}_i)}$$

where $\mathbf{x} \in B^n$.

A cube function is an $\wedge$-product of factors like x_i and x_i'. It has output 1 inside the cube and 0 outside.

Cubes and cube functions (Ex)

Example

Let

$$\mathcal{C} = (\{0,1\},\{0\},\{1\}) = (B,0,1)$$

Then, the correponding cube function is

$$\begin{aligned} c^{\mathcal{C}}(\mathbf{x}) &= c^{(B,0,1)}(x_1,x_2,x_3) \\ &= x_1^{(B)} \wedge x_2^{(0)} \wedge x_3^{(1)} \\ &= 1 \wedge x_2' \wedge x_3 \\ &= x_2' x_3 \end{aligned}$$

View the function in a table

$x_1 x_2 x_3$	$c^{\mathcal{C}}(\mathbf{x})$
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	0

Minterms

Definition

A point (corner) in B^n is called a vertex. It is a cube with only 0s and 1s.
A minterm is a cube function of a vertex, and corresponds to one 1 in the function.
The minterm of the vertex $\mathcal{V}$ is denoted

$$m_{\mathcal{V}} = c^{\mathcal{V}}(\mathbf{x}) = x_1^{(v_1)} \wedge \cdots \wedge x_n^{(v_n)}$$

It is an $\wedge$-product of all variables, with or without $'$.

Minterms (Ex)

Example

Define $f \in B_3^*$ as

$x_1x_2x_3$	f
0 0 0	1
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	-
1 1 0	-
1 1 1	1

The on-set is $f^{-1}(1) = \{0, 2, 3, 7\}$ and the corresponding minterms

$$m_0 = x_1^{(0)}x_2^{(0)}x_3^{(0)} = x_1'x_2'x_3'$$
$$m_2 = x_1^{(0)}x_2^{(1)}x_3^{(0)} = x_1'x_2x_3'$$
$$m_3 = x_1^{(0)}x_2^{(1)}x_3^{(1)} = x_1'x_2x_3$$
$$m_7 = x_1^{(1)}x_2^{(1)}x_3^{(1)} = x_1x_2x_3$$

Disjunctive normal form

DNF

All Boolean functions $f(\mathbf{x}) \in B_n$ can be written on disjunctive normal form (DNF):

$$f(\mathbf{x}) = \bigvee_{\mathbf{a} \in f^{-1}(1)} m_{\mathbf{a}}$$

It is the $\vee$-sum of the minterms corresponding to the on-set.

Maxterms

Definition

The dual of a minterm is called a maxterm:

$$M_{\mathcal{V}} = \left(c^{(\mathcal{V})}(\mathbf{x})\right)' = x_1^{(v_1')} \vee \cdots \vee x_n^{(v_n')}, \quad \mathbf{a} \in B^n$$

It is the inverse cube function of a vertex, and corresponds to a 0 in the function.

Remark on min- and maxterms

A minterm describes a 1 in the function truth table, while a maxterm describes a 0. Hence, a maxterm can be derived as the inverse of the corresponding minterm.

$$M_{\mathcal{V}} = (m_{\mathcal{V}})' = (x_1^{(v_1)} \wedge \cdots \wedge x_n^{(v_n)})' = x_1^{(v_1')} \vee \cdots \vee x_n^{(v_n')}$$

Maxterms (Ex)

Example ((cont'd))

Define $f \in B_3^*$ as

$x_1 x_2 x_3$	f
0 0 0	1
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	-
1 1 0	-
1 1 1	1

The off-set is $f^{-1}(0) = \{1, 4\}$ and the corresponding maxterms

$$M_1 = \left(c^{(001)}(\mathbf{x})\right)' = (x_1' x_2' x_3)'$$
$$= x_1 \vee x_2 \vee x_3'$$

$$M_4 = \left(c^{(100)}(\mathbf{x})\right)' = (x_1 x_2' x_3')'$$
$$= x_1' \vee x_2 \vee x_3$$

Conjunctive normal form

CNF

All Boolean functions $f(\mathbf{x}) \in B_n$ can be written on conjunctive normal form (CNF):

$$f(\mathbf{x}) = \bigwedge_{\mathbf{a} \in f^{-1}(0)} M_{\mathbf{a}}$$

It is the $\wedge$-product of the maxterms corresponding to the off-set.

DNF and CNF (Ex)

Example (cont'd)

Consider the function f with

$$f^{-1}(1) = \{0, 2, 3, 7\} \text{ and } f^{-1}(0) = \{1, 4\}$$

- In DNF it is realized as

$$\begin{aligned} f(\mathbf{x}) = \bigvee_{\mathbf{a} \in f^{-1}(1)} m_{\mathbf{a}} &= m_0 \vee m_2 \vee m_3 \vee m_7 \\ &= x_1'x_2'x_3' \vee x_1'x_2x_3' \vee x_1'x_2x_3 \vee x_1x_2x_3 \end{aligned}$$

- In CNF it is realized as

$$f(\mathbf{x}) = \bigwedge_{\mathbf{a} \in f^{-1}(0)} M_{\mathbf{a}} = M_1 \wedge M_4 = (x_1 \vee x_2 \vee x_3')(x_1' \vee x_2 \vee x_3)$$

Conversion to DNF

In the previous examples we had the on-set of the function given. If we do not have it, we can use the following method.

Example

Express $g(\mathbf{x}) = x_2 \vee x_1'x_3'$ in DNF.

Option 1: Use $1 = a \vee a'$ to insert the missing variables:

$$
\begin{aligned}
g(\mathbf{x}) &= x_2 \vee x_1'x_3' \\
&= (1 \wedge x_2 \wedge 1) \vee (x_1' \wedge 1 \wedge x_3') \\
&= (x_1 \vee x_1')x_2(x_3 \vee x_3') \vee x_1'(x_2 \vee x_2')x_3' \\
&= x_1x_2x_3 \vee x_1'x_2x_3 \vee x_1x_2x_3' \vee x_1'x_2x_3' \vee x_1'x_2'x_3'
\end{aligned}
$$

Option 2: Find the on-set $g^{-1}(1)$, for example with a truth table, and write the minterms directly as before.

Conversion to CNF

Example (cont'd)

Show that $g(\mathbf{x}) = (x_1' \vee x_2)(x_2 \vee x_3')$.
Write the function in CNF by using $0 = a \wedge a'$,

$$\begin{aligned}
\hat{g}(\mathbf{x}) &= (x_1' \vee x_2)(x_2 \vee x_3') \\
&= (x_1' \vee x_2 \vee 0)(0 \vee x_2 \vee x_3') \\
&= (x_1' \vee x_2 \vee (x_3 \wedge x_3'))((x_1 \wedge x_1') \vee x_2 \vee x_3') \\
&= (x_1' \vee x_2 \vee x_3)(x_1' \vee x_2 \vee x_3')(x_1 \vee x_2 \vee x_3') \\
&= M_4 \wedge M_5 \wedge M_1
\end{aligned}$$

$\Rightarrow \hat{g}^{-1}(0) = \{1, 4, 5\}$.
Since $g^{-1}(1) \cap \hat{g}^{-1}(0) = \emptyset$ and $g^{-1}(1) \cup \hat{g}^{-1}(0) = \mathbb{Z}_8$ we see that g and $\hat{g}$ are equal.

Reed-Muller canonical form (RMF)

Theorem (4.6)

All Boolean functions $f(\mathbf{x}) \in B_n$ can be written with the (ring) operations $+$ and $\times$, by the Reed-Muller canonical form (RMF),

$$f(\mathbf{x}) = \bigoplus_{j=0}^{2^n-1} a_j \bigotimes_{i \in I_n(j)} x_i$$

where $a_j \in B$ and $I_n(j)$ is an index function.
Here $\oplus$ and $\otimes$ are modulo 2 addition and multiplication.

Derivation of RMF

Four ways to derive the RMF from a Boolean expression:

- Use the definition of the Boolean operations:

$$a \wedge b = a \cdot b$$
$$a \vee b = a \oplus b \oplus ab$$
$$a' = 1 \oplus a$$

- Use deMorgan's law to get rid of $\vee$, then use $a' = 1 \oplus a$.
- Write the function in DNF. Then use that

$$m_{\mathbf{i}} \vee m_{\mathbf{j}} = m_{\mathbf{i}} \oplus m_{\mathbf{j}} \oplus \underbrace{m_{\mathbf{i}} \cdot m_{\mathbf{j}}}_{=0,\ i \neq j} = m_{\mathbf{i}} \oplus m_{\mathbf{j}}$$

  and $a' = 1 \oplus a$.
- Reed-Muller transform

RMF (Ex)

Convert $g(\mathbf{x}) = x_1x_3' \vee x_1x_2$ to RMF.

Definition

$$\begin{aligned} g(\mathbf{x}) &= x_1x_3' \vee x_1x_2 \\ &= x_1(1 \oplus x_3) \oplus x_1x_2 \oplus x_1(1 \oplus x_3)x_1x_2 \\ &= x_1 \oplus x_1x_3 \oplus x_1x_2 \oplus x_1x_2 \oplus x_1x_2x_3 \\ &= x_1 \oplus x_1x_3 \oplus x_1x_2x_3 \end{aligned}$$

deMorgan

$$\begin{aligned} g(\mathbf{x}) &= x_1x_3' \vee x_1x_2 = ((x_1x_3')'(x_1x_2)')' \\ &= 1 \oplus (1 \oplus x_1(1 \oplus x_3))(1 \oplus x_1x_2) \\ &= 1 \oplus 1 \oplus x_1x_2 \oplus x_1 \oplus x_1x_2 \oplus x_1x_3 \oplus x_1x_2x_3 \\ &= x_1 \oplus x_1x_3 \oplus x_1x_2x_3 \end{aligned}$$

DNF

$$\begin{aligned} g(\mathbf{x}) &= x_1x_3' \vee x_1x_2 \\ &= x_1x_2x_3' \vee x_1x_2'x_3' \vee x_1x_2x_3 \\ &= x_1x_2(1 \oplus x_3) \oplus x_1(1 \oplus x_2)(1 \oplus x_3) \oplus x_1x_2x_3 \\ &= x_1x_2 \oplus x_1x_2x_3 \oplus x_1 \oplus x_1x_2 \oplus x_1x_3 \oplus x_1x_2x_3 \oplus x_1x_2x_3 \\ &= x_1 \oplus x_1x_3 \oplus x_1x_2x_3 \end{aligned}$$

Difference in notation

- Course book (also KTH, Chalmers, most old literature):
  - DNF = OR-sum of minterms
  - disjunctive form = any OR-sum of AND-expressions
  - minimal disjunctive form = an OR-sum of AND-expressions with minimum number of AND-expressions
- Wikipedia (also logic literature, D1 course, etc.)
  - full DNF = OR-sum of minterms
  - DNF = any OR-sum of AND-expressions
  - minimal DNF = an OR-sum of AND-expressions with minimum number of AND-expressions