LUND UNIVERSITY

Electrical and Information Technology

# Digitalteknik

## Solutions to selected problems

**Contents**

## 1.1

$\mathcal{U} = \{\text{ice}, \text{water}, \text{steam}\} \times [-273, \infty)$
$\mathcal{B} = (\{\text{ice}\} \times [-273, 0)) \cup (\{\text{water}\} \times [0, 100)) \cup (\{\text{steam}\} \times [100, \infty))$

## 1.2

The universe consists of all conceivable force/position vectors; that is $\mathcal{U} = \mathbb{R}^3 \times \mathbb{R}^3$.

The behavior is

$$\mathcal{B} = \{(\boldsymbol{F}, \boldsymbol{r}) \in \mathbb{R}^3 \times \mathbb{R}^3 | (1.20) \text{ satisfied}\}$$

The behavioral equation describes a static situation. Hence, we do not need to specify the time axis $\mathcal{T}$.

## 1.3

$$\begin{cases} GNP(t) = C(t) + I(t) + G(t) \\ C(t) = cGNP(t-1) \\ I(t) = i(C(t) - C(t-1)) \end{cases} \tag{1}$$

$\Rightarrow$

$$GNP(t) = c(i+1)GNP(t-1) - ciGNP(t-2) + G(t). \tag{2}$$

Thus,

$$\begin{cases} \mathcal{T} = \mathbb{Z} \\ \mathcal{W} = \mathbb{R}^2 \quad \text{Manifest variables: } GNP(t) \text{ and } G(t) \\ \mathcal{L} = \mathbb{R}^2 \quad \text{Latent variables: } C(t) \text{ and } I(t) \\ \mathcal{B}_f = \{w_f : \mathcal{T} \to (\mathcal{W} \times \mathcal{L}) | \text{ such that (1) is satisfied}\} \\ \mathcal{B} = \{w : \mathcal{T} \to \mathcal{W} | \text{ such that (2) is satisfied}\} \end{cases}$$

## 1.4

We are interested in the relation between the voltage $V$ and the current $I$ and choose these as the manifest variables. Take $\mathcal{W} = \mathbb{R}^2$ as the signal space and $\mathcal{T} = \mathbb{R}$ as the time axis. Let the currents $I_2$ through resistor $R_2$ and $I_3$ through resistor $R_3$ be our two auxiliary variables that help us to formulate the relation between $V$ and $I$. $\mathcal{L} = \mathbb{R}^2$ is our latent variable space.

By Kirchoff's voltage law we know that

$$R_2 I_2 = R_3 I_3 \tag{3}$$

$$\text{and} \quad V = R_1 I + R_2 I_2 = R_1 I + R_3 I_3 \tag{4}$$

Kirchoff's current law gives us

$$I = I_2 + I_3 \tag{5}$$

Using (3) and (5) we obtain

$$R_2 I_2 = R_3 (I - I_2)$$

$$\text{and hence} \quad I_2 = \frac{R_3}{R_2 + R_3} I \tag{6}$$

Now (4) and (6) yield

$$V = R_1 I + \frac{R_2 R_3}{R_2 + R_3} I = \left(R_1 + \frac{R_2 R_3}{R_2 + R_3}\right) I \tag{7}$$

We have eliminated our latent variables $I_2$ and $I_3$ and obtained the behavioral equation (7). The behavior of our model is

$$\mathcal{B} = \{w : \mathcal{T} \to \mathbb{R}^2 | (V, I) \quad \text{satisfies (7)}\} \tag{8}$$

**1.5**

a) $\ldots(b, B);(a, C);(a, C);(b, C);(a, A);(b, C)\ldots$

b) $\ldots(b, B);(a, B);(b, C);(b, C);(b, C);(a, A)\ldots$

c) Let

$$w_1 : \mathcal{T} \to \mathcal{S} = \{a, b\}$$
$$w_2 : \mathcal{T} \to \mathcal{M} = \{A, B, C\}$$

then $(w_1, w_2)$ is the unique input/output partion.

## 2.1

(a) $t_0 + 1$; The edge points at the state which the system updates to when it is clocked. This is the *next state* function, $\delta(s, i)$.

(b) $t_0$; The label on the edge is given directly as output. This is the *output* function, $\lambda(s, i)$.

## 2.2

a) False. The number of entering edges can be different for different states. There can for example be an edge from every state that resets the system to the starting state.

b) False. See *a*.

c) False. There can be more than one edge from one state to another.

d) False. See *e*.

e) True. The next state function is a function of both the state and the input. In every state, each input value corresponds to an edge leaving the state.

f) False. The statement is not true if the number of states is less than the number of inputs.

## 2.3

a) The statement means that independent of the current state, the input 001 results in the output 1. This is true for all three graphs since two 0:s drives the machine to state $s_2$. Then a 1 will output a 1.

b) This means that independent of the current state, the only way to get output 1 is to give 001 as input. In the first graph we can only get to state $s_2$ by giving two 0:s. The only output 1 is given from state $s_2$ with input 1. Hence, it is true for the first graph.

The second graph always gives output 1, so the statement is not true. In the third graph we can also get output 1 for input 101, hence, the statement is not true.

c) *If* and *only if* means that both *a* and *b* should be true. That gives graph 1.

## 2.4

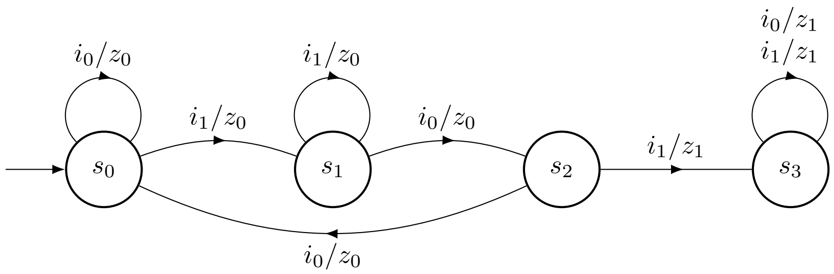
(a) In the first graph we can get stuck in state $s_4$, which gives a 1 out independent of the input. In the second graph, state $s_3$ means that the three most resent inputs are $i_0 i_1 i_0$. Then input $i_1$ will generate output $z_1$. This is the only way to get $z_1$ and, hence, the statement is true for the second graph. The third graph will generate a $z_1$ for every even consecutive occurrence of $i_0 i_1$. Therefore, $i_0 i_1 i_0 i_1 i_0 i_1$ (where the four most resent inputs are $i_0 i_1 i_0 i_1$ will not generate $z_1$.

Concluding the above, we see that it is only the second graph that fulfills the statement.

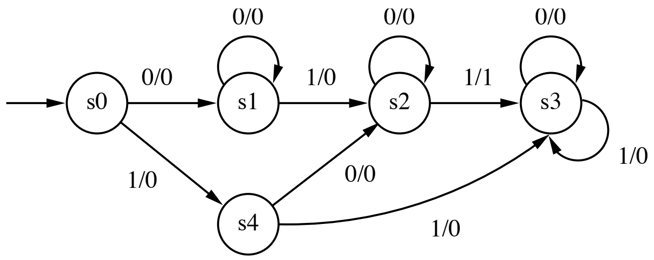
(b) In the first graph input $i_0 i_1 i_0 i_1$ will lead to state $s_4$, which will generate $z_1$ for ever. Hence, the statement is true for the first graph. For the other two it is false.

(c) The statement means that the output becomes $z_1$ after the first occurrence of $i_0 i_1 i_0 i_1$, and stays that way until the next occurrence when it is reset to $z_0$. This is not true for any of the three graphs.

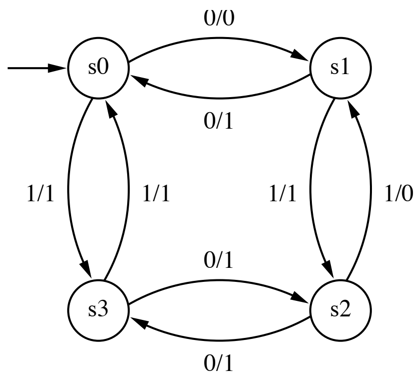
## 2.5



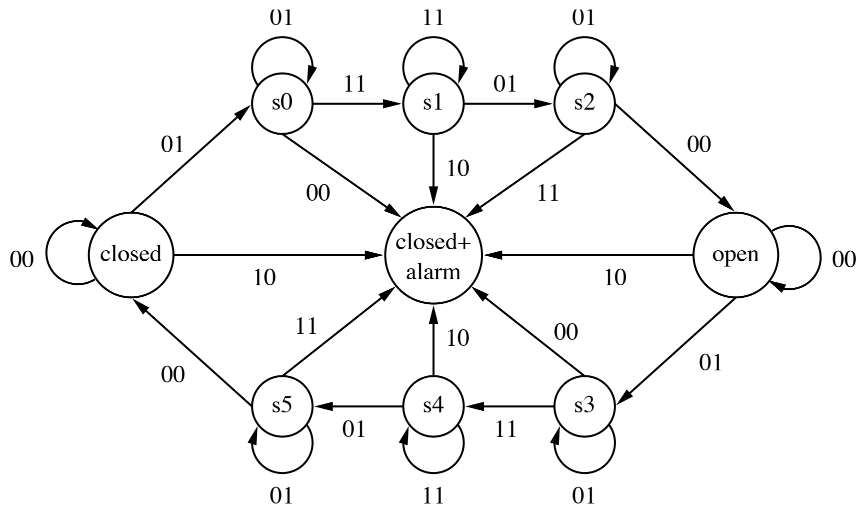
**2.6**




**2.8**

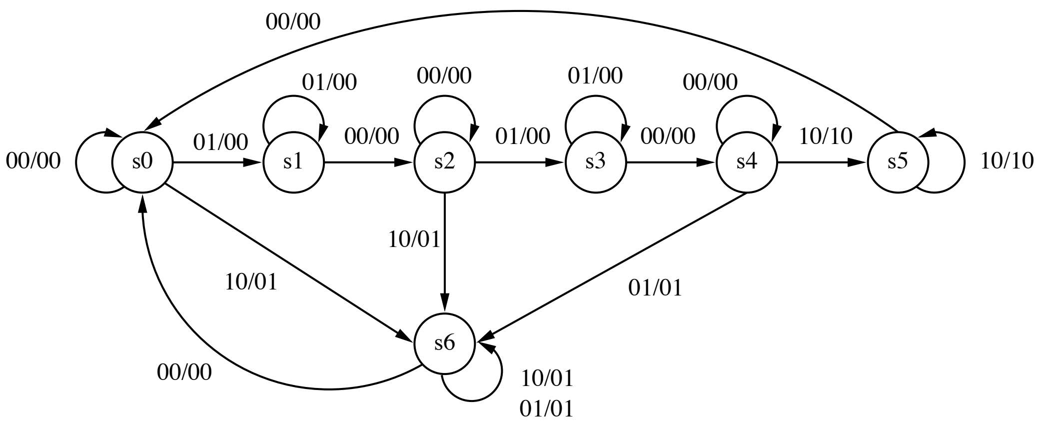



**2.11**



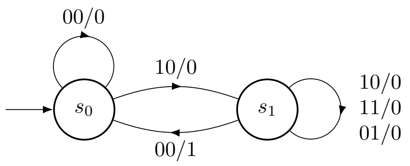

**2.12**

B1,B2/pony,rocking-horse

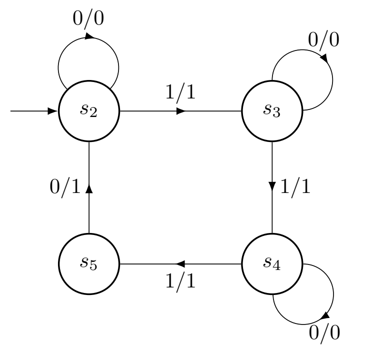


## 2.13

L1,L2/x

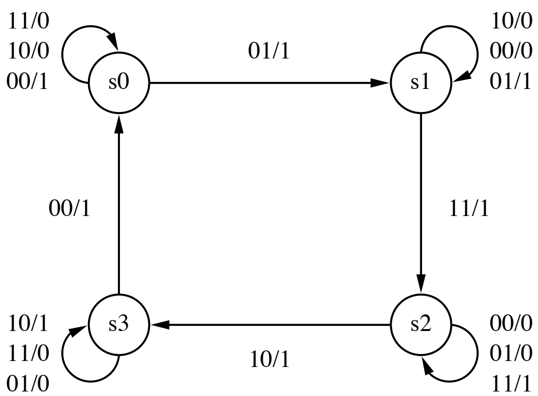



x/count (increment for each 1)



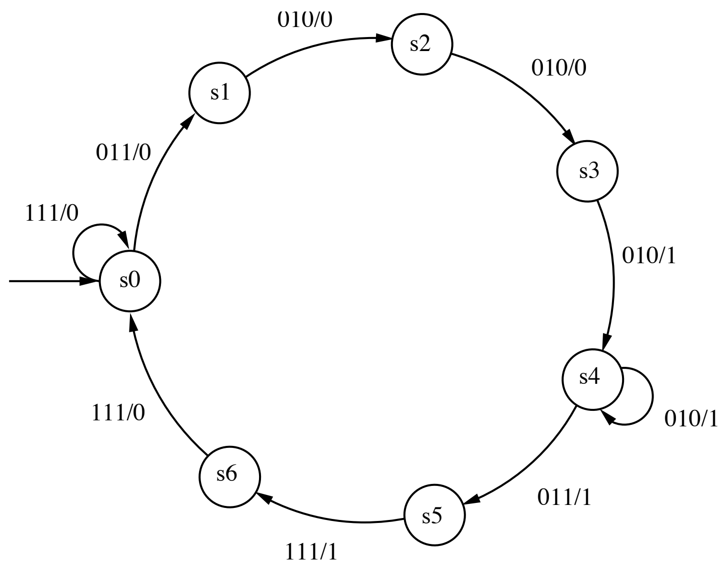

Alternative solution:

L1,L2/count (increment for each $0 \to 1$)




## 2.14

nIOR,nWRITE,nAddrSTB/nWAIT



## 2.16

(a)

| $x_1x_2x_3$ | $x_1 \vee x_2$ | $x_2 \vee x_3$ | $(x_1 \vee x_2) \vee x_3$ | $x_1 \vee (x_2 \vee x_3)$ |
|---|---|---|---|---|
| 0 0 0 | 0 | 0 | 0 | 0 |
| 0 0 1 | 0 | 1 | 1 | 1 |
| 0 1 0 | 1 | 1 | 1 | 1 |
| 0 1 1 | 1 | 1 | 1 | 1 |
| 1 0 0 | 1 | 0 | 1 | 1 |
| 1 0 1 | 1 | 1 | 1 | 1 |
| 1 1 0 | 1 | 1 | 1 | 1 |
| 1 1 1 | 1 | 1 | 1 | 1 |

## 2.17

a) Choose $f_1(x_1, x_2, x_3) = (x_1 \bar{\vee} x_2) \bar{\vee} x_3$ and $f_2(x_1, x_2, x_3) = x_1 \bar{\vee} (x_2 \bar{\vee} x_3)$. Then we can make a list $f_1(x_1, x_2, x_3)$ and $f_2(x_1, x_2, x_3)$ for all possible input 3-tuples $x_1\, x_2\, x_3$.

| $x_1x_2x_3$ | $f_1(x_1, x_2, x_3)$ | $f_2(x_1, x_2, x_3)$ |
|---|---|---|
| 0 0 0 | 0 | 0 |
| 0 0 1 | 1 | 1 |
| 0 1 0 | 1 | 1 |
| 0 1 1 | 0 | 0 |
| 1 0 0 | 1 | 1 |
| 1 0 1 | 0 | 0 |
| 1 1 0 | 0 | 0 |
| 1 1 1 | 1 | 1 |

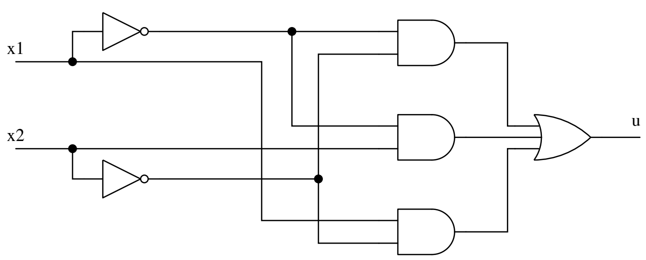
From the table we see that $f_1(x_1, x_2, x_3)$ and $f_2(x_1, x_2, x_3)$ are equal. Thus, the XOR operation is associative.

b) The XOR function for several inputs gives a 1 as output if exactly one input is 1 and the other 0.

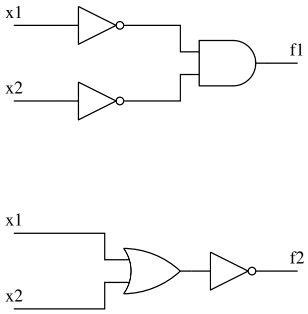
Take $x_1 = x_2 = x_3 = 1$. Then $x_1 \bar{\vee} x_2 \bar{\vee} x_3 = 0$. It follows then that the three-input XOR function is not equal to $f_1(x_1, x_2, x_3)$.

## 2.18

| $x_1x_2$ | $x_1'x_2'$ | $x_1' \wedge x_2'$ | $x_1' \wedge x_2$ | $x_1 \wedge x_2'$ | $u$ |
|---|---|---|---|---|---|
| 0 0 | 1 1 | 1 | 0 | 0 | 1 |
| 0 1 | 1 0 | 0 | 1 | 0 | 1 |
| 1 0 | 0 1 | 0 | 0 | 1 | 1 |
| 1 1 | 0 0 | 0 | 0 | 0 | 0 |

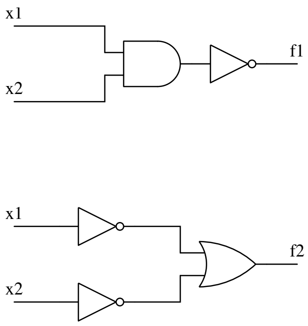


## 2.19



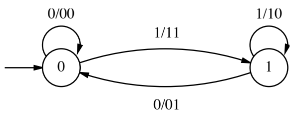

| $x_1 x_2$ | $f_1$ | $f_2$ |
|---|---|---|
| 0 0 | 1 | 1 |
| 0 1 | 0 | 0 |
| 1 0 | 0 | 0 |
| 1 1 | 0 | 0 |

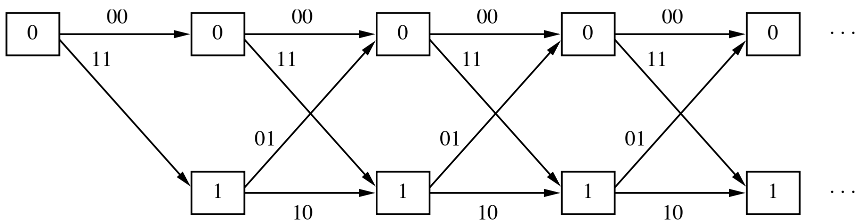
## 2.20




| $x_1 x_2$ | $f_1$ | $f_2$ |
|---|---|---|
| 0 0 | 1 | 1 |
| 0 1 | 1 | 1 |
| 1 0 | 1 | 1 |
| 1 1 | 0 | 0 |

## 2.21

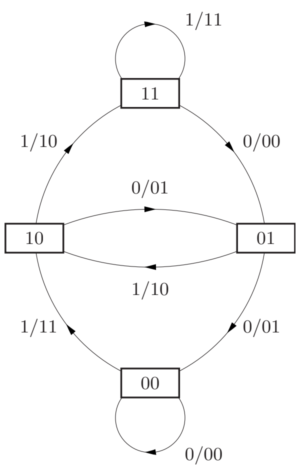


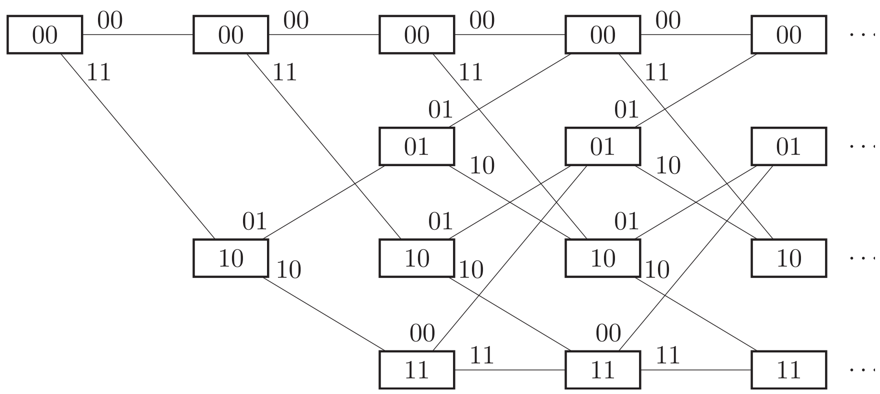




the smallest number of binary digits by which any two sequences differ is $d_{\text{free}} = 3$
$\Rightarrow$ we can always correct 1 error

## 2.22

We only give the trellis and the state transition graph. The calculation of the free distance is omitted.



## 2.23

(1) The figure gives that

$$\begin{cases} u = x \vee q' \\ q^+ = (x \wedge q) \vee (x' \wedge q') \end{cases}$$

| $qx$ | $q^+$ | $u$ |
|---|---|---|
| 00 | 1 | 1 |
| 01 | 0 | 1 |
| 10 | 0 | 0 |
| 11 | 1 | 1 |

OK, with $i_0 = 1$, $i_1 = 0$, $z_0 = 0$, $z_1 = 1$, $s_0 = 0$, and $s_1 = 1$.

(2) The figure gives that

$$\begin{cases} u = x \wedge q' \\ q^+ = (x \vee q) \wedge (x' \vee q') \end{cases}$$

| $qx$ | $q^+$ | $u$ |
|---|---|---|
| 00 | 0 | 0 |
| 01 | 1 | 1 |
| 10 | 1 | 0 |
| 11 | 0 | 0 |

OK, with $i_0 = 0$, $i_1 = 1$, $z_0 = 1$, $z_1 = 0$, $s_0 = 1$, and $s_1 = 0$.

(3) The figure gives that

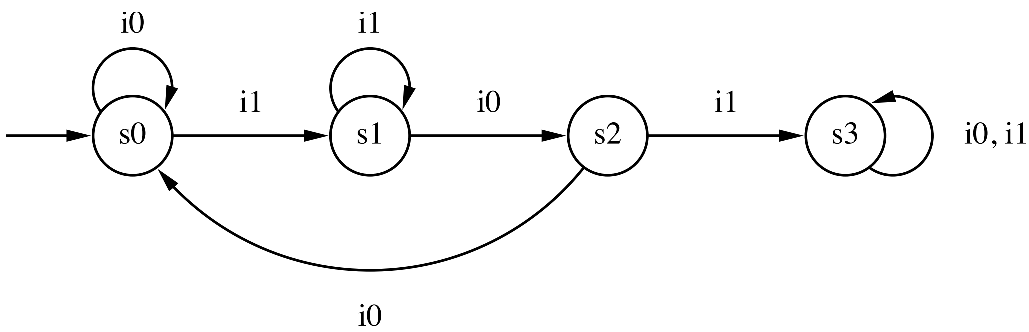
$$\begin{cases} u = x' \vee q \\ q^+ = x \oplus q \end{cases}$$

| $qx$ | $q^+$ | $u$ |
|---|---|---|
| 00 | 0 | 1 |
| 01 | 1 | 0 |
| 10 | 1 | 1 |
| 11 | 0 | 1 |

OK, with $i_0 = 0$, $i_1 = 1$, $z_0 = 0$, $z_1 = 1$, $s_0 = 1$, and $s_1 = 0$.

Hence, all of the sequential circuits are realizations of the graph.

## 2.24




Use the following encoding.

| | $s_0$ | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|---|
| $q_0q_1$ | 00 | 01 | 11 | 10 |

| | $i_0$ | $i_1$ |
|---|---|---|
| $x$ | 0 | 1 |

| | $z_0$ | $z_1$ |
|---|---|---|
| $u$ | 0 | 1 |

This gives the state transition table and output table

| $q_0q_1x$ | $q_0^+q_1^+$ | $q_0q_1x$ | $q_0^+q_1^+$ |
|---|---|---|---|
| 00 0 | 00 | 10 0 | 10 |
| 00 1 | 01 | 10 1 | 10 |
| 01 0 | 11 | 11 0 | 00 |
| 01 1 | 01 | 11 1 | 10 |

| $q_0q_1$ | $u$ |
|---|---|
| 00 | 0 |
| 01 | 0 |
| 10 | 1 |
| 11 | 0 |

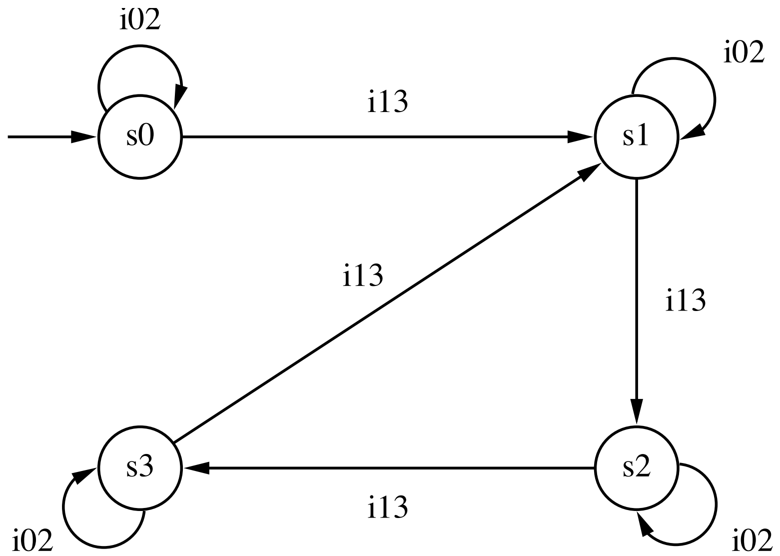
It can be realized by the following equations

$$
\begin{aligned}
u &= q_0 q_1' \\
q_0^+ &= q_0' q_1 x' \vee q_0 q_1' x' \vee q_0 q_1' x \vee q_0 q_1 x \\
q_1^+ &= q_0' q_1' x \vee q_0' q_1 x' \vee q_0' q_1 x
\end{aligned}
$$

An alternative (minimal) solution can be found as

$$
\begin{aligned}
u &= q_0 q_1' \\
q_0^+ &= q_0 q_1' \vee q_0 x \vee q_0' q_1 x' \\
q_1^+ &= q_0' q_1 \vee q_0' x
\end{aligned}
$$

**2.25**




Use the following encoding.

| | $s_0$ | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|---|
| $q_0q_1$ | 00 | 01 | 11 | 10 |

| | $i_0i_2$ | $i_1i_3$ |
|---|---|---|
| $x$ | 0 | 1 |

| | $z_0$ | $z_1$ |
|---|---|---|
| $y$ | 0 | 1 |

This gives the state transition table and output table

| $q_0q_1x$ | $q_0^+q_1^+$ | $q_0q_1x$ | $q_0^+q_1^+$ |
|---|---|---|---|
| 00 0 | 00 | 10 0 | 10 |
| 00 1 | 01 | 10 1 | 01 |
| 01 0 | 01 | 11 0 | 11 |
| 01 1 | 11 | 11 1 | 10 |

| $q_0q_1$ | $y$ |
|---|---|
| 00 | 0 |
| 01 | 0 |
| 10 | 1 |
| 11 | 0 |

It can be realized by the following equations

$$
\begin{aligned}
q_0^+ &= q_0' q_1 x \vee q_0 q_1' x' \vee q_0 q_1 x' \vee q_0 q_1 x \\
q_1^+ &= q_0' q_1' x \vee q_0' q_1 x' \vee q_0' q_1 x \vee q_0 q_1' x \vee q_0 q_1 x' \\
y &= q_0 q_1'
\end{aligned}
$$

An alternative (minimal) solution can be found as

$$
\begin{aligned}
q_0^+ &= q_0 x' \vee q_1 x \\
q_1^+ &= q_1 x' \vee q_0' x \vee q_1' x \\
y &= q_0 q_1'
\end{aligned}
$$

**2.26**

Use for example

| state | $q_1q_2$ |
|---|---|
| $s_0$ | 00 |
| $s_1$ | 01 |
| $s_2$ | 11 |
| $s_3$ | 10 |

This leads to the following functional table

| $q_1q_2\,i$ | $q_1^+q_2^+z$ |
|---|---|
| 0 0 0 | 0 1 0 |
| 0 0 1 | 1 0 1 |
| 0 1 0 | 0 0 1 |
| 0 1 1 | 1 1 1 |
| 1 0 0 | 1 1 1 |
| 1 0 1 | 0 0 1 |
| 1 1 0 | 1 0 1 |
| 1 1 1 | 0 1 0 |

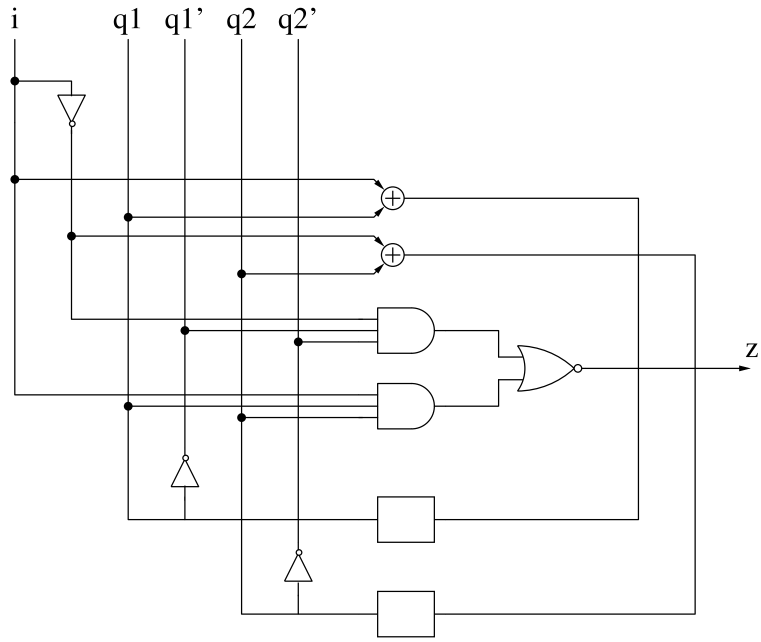
The state transition functions and the output function can be realized as

$$
\begin{aligned}
q_1^+ &= q_1'q_2'i \vee q_1'q_2 i \vee q_1 q_2' i' \vee q_1 q_2 i' \\
q_2^+ &= q_1'q_2'i' \vee q_1'q_2 i \vee q_1 q_2' i' \vee q_1 q_2 i \\
z &= (z')' = (q_1'q_2'i' \vee q_1 q_2 i)'
\end{aligned}
$$

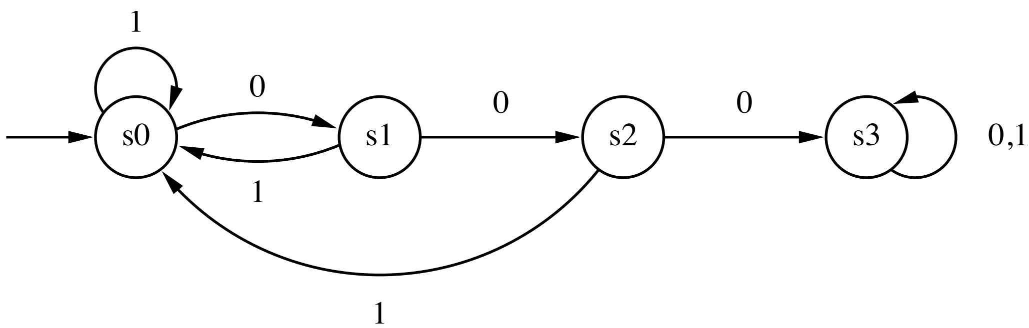
Another way to realize the state transition functions is with

$$
\begin{aligned}
q_1^+ &= q_1 \oplus i \\
q_2^+ &= q_2 \oplus i'
\end{aligned}
$$

The realization then becomes




**2.27**




Use the following state encoding

| state | $q_1 q_2$ |
|---|---|
| $s_0$ | 00 |
| $s_1$ | 01 |
| $s_2$ | 11 |
| $s_3$ | 10 |

We get the state transition table

| $q_0 q_1 x$ | $q_0^+ q_1^+ u$ |
|---|---|
| 00 0 | 01 0 |
| 00 1 | 00 0 |
| 01 0 | 11 0 |
| 01 1 | 00 0 |
| 10 0 | 10 1 |
| 10 1 | 10 1 |
| 11 0 | 10 0 |
| 11 1 | 00 0 |

Functions:

$$
\begin{aligned}
q_0^+ &= q_0' q_1 x' \vee q_0 q_1' x' \vee q_0 q_1' x \vee q_0 q_1 x' \\
q_1^+ &= q_0' q_1' x' \vee q_0' q_1 x' \\
u &= q_0 q_1'
\end{aligned}
$$

Alternative (minimal) functions

$$
\begin{aligned}
q_0^+ &= q_1 x' \vee q_0 q_1' \\
q_1^+ &= q_0' x' \\
y &= q_0 q_1'
\end{aligned}
$$

## 2.28

In the alternative solution of 2.11 we need four states that we, for example, can encode as

| state | $q_1 q_2$ |
|---|---|
| $s_0$ | 00 |
| $s_1$ | 01 |
| $s_2$ | 11 |
| $s_3$ | 10 |

This results in the functional table

| $q_1 q_2$ | $i_1 i_2$ | $q_1^+ q_2^+$ | $z$ |
|---|---|---|---|
| 0 0 | 0 0 | 0 0 | 1 |
| 0 0 | 0 1 | 0 1 | 1 |
| 0 0 | 1 0 | 0 0 | 0 |
| 0 0 | 1 1 | 0 0 | 0 |
| 0 1 | 0 0 | 0 1 | 0 |
| 0 1 | 0 1 | 0 1 | 1 |
| 0 1 | 1 0 | 0 1 | 0 |
| 0 1 | 1 1 | 1 1 | 1 |
| 1 0 | 0 0 | 0 0 | 1 |
| 1 0 | 0 1 | 1 0 | 0 |
| 1 0 | 1 0 | 1 0 | 1 |
| 1 0 | 1 1 | 1 0 | 0 |
| 1 1 | 0 0 | 1 1 | 0 |
| 1 1 | 0 1 | 1 1 | 0 |
| 1 1 | 1 0 | 1 0 | 1 |
| 1 1 | 1 1 | 1 1 | 1 |

This can be expressed, for example, by

$$
\begin{aligned}
q_1^+ &= q_2 i_1 i_2 \vee q_1 q_2 \vee q_1 i_2 \vee q_1 i_1 \\
q_2^+ &= q_1' i_1' i_2 \vee q_1' q_2 \vee q_2 i_1' \vee q_2 i_2 \\
z &= q_1' q_2' i_1' \vee q_1' q_2 i_2 \vee q_1 q_2 i_2 \vee q_1 q_2' i_2'
\end{aligned}
$$

(The figure of the realization is omitted!)

### 3.1

(a) $45 = 6 \cdot 7 + 3 \Rightarrow R_7(45) = 3$

(b) $-45 = -7 \cdot 7 + 4 \Rightarrow R_7(-45) = 4$

(c) $R_7(45 + 63) = R_7(45 + 9 \cdot 7) \overset{\text{Th.3.2}}{=} R_7(45) = 3$

### 3.2

(i)

$$\begin{aligned} R_d(n_1 + n_2) &= R_d\Big(\overbrace{(q_1 d + R_d(n_1))}^{n_1} + \overbrace{(q_2 d + R_d(n_2))}^{n_2}\Big) \\ &= R_d\Big((q_1 + q_2)d + R_d(n_1) + R_d(n_2)\Big) \\ &\overset{\text{Th.3.2}}{=} R_d\Big(R_d(n_1) + R_d(n_2)\Big) \quad . \end{aligned}$$

(ii)

$$\begin{aligned} R_d(n_1 n_2) &= R_d\Big(\overbrace{(q_1 d + R_d(n_1))}^{n_1}\overbrace{(q_2 d + R_d(n_2))}^{n_2}\Big) \\ &= R_d\Big(q_1 q_2 d^2 + q_1 d R_d(n_2) + R_d(n_1) q_2 d + R_d(n_1) R_d(n_2)\Big) \\ &= R_d\Big(\underbrace{(q_1 q_2 d + q_1 R_d(n_2) + R_d(n_1) q_2}_{\text{integer}})d + R_d(n_1) R_d(n_2)\Big) \\ &\overset{\text{Th.3.2}}{=} R_d\Big(R_d(n_1) R_d(n_2)\Big) \quad . \end{aligned}$$

### 3.3

(a) Let $n_1 = 1946$ and $n_2 = 1956$. Euclid's algorithm gives

$$\begin{aligned} 1956 &= 1 \cdot 1946 + 10 \\ 1946 &= 194 \cdot 10 + 6 \\ 10 &= 1 \cdot 6 + 4 \\ 6 &= 1 \cdot 4 + 2 \\ 4 &= 2 \cdot 2 + 0 \Rightarrow \gcd(1956, 1946) = 2. \end{aligned}$$

Then, get Bezout's identity as

$$\begin{aligned} 2 &= 6 - 4 \\ &= 6 - (10 - 6) = -10 + 2 \cdot 6 \\ &= -10 + 2(1946 - 194 \cdot 10) = 2 \cdot 1946 - 389 \cdot 10 \\ &= 2 \cdot 1946 - 389(1956 - 1946) = -389 \cdot 1956 + 391 \cdot 1946. \end{aligned}$$

Alternatively, this can be solved with Euclid's extended algorithm as

| $i$ | $r_i$ | $q_i$ | $s_i$ | $t_i$ |
|---|---|---|---|---|
| -2 | 1956 | | 1 | 0 |
| -1 | 1946 | | 0 | 1 |
| 0 | 10 | 1 | 1 | -1 |
| 1 | 6 | 194 | -194 | 195 |
| 2 | 4 | 1 | 195 | -196 |
| 3 | **2** | 1 | **-389** | **391** |
| 4 | 0 | | | |

Which gives that $\gcd(1956, 1946) = 2 = -389 \cdot 1956 + 391 \cdot 1946$.

(b) Let $n_1 = 1870$ and $n_2 = 222$. Use Euclid's extended algorithm to get

| $i$ | $r_i$ | $q_i$ | $s_i$ | $t_i$ |
|---|---|---|---|---|
| -2 | 1870 | | 1 | 0 |
| -1 | 222 | | 0 | 1 |
| 0 | 94 | 8 | 1 | -8 |
| 1 | 34 | 2 | -2 | 17 |
| 2 | 26 | 2 | 5 | -42 |
| 3 | 8 | 1 | -7 | 59 |
| 4 | **2** | 3 | **26** | **-219** |
| 5 | 0 | | | |

Hence, $\gcd(1870, 222) = 2 = 26 \cdot 1870 - 219 \cdot 222$.

(c) Let $n_1 = 561$ and $n_2 = 341$. Use Euclid's extended algorithm to get

| $i$ | $r_i$ | $q_i$ | $s_i$ | $t_i$ |
|---|---|---|---|---|
| -2 | 561 | | 1 | 0 |
| -1 | 341 | | 0 | 1 |
| 0 | 220 | 1 | 1 | -1 |
| 1 | 121 | 1 | -1 | 2 |
| 2 | 99 | 1 | 2 | -3 |
| 3 | 22 | 1 | -3 | 5 |
| 4 | **11** | 4 | **14** | **-23** |
| 5 | 0 | | | |

Hence, $\gcd(561, 341) = 11 = 14 \cdot 561 - 23 \cdot 341$.

## 3.4

(a) Use Euclid's extended algorithm on $n_1 = 73$ and $n_2 = 11$:

| $i$ | $r_i$ | $q_i$ | $s_i$ | $t_i$ |
|---|---|---|---|---|
| -2 | 73 | - | 1 | 0 |
| -1 | 11 | - | 0 | 1 |
| 0 | 7 | 6 | 1 | -6 |
| 1 | 4 | 1 | -1 | 7 |
| 2 | 3 | 1 | 2 | -13 |
| 3 | 1 | 1 | -3 | 20 |
| 4 | 0 | 3 | - | - |

$\gcd(73, 11) = 1 = -3 \cdot 73 + 20 \cdot 11 \Rightarrow (x, y) = (20, -3)$.

(b) Divide the equation by two and solve $17x + 3y = 1$ instead. Use Euclid's extended algorithm:

| $i$ | $r_i$ | $q_i$ | $s_i$ | $t_i$ |
|---|---|---|---|---|
| -2 | 17 | - | 1 | 0 |
| -1 | 3 | - | 0 | 1 |
| 0 | 2 | 5 | 1 | -5 |
| 1 | 1 | 1 | -1 | 6 |
| 2 | 0 | 2 | - | - |

$\gcd(17, 3) = 1 = -1 \cdot 17 + 6 \cdot 3 \Rightarrow (x, y) = (-1, 6)$.

(c) Multiply the result in (b) by two. Hence, $(x, y) = (-2, 12)$.

(d) Since $\gcd(34, 6) = 2$ does not divide 3, there is no solution.

**3.5**

We want to find $x$ and $y$ such that

$$3x + 7y = 5. \tag{9}$$

Since $\gcd(3,7) = 1$, we know that $\gcd(3 \cdot 5, 7 \cdot 5) = 5$ and Euclid's algorithm will yield to a solution to (9).

$$\begin{cases} 35 = 2 \cdot 15 + 5 \\ 15 = 3 \cdot 5 + 0 \end{cases}$$

We see that $5 = 35 - 2 \cdot 15 = 5 \cdot 7 - 10 \cdot 3$. Hence we can time $5$ minutes as the time from the tenth turning of the 3-minute hourglass to the fifth turning of the 7-minute hourglass. Note that we will have to wait 30 minutes before we can start boiling.

There is an alternative, more efficient solution. Another solution to (9) is $x = -3$, $y = 2$. In other words, we can time $5$ minutes as the time from the third turning of the 3-minute hourglass to the second turning of the 7-minute hourglass. This way we can start boiling the egg after 9 minutes.

**3.7**

Ring $(\mathbb{Z}_{12}, \oplus, \otimes)$.

(a) $5 \oplus a = 0 \Leftrightarrow 5 + a \equiv 0 \mod 12 \Rightarrow a = 7$.

(b) $5 \otimes a = 1 \Leftrightarrow 5 \cdot a \equiv 1 \mod 12 \Rightarrow a = 5$ (since $5 \cdot 5 = 25 = 1 + 2 \cdot 12$).

(c) $2/5 = 2 \otimes 5^{-1} = 2 \otimes 5 = 10$. Then $5 \otimes (2/5) = 5 \cdot 10 \mod 12 = 2$.

(d) By Thm. 3.10, the units are all elements $u$ such that $\gcd(12, u)$=1, i.e., $U = \{1, 5, 7, 11\}$.

**3.8**

Ring $(\mathbb{Z}_{35}, \oplus, \otimes)$.

(a) $22 \oplus a = 0 \Leftrightarrow 22 + a \equiv 0 \mod 35 \Rightarrow a = 13$.

(b) We want to find $a$ such that $2 \cdot a \equiv 1 \mod 35$, i.e., such that $2 \cdot a + 35 \cdot b = 1$. We can solve this by Euclid's extended algorithm, but here it is easily seen that $a = 18$, $b = -1$ is a solution. Hence $2^{-1} = 18$ in $\mathbb{Z}_{35}$.

(c) $7/2 = 7 \otimes 2^{-1} = 7 \otimes 18 = 126 \mod 35 = 21$. Then $2 \otimes (7/2) = 2 \cdot 21 \mod 35 = 7$.

(d) By Thm. 3.10, the units are all elements $u$ such that $\gcd(35, u)$=1, i.e.,
$U = \{u \in \mathbb{Z}_{35} \backslash \{0\} \mid 5 \not| u$ and $7 \not| u\}$. ($\not|$means "does not divide".)

**3.9**

Ring $(\mathbb{Z}_{11}, \oplus, \otimes)$.

(a) $1 \oplus a = 0 \Leftrightarrow 1 + a \equiv 0 \mod 11 \Rightarrow a = 10$.

(b) $5 \otimes a = 1 \Leftrightarrow 5 \cdot a \equiv 1 \mod 11 \Rightarrow a = 9$ (since $5 \cdot 9 = 45 = 1 + 4 \cdot 11$).

(c) $-1/5 = -1 \otimes 5^{-1} = 10 \otimes 9 = 90 \mod 11 = 2$. Then $5 \otimes (-1/5) = 5 \cdot 2 = 10 = -1$.

(d) Since 11 is a prime, all elements except 0 are units.

**3.11**

Ring $(\mathbb{Z}_3, \oplus, \otimes)$. Solve

$$\begin{cases} x \oplus 2y & = & 1 & (1) \\ x \oplus y & = & 0 & (2) \end{cases}$$

(1) $\oplus$ (2) $\Rightarrow 2x \oplus 3y = 1 \Leftrightarrow 2x = 1$ (since $3 = 0 \mod 3$). Hence $x = 2^{-1} = 2$, and $y = -2 = 1$.

**3.12**

Ring $(\mathbb{Z}_5, \oplus, \otimes)$. Solve

$$\begin{cases} x \oplus 2y & = & 1 & (1) \\ x \oplus y & = & 0 & (2) \end{cases}$$

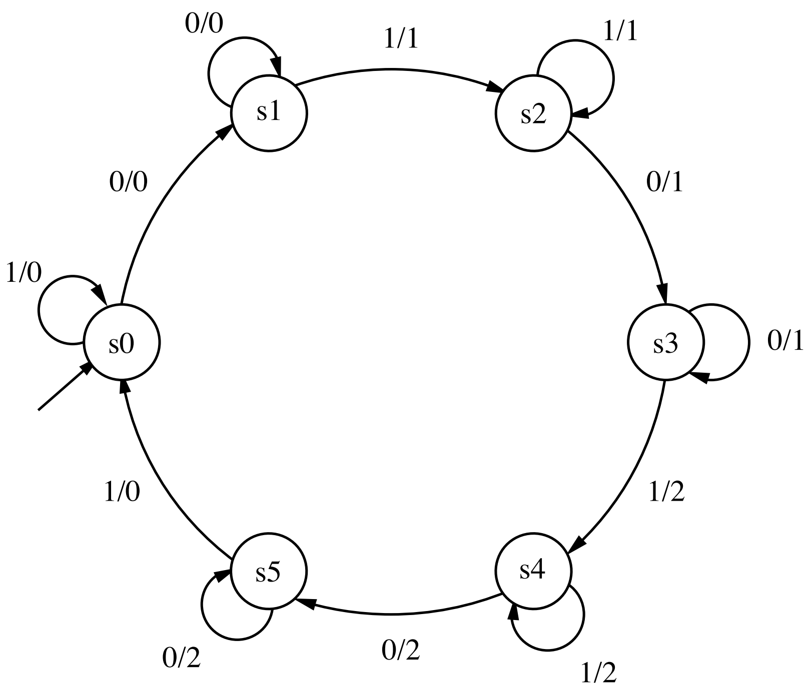
$(1) - (2) \Rightarrow y = 1 \Rightarrow x = -1 = 4.$

**3.13**

We have direcly that $a \cdot 0 = 0 \cdot a = 0$ and $a \cdot 1 = 1 \cdot a = a$. What is left is $2 \cdot 2$, $2 \cdot 3$, $3 \cdot 2$, and $3 \cdot 3$. Since all elements in a field are units it follows that $a \cdot b = a \cdot c \Rightarrow b = c$ and, hence, all rows and all columns must contain each element exactly once. Therefore, $2 \cdot 2 = 3$ and $3 \cdot 3 = 2$. We complete the table by $2 \cdot 3 = 1$ and $3 \cdot 2 = 1$.

| $\cdot$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 3 | 1 |
| 3 | 0 | 3 | 1 | 2 |

**3.14**



### 3.15

To decrypt the message by exhaustive search we write the complete alphabet below each letter. In this example, it is already after a few letters possible to guess the key that delivers the correct message, since the other alternatives hardly make any sense.

```
    YREEZSRCZJTEDZEXREUYVYRJVCVGYREKJ
    ZSFFATSD
    ATGGBUTE
    BUHHCVUF
    CVIIDWVG
    DWJJEXWH
    EXKKFYXI
    FYLLGZYJ
    GZMMHAZK
--> HANNIBALISCNMINGANDHEHASELEPHANTS
    IBOOJCBM
    JCPPKDCN
    KDQQLEDO
    LERRMFEP
    MFSSNGFQ
    NGTTOHGR
    OHUUPIHS
    PIVVQJIT
    QJWWRKJU
    RKXXSLKV
    SLYYTMLW
    TMZZUNMX
    UNAAVONY
    VOBBWPOZ
    WPCCXQPA
    XQDDYRQB
```

Looking at the first 8 letters we recognize the row with key $k = 9$ to be the message and decrypt the remaining letters with this key.
Observe that there is an error in the encrypted sequence: the 12th letter should be an F instead of an E. Then we get the message "Hannibal is coming and he has elephants".

### 3.17

Write $26 = 2 \cdot 13$. Then (3.124) gives $\varphi(26) = 26(1 - 1/2)(1 - 1/13) = 12$.
By Theorem 3.13 we get $R_{26}(3^{12}) = R_{26}(3^{\varphi(26)}) = 1$.

### 3.18

Since $100 = 64 + 32 + 4$ we can write $R_{34}(12^{100}) = R_{34}(12^{64} \cdot 12^{32} \cdot 12^4) = R_{34}(R_{34}(12^{64})R_{34}(12^{32})R_{34}(12^4))$.

$$\begin{aligned}
12^1 &\equiv 12 \pmod{34} \\
12^2 &\equiv 8 \pmod{34} \\
12^4 &= 12^2 \cdot 12^2 \equiv 8 \cdot 8 \pmod{34} \equiv 30 \pmod{34} \\
12^8 &= 12^4 \cdot 12^4 \equiv 30 \cdot 30 \pmod{34} \equiv 16 \pmod{34} \\
12^{16} &= 12^8 \cdot 12^8 \equiv 16 \cdot 16 \pmod{34} \equiv 18 \pmod{34} \\
12^{32} &= 12^{16} \cdot 12^{16} \equiv 18 \cdot 18 \pmod{34} \equiv 18 \pmod{34} \\
12^{64} &= 12^{32} \cdot 12^{32} \equiv 18 \cdot 18 \pmod{34} \equiv 18 \pmod{34}
\end{aligned}$$

$R_{34}(12^{100}) = R_{34}(18 \cdot 18 \cdot 30) = 30$.

### 3.19

$m = 143 = 11 \cdot 13 \Rightarrow p = 11, q = 13$. $\varphi(m) = \varphi(p)\varphi(q) = 10 \cdot 12 = 120$.
To find $d$, i.e. the inverse of $e$, we use Euclid's extended algorithm and Bezout's identity to solve $\gcd(120, 23) = s \cdot 120 + t \cdot 23 = 1$.

| $i$ | $r_i$ | $q_i$ | $s_i$ | $t_i$ |
|---|---|---|---|---|
| -2 | 120 | – | 1 | 0 |
| -1 | 23 | – | 0 | 1 |
| 0 | 5 | 5 | 1 | -5 |
| 1 | 3 | 4 | -4 | 21 |
| 2 | 2 | 1 | 5 | -26 |
| 3 | **1** | 1 | **-9** | **47** |
| 4 | 0 | | | |

$\Rightarrow 1 = -9 \cdot 120 + 47 \cdot 23 \Rightarrow d = 47 = e^{-1}$.

Next, we have $D(C) = C^d = M \Rightarrow M \equiv 9^{47} \pmod{143}$.
Write $47 = 32 + 8 + 4 + 2 + 1$

$$\begin{aligned}
9^1 &\equiv 9 \pmod{143}\\
9^2 &\equiv 81 \pmod{143}\\
9^4 &\equiv 81 \cdot 81 \pmod{143} \equiv 126 \pmod{143}\\
9^8 &\equiv 126 \cdot 126 \pmod{143} \equiv 3 \pmod{143}\\
9^{32} = (9^8)^4 &\equiv 3^4 \pmod{143} \equiv 81 \pmod{143}
\end{aligned}$$

$$M = 9 \cdot 81 \cdot 126 \cdot 3 \cdot 81 \equiv 3 \cdot 9 \cdot \underbrace{\underbrace{81 \cdot 81}_{126} \cdot 126}_{3} \pmod{143} \equiv 3 \cdot 9 \cdot 3 \pmod{143} = 81 \pmod{143}.$$

### 3.21

Since $a' = 1 + a$ and $a \vee b = a + b + ab$
$\Rightarrow a \vee b = a + b + \underbrace{ab}_{0} = 1 \Rightarrow b = 1 + a = a'$.

### 3.22

$$a \vee a'b = a \vee (1+a)b = a \vee (b + ab) = a + b + ab + a(b + ab) = a + b + \underbrace{ab + ab}_{\substack{0\\ \text{Th. 3.16}}} + \underbrace{aab}_{\substack{ab\\ \text{(idempotence)}}} = a + b + ab = a \vee b.$$

### 3.23

$$a(a' \vee b) = a(a' + b + a'b) = \underbrace{aa'}_{0} + ab + \underbrace{aa'}_{0} b = ab.$$

### 3.24

(iii) $= (a \vee b)(a' \vee c) = \underbrace{aa'}_{0} \vee ac \vee a'b \vee bc =$ (ii) $\overset{\text{consensus}}{=} ac \vee a'b =$ (iv).

(i) $= (a \vee b)(a' \vee c)(b \vee c) = \underbrace{\underbrace{aa'}_{0}(b \vee c)}_{0} \vee \underbrace{acb \vee acc}_{ac} \vee \underbrace{a'bb \vee a'bc}_{\substack{a'b\\ \text{absorption}}} \vee \underbrace{bcb \vee bcc}_{bc} = ac \vee a'b \vee bc =$ (ii).

$\Rightarrow$ all expressions are equal.

## 3.25

(a) $B = \{0, 1, 5, 6\}$.

(b) Scetch of proof: Check that $(B, +, \cdot)$ is a ring (check the conditions of the definition) and that all elements are idempodent.

(c) From (3.173)-(3.175) together with (3.213) and (3.214) we obtain

$$\begin{aligned} a \wedge b &= a \cdot b = a \otimes b \\ a \vee b &= (a+b) + ab = (a \oplus b \oplus 8ab) \oplus ab \oplus 8(a \oplus b \oplus 8ab)ab = a \oplus b \oplus 89ab \\ &= a \oplus b \oplus 9ab \\ a' &= 1 + a = 1 \oplus a \oplus 8a = 1 \oplus 9a \end{aligned}$$

| $\wedge$ | 0 | 1 | 5 | 6 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 5 | 6 |
| 5 | 0 | 5 | 5 | 0 |
| 6 | 0 | 6 | 0 | 6 |

| $\vee$ | 0 | 1 | 5 | 6 |
|---|---|---|---|---|
| 0 | 0 | 1 | 5 | 6 |
| 1 | 1 | 1 | 1 | 1 |
| 5 | 5 | 1 | 5 | 1 |
| 6 | 6 | 1 | 1 | 6 |

| $'$ | |
|---|---|
| 0 | 1 |
| 1 | 0 |
| 5 | 6 |
| 6 | 5 |

## 3.27

$M$ is a subset of $N$ if and only if the intersection of $M$ and $N$ is equal to $M$, i.e. $M \subseteq N \Leftrightarrow M \cap N = M$.

$$\begin{aligned} M \cap N &= M \\ M' \cup (M \cap N) &= \underbrace{M' \cup M}_{U} \\ \underbrace{(M' \cup M)}_{U} \cap (M' \cup N) &= U \\ M' \cup N &= U \end{aligned}$$

Hence, $M \subseteq N \Leftrightarrow M' \cup N = U$.

## 3.28

$$\begin{aligned} (A' \cap B' \cap C) \cup C' &= (A' \cup C') \cap (B' \cup C') \cap \underbrace{(C \cup C')}_{U} \\ &= \big((A' \cup C')' \cup (B' \cup C')'\big)' \\ &= \big((A \cap C) \cup (B \cap C)\big)' \ . \end{aligned}$$

## 3.29

Assume the following notation

$S$ = "Sweden will win the World Cup in hockey"
$R$ = "the Russians will be surprised"
$H$ = "Silvia will be happy"

"If Sweden will win the World Cup in hockey, then the Russians will be surprised": $S \Rightarrow R$
"If the Russians will be surprised, then Silvia will be happy": $R \Rightarrow H$
These two can be combined to

$$f(S, R, H) = (S \Rightarrow R)(R \Rightarrow H) = (S' \vee R)(R' \vee H) = S'R' \vee S'H \vee RH \ .$$

The complement of $f$ is

$$f'(S, R, H) = (S \vee R)(S \vee H')(R' \vee H') = (R' \vee H')(R \vee S) \ .$$

(i) $f \Rightarrow H = f' \vee H = R'S \vee H'R \vee H'S \vee H = \big((R \vee S')(H \vee R')(H \vee S')H'\big)' = (S'R'H')'$.

(ii) $f \Rightarrow (R' \Rightarrow H') = f' \vee (R \vee H') = R'S \vee H'R \vee H'S \vee R \vee H' = R'S \vee R \vee H' = \big((R \vee S')R'H\big)' = (S'R'H)'$.

(iii) $f \Rightarrow (H' \Rightarrow S') = f' \vee (H \vee S') = R'S \vee H'R \vee H'S \vee \underbrace{H \vee S'}_{(H'S)'} = R'S \vee H'R \vee 1 = 1$.

The following table summarizes the results:

| $SRH$ | $ff'$ | $f' \vee H$ | $f' \vee R \vee H'$ | $f' \vee H \vee S'$ |
|---|---|---|---|---|
| 0 0 0 | 1 0 | 0 | 1 | 1 |
| 0 0 1 | 1 0 | 1 | 0 | 1 |
| 0 1 0 | 0 1 | 1 | 1 | 1 |
| 0 1 1 | 1 0 | 1 | 1 | 1 |
| 1 0 0 | 0 1 | 1 | 1 | 1 |
| 1 0 1 | 0 1 | 1 | 1 | 1 |
| 1 1 0 | 0 1 | 1 | 1 | 1 |
| 1 1 1 | 1 0 | 1 | 1 | 1 |

$\Rightarrow$ only sentence (iii) follows from $f$.

## 3.30

(i) $(P \Rightarrow QR)' = (P' \vee QR)' = P(Q' \vee R') = PQ' \vee PR'$.

(ii) $P' \Rightarrow Q \vee R = P \vee Q \vee R$.

(iii) $P \Rightarrow (QR)' = P' \vee (QR)' = P' \vee Q' \vee R'$.

>From this we obtain the following table

| $PQR$ | (i) | (ii) | (iii) |
|---|---|---|---|
| 0 0 0 | 0 | 0 | 1 |
| 0 0 1 | 0 | 1 | 1 |
| 0 1 0 | 0 | 1 | 1 |
| 0 1 1 | 0 | 1 | 1 |
| 1 0 0 | 1 | 1 | 1 $\leftarrow$ |
| 1 0 1 | 1 | 1 | 1 $\leftarrow$ |
| 1 1 0 | 1 | 1 | 1 $\leftarrow$ |
| 1 1 1 | 0 | 1 | 0 |

All three expressions have different meanings. For the three assignments $PQ'R'$, $PQ'R$ and $PQR'$ all three expressions are true.

## 3.31

| | | | |
|---|---|---|---|
| (a) | (1) | $(a \Rightarrow b)$ | $A$ |
| | (2) | $(a \Rightarrow b')$ | $A$ |
| | (3) | $a$ | $A$ |
| | (4) | $b$ | 1,3,$MPP$ |
| | (5) | $b'$ | 2,3,$MPP$ |
| | (6) | $b \wedge b'$ | 4,5,$\wedge I$ |
| | (7) | $a'$ | 3,6,$RAA$ |

| | | | |
|---|---|---|---|
| (b) | (1) | $a$ | $A$ |
| | (2) | $b$ | $A$ |
| | (3) | $ab$ | 1,2,$\wedge I$ |
| | (4) | $b \Rightarrow ab$ | 2,3,$CP$ |

| | | | |
|---|---|---|---|
| (c) | (1) | $(a \Rightarrow b)(c \Rightarrow b)$ | $A$ |
| | (2) | $(a \Rightarrow b)$ | $\wedge E$ |
| | (3) | $(c \Rightarrow b)$ | $\wedge E$ |
| | (4) | $a \vee c$ | $A$ |
| | (5) | $b$ | 2,3,4,$\vee E$ |
| | (6) | $(a \vee c) \Rightarrow b$ | 4,5,$CP$ |

## 3.32

(a) Let $c = (a \vee b)$ and $d = (a \Rightarrow b')$. Then we have

| $ab$ | $c$ | $d$ | $c \Rightarrow d$ |
|---|---|---|---|
| 00 | 0 | 1 | 1 |
| 01 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 |
| 11 | 1 | 0 | 0 |

(b) Let $c = a'$ and $d = (a \Leftrightarrow b)$.

| $ab$ | $c$ | $d$ | $c \vee d$ |
|---|---|---|---|
| 00 | 1 | 1 | 1 |
| 01 | 1 | 0 | 1 |
| 10 | 0 | 0 | 0 |
| 11 | 0 | 1 | 1 |

(c) Let $d = (b \wedge c')$ and $f = (a \Rightarrow c')$. Then let $e = (a \Rightarrow d)$ to complete the table:

| $abc$ | $d$ | $e$ | $f$ | $e \Leftrightarrow f$ |
|---|---|---|---|---|
| 000 | 0 | 1 | 1 | 1 |
| 001 | 0 | 1 | 1 | 1 |
| 010 | 1 | 1 | 1 | 1 |
| 011 | 0 | 1 | 1 | 1 |
| 100 | 0 | 0 | 1 | 0 |
| 101 | 0 | 0 | 0 | 1 |
| 110 | 1 | 1 | 1 | 1 |
| 111 | 0 | 0 | 0 | 1 |

The answers are given in the right most columns.

## 3.33

(i) $ab \vee a' \vee b' = ab \vee (ab)' = 1$.
true for all $a, b$.

(ii) $ab' \vee a'b$.
not true for all $a, b$.

(iii) $(a \vee b)(a' \vee b)(a \vee b') = (ab \vee ba' \vee b)(a \vee b') = ab \vee ba = ab$.
not true for all $a, b$.

| $ab$ | (i) | (ii) | (iii) |
|---|---|---|---|
| 00 | 1 | 0 | 0 |
| 01 | 1 | 1 | 0 |
| 10 | 1 | 1 | 0 |
| 11 | 1 | 0 | 1 |

Only (i) is a tautology.

## 4.1

a) $$\begin{aligned} x_1 \vee x_1'x_2 &= x_1 \overbrace{(x_2 \vee x_2')}^{=1} \vee x_1'x_2 = x_1x_2 \vee x_1x_2' \vee x_1'x_2 \\ &= x_1x_2(x_3 \vee x_3') \vee x_1x_2'(x_3 \vee x_3') \vee x_1'x_2(x_3 \vee x_3') \\ &= x_1x_2x_3 \vee x_1x_2x_3' \vee x_1x_2'x_3 \vee x_1x_2'x_3' \vee x_1'x_2x_3 \vee x_1'x_2x_3' \\ &= m_{111} \vee m_{110} \vee m_{101} \vee m_{100} \vee m_{011} \vee m_{010} = m_7 \vee m_6 \vee m_5 \vee m_4 \vee m_3 \vee m_2. \end{aligned}$$

b) $$\begin{aligned} x_1x_2' \vee x_1x_2 \vee x_1x_3 &= x_1x_2'(x_3 \vee x_3') \vee x_1x_2(x_3 \vee x_3') \vee x_1(x_2 \vee x_2')x_3 \\ &= x_1x_2'x_3 \vee x_1x_2'x_3' \vee x_1x_2x_3 \vee x_1x_2x_3' \vee x_1x_2x_3 \vee x_1x_2'x_3 \\ &= x_1x_2'x_3 \vee x_1x_2'x_3' \vee x_1x_2x_3 \vee x_1x_2x_3' \\ &= m_{101} \vee m_{100} \vee m_{111} \vee m_{110} = m_5 \vee m_4 \vee m_7 \vee m_6. \end{aligned}$$

c) $$\begin{aligned} (x_1 \vee x_2 \vee x_3)(x_1x_2 \vee x_1'x_3)' &= (x_1 \vee x_2 \vee x_3)((x_1x_2)'(x_1'x_3)') \\ &= (x_1 \vee x_2 \vee x_3)((x_1' \vee x_2')(x_1 \vee x_3')) \\ &= (x_1 \vee x_2 \vee x_3)(x_1'x_3' \vee x_1x_2' \vee x_2'x_3') \\ &= x_1x_2' \vee x_1x_2'x_3' \vee x_1'x_2x_3' \vee x_1x_2'x_3 \\ &= x_1x_2'x_3 \vee x_1x_2'x_3' \vee x_1x_2'x_3' \vee x_1'x_2x_3' \vee x_1x_2'x_3 \\ &= x_1x_2'x_3 \vee x_1x_2'x_3' \vee x_1'x_2x_3' \\ &= m_{101} \vee m_{100} \vee m_{010} = m_5 \vee m_4 \vee m_2. \end{aligned}$$

d) $$\begin{aligned} x_1x_2x_3 \vee (x_1 \vee x_1x_2 \vee x_2x_3 \vee x_1x_3) &= x_1x_2x_3 \vee x_1(x_2 \vee x_2')(x_3 \vee x_3') \vee x_1x_2(x_3 \vee x_3') \\ &\quad \vee (x_1 \vee x_1')x_2x_3 \vee x_1(x_2 \vee x_2')x_3 \\ &= x_1x_2x_3 \vee x_1x_2x_3 \vee x_1x_2x_3' \vee x_1x_2'x_3 \vee x_1x_2'x_3' \vee x_1x_2x_3 \vee x_1x_2x_3' \\ &\quad \vee x_1x_2x_3 \vee x_1'x_2x_3 \vee x_1x_2x_3 \vee x_1x_2'x_3 \\ &= x_1x_2x_3 \vee x_1x_2x_3' \vee x_1x_2'x_3 \vee x_1x_2'x_3' \vee x_1'x_2x_3 \\ &= m_{111} \vee m_{110} \vee m_{101} \vee m_{100} \vee m_{011} = m_7 \vee m_6 \vee m_5 \vee m_4 \vee m_3. \end{aligned}$$

## 4.2

a) $x_1 \vee x_1'x_2 = x_1x_2 \vee x_1x_2' \vee x_1'x_2$.

b) $x_1x_2' \vee x_1x_2 \vee x_1x_3 = x_1(x_2' \vee x_2) \vee x_1x_3 = x_1 \vee x_1x_3 = x_1$

c) The same solution as in 4.1c.

d) The same solution as in 4.1d.

## 4.3

a) Compare to the result in 4.1a).

$$f(\mathbf{x_1})_{CNF} = \bigwedge_{\mathbf{a} \in f^{-1}(0)} M_{\mathbf{a}}(\mathbf{x_1}) = M_1 \wedge M_0 = M_{001} \wedge M_{000} = (x_1 \vee x_2 \vee x_3')(x_1 \vee x_2 \vee x_3).$$

b) Compare to the result in 4.1b).

$$f(\mathbf{x_1})_{CNF} = M_0 \wedge M_1 \wedge M_2 \wedge M_3 = (x_1 \vee x_2 \vee x_3)(x_1 \vee x_2 \vee x_3')(x_1 \vee x_2' \vee x_3)(x_1 \vee x_2' \vee x_3').$$

c) Compare to the result in 4.1c).

$$f(\mathbf{x_1})_{CNF} = M_0 \wedge M_1 \wedge M_3 \wedge M_6 \wedge M_7 = (x_1 \vee x_2 \vee x_3)(x_1 \vee x_2 \vee x_3')(x_1 \vee x_2' \vee x_3')(x_1' \vee x_2' \vee x_3)(x_1' \vee x_2' \vee x_3').$$

d) Compare to the result in 4.1d).

$$f(\mathbf{x_1})_{CNF} = M_0 \wedge M_1 \wedge M_2 = (x_1 \vee x_2 \vee x_3)(x_1 \vee x_2 \vee x_3')(x_1 \vee x_2' \vee x_3).$$

### 4.4

a) $f(x_1) = M_0 = M_{00} = x_1 \vee x_2$.

b) The same solution as in 4.2b.

c) The same solution as in 4.3c.

d) The same solution as in 4.3d.

### 4.5

a) $f(x_1, x_2) = x_1 \vee x_1' x_2 = x_1 \oplus x_1' x_2 \oplus \overbrace{x_1 x_1' x_2}^{=0} = x_1 \oplus (1 \oplus x_1) x_2 = x_1 \oplus x_2 \oplus x_1 x_2$,
or, alternatively,

$$\begin{aligned} f(x_1, x_2) &= x_1 \vee x_1' x_2 = 1 \oplus (x_1 \vee x_1' x_2)' \\ &= 1 \oplus x_1'(x_1' x_2)' \\ &= 1 \oplus (1 \oplus x_1)(1 \oplus (1 \oplus x_1) x_2) \\ &= 1 \oplus 1 \oplus x_2 \oplus x_1 x_2 \oplus x_1 \oplus x_1 x_2 \oplus x_1 x_2 = x_1 \oplus x_2 \oplus x_1 x_2. \end{aligned}$$

b) $f(x_1, x_2, x_3) = x_1$.

c) $f(x_1, x_2, x_3) = x_1 \oplus x_2 \oplus x_2 x_3 \oplus x_1 x_2 x_3$.

d) $f(x_1, x_2, x_3) = x_1 \oplus x_2 x_3 \oplus x_1 x_2 x_3$.

### 4.6

Use deMorgan's laws to get

a) $f(x_1, \ldots, x_n) = \bigvee_{\boldsymbol{a} \in f^{-1}(1)} x_1^{(a_1)} \wedge \cdots \wedge x_n^{(a_n)} = \left( \bigwedge_{\boldsymbol{a} \in f^{-1}(1)} \left( x_1^{(a_1)} \wedge \cdots \wedge x_n^{(a_n)} \right)' \right)'$

b) $f(x_1, \ldots, x_n) = \bigwedge_{\boldsymbol{a} \in f^{-1}(0)} x_1^{(a_1')} \vee \cdots \vee x_n^{(a_n')} = \left( \bigvee_{\boldsymbol{a} \in f^{-1}(0)} \left( x_1^{(a_1')} \vee \cdots \vee x_n^{(a_n')} \right)' \right)'$

### 4.7

| $x_1 x_2 x_3$ | $f$ |
|---|---|
| 0 0 0 | 0 |
| 0 0 1 | 0 |
| 0 1 0 | 0 |
| 0 1 1 | 1 |
| 1 0 0 | 0 |
| 1 0 1 | 1 |
| 1 1 0 | 1 |
| 1 1 1 | 1 |

$$\Rightarrow \quad \begin{aligned} f(x_1, x_2, x_3) &= x_1' x_2 x_3 \vee x_1 x_2' x_3 \vee x_1 x_2 x_3' \vee x_1 x_2 x_3 \\ &= (x_1 \vee x_2 \vee x_3)(x_1 \vee x_2 \vee x_3')(x_1 \vee x_2' \vee x_3)(x_1' \vee x_2 \vee x_3) \end{aligned}$$

## 4.8

(a)

| $x_1 \backslash x_2x_3$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 |

(b)

$$
\begin{aligned}
&\text{DNF}: f = x_1'x_2'x_3' \vee x_1'x_2x_3' \vee x_1'x_2x_3 \vee x_1x_2'x_3 \vee x_1x_2x_3\\
&\text{CNF}: f = (x_1'x_2'x_3 \vee x_1x_2'x_3' \vee x_1x_2x_3')' = (x_1 \vee x_2 \vee x_3')(x_1' \vee x_2 \vee x_3)(x_1' \vee x_2' \vee x_3)\\
&\text{RSE}: \left\{ \begin{array}{rcl}
x_1'x_2'x_3' & = & (1 \oplus x_1)(1 \oplus x_2)(1 \oplus x_3)\\
 & & = 1 \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_1x_2 \oplus x_1x_3 \oplus x_1x_3 \oplus x_1x_2x_3\\
x_1'x_2x_3' & = & x_2 \oplus x_1x_2 \oplus x_2x_3 \oplus x_1x_2x_3\\
x_1'x_2x_3 & = & x_2x_3 \oplus x_1x_2x_3\\
x_1x_2'x_3 & = & x_1x_3 \oplus x_1x_2x_3
\end{array} \right.
\end{aligned}
$$

Hence,

$$f = 1 \oplus x_1 \oplus x_3 \oplus x_2x_3 \oplus x_1x_2x_3$$

## 4.9

$$
\begin{aligned}
f^{-1}(1) &= \{(0000), (0010), (0011), (0101), (0111), (1000)\}\\
&= \{0, 2, 3, 5, 7, 8\}
\end{aligned}
$$

$$
\begin{aligned}
f^{-1}(0) &= \{(0001), (0100), (0110), (1001), (1111)\}\\
&= \{1, 4, 6, 9, 15\}
\end{aligned}
$$

$$
\begin{aligned}
f^{-1}(-) &= \{(1010), (1011), (1100), (1101), (1110)\}\\
&= \{10, 11, 12, 13, 14\}
\end{aligned}
$$

## 4.11

(a)

$$
\begin{array}{rcl}
f(x_1, x_2, x_3) = x_1 \vee x_1'x_2 & = & x_1(x_2 \vee x_2')(x_3 \vee x_3') \vee x_1'x_2(x_3 \vee x_3')\\
 & = & x_1(x_2x_3 \vee x_2x_3' \vee x_2'x_3 \vee x_2'x_3') \vee x_1'x_2x_3 \vee x_1'x_2x_3'\\
 & = & x_1x_2x_3 \vee x_1x_2x_3' \vee x_1x_2'x_3 \vee x_2'x_3' \vee x_1'x_2x_3 \vee x_1'x_2x_3'\\
\Rightarrow & & f^{-1}(1) = \{7, 6, 5, 4, 3, 2\}
\end{array}
$$

All minterms are implicants of three variables. We also have the following seven implicants of two variables:

$$
\begin{array}{rcl}
x_1x_2x_3 \vee x_1x_2x_3' & = & x_1x_2\\
x_1x_2x_3 \vee x_1x_2'x_3 & = & x_1x_3\\
x_1x_2x_3 \vee x_1'x_2x_3 & = & x_2x_3\\
x_1x_2x_3' \vee x_1x_2'x_3' & = & x_1x_3'\\
x_1x_2x_3' \vee x_1'x_2x_3' & = & x_2x_3'\\
x_1x_2'x_3 \vee x_1x_2'x_3' & = & x_1x_2'\\
x_1'x_2x_3 \vee x_1'x_2x_3' & = & x_1'x_2
\end{array}
$$

as well as two implicants of one variable:

$$
\begin{aligned}
x_1x_2 \vee x_1x_2' &= x_1 \\
x_1x_2 \vee x_1'x_2 &= x_2 \\
(x_1x_3 \vee x_1x_3' &= x_1) \\
(x_2x_3 \vee x_2x_3' &= x_2)
\end{aligned}
$$

(b) $f(x_1, x_2, x_3) = x_1x_2' \vee x_1x_2 \vee x_1x_3$
Minterms: $x_1x_2x_3, x_1x_2x_3', x_1x_2'x_3, x_1x_2'x_3'$

$$
\text{Implicants} : \begin{cases} \text{minterms} \\ x_1x_2', x_1x_3, x_1x_2, x_1x_3' \\ x_1 \end{cases}
$$

(c)

$$
\begin{aligned}
f(x_1, x_2, x_3) &= (x_1 \vee x_2 \vee x_3)(x_1x_2 \vee x_1'x_3)' = (x_1 \vee x_2 \vee x_3)(x_1x_2)'(x_1'x_3)' \\
&= (x_1 \vee x_2 \vee x_3)(x_1' \vee x_2')(x_1 \vee x_3') = x_1x_2' \vee x_1x_2'x_3' \vee x_1'x_2x_3' \vee x_1x_2'x_3 \\
&= x_1x_2'x_3' \vee x_1x_2'x_3 \vee x_1'x_2x_3'
\end{aligned}
$$

Minterms: $x_1x_2'x_3', x_1x_2'x_3, x_1'x_2x_3'$
Implicants: minterms, $x_1x_2'$

d) $f(x_1, x_2, x_3) = x_1x_2x_3 \vee (x_1 \vee x_2)(x_1 \vee x_3) = x_1x_2x_3 \vee x_1 \vee x_1x_3 \vee x_1x_2 \vee x_2x_3 = x_1 \vee x_2x_3$
Minterms: $x_1'x_2x_3, x_1x_2'x_3', x_1x_2'x_3, x_1x_2x_3', x_1x_2x_3$
Implicants: minterms, $x_1x_2', x_1x_2, x_1x_3', x_1x_3, x_2x_3, x_1$

## 4.12

The functions are split into two sub-functions each

$$
\begin{aligned}
f_1 &= \underbrace{(x_3' \vee x_2')}_{f_1^{(1)}} \wedge \underbrace{(x_3 \vee x_1'x_2 \vee x_1'x_4)}_{f_1^{(2)}} \\
f_2 &= \underbrace{\big(x_2x_3(x_1 \vee x_4)\big)}_{f_2^{(1)}} \vee \underbrace{\big(x_1'(x_2 \vee x_3 \vee x_4)(x_2 \vee x_3')\big)}_{f_2^{(2)}} \\
f_3 &= \underbrace{(x_1'x_2 \vee x_2x_3' \vee x_1'x_4)}_{f_3^{(1)}} \oplus \underbrace{(x_1x_2 \vee x_1x_4 \vee x_2'x_3x_4)}_{f_3^{(2)}}
\end{aligned}
$$

These functions can easily be visualized in Karnaugh maps:

| $f_1^{(1)}$ ($x_1x_2$ \ $x_3x_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | 1 | 1 | 1 |
| 01 | 1 | 1 | 0 | 0 |
| 11 | 1 | 1 | 0 | 0 |
| 10 | 1 | 1 | 1 | 1 |

| $f_1^{(2)}$ ($x_1x_2$ \ $x_3x_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 1 | 1 |
| 01 | 1 | 1 | 1 | 1 |
| 11 | 0 | 0 | 1 | 1 |
| 10 | 0 | 0 | 1 | 1 |

| $f_1 = f_1^{(1)} \wedge f_1^{(2)}$ ($x_1x_2$ \ $x_3x_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 1 | 1 |
| 01 | 1 | 1 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 1 | 1 |

| $f_2^{(1)}$ ($x_1x_2$ \ $x_3x_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 0 | 0 | 0 |
| 01 | 0 | 0 | 1 | 0 |
| 11 | 0 | 0 | 1 | 1 |
| 10 | 0 | 0 | 0 | 0 |

| $f_2^{(2)}$ ($x_1x_2$ \ $x_3x_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 0 | 0 |
| 01 | 1 | 1 | 1 | 1 |
| 11 | 0 | 0 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 |

| $f_2 = f_2^{(1)} \vee f_2^{(2)}$ ($x_1x_2$ \ $x_3x_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 0 | 0 |
| 01 | 1 | 1 | 1 | 1 |
| 11 | 0 | 0 | 1 | 1 |
| 10 | 0 | 0 | 0 | 0 |

| $f_3^{(1)}$ ($x_1x_2$ \ $x_3x_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 1 | 0 |
| 01 | 1 | 1 | 1 | 1 |
| 11 | 1 | 1 | 0 | 0 |
| 10 | 0 | 0 | 0 | 0 |

| $f_3^{(2)}$ ($x_1x_2$ \ $x_3x_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 0 | 1 | 0 |
| 01 | 0 | 0 | 0 | 0 |
| 11 | 1 | 1 | 1 | 1 |
| 10 | 0 | 1 | 1 | 0 |

| $f_3 = f_3^{(1)} \oplus f_3^{(2)}$ ($x_1x_2$ \ $x_3x_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 0 | 0 |
| 01 | 1 | 1 | 1 | 1 |
| 11 | 0 | 0 | 1 | 1 |
| 10 | 0 | 1 | 1 | 0 |

Then we see that all three expressions realize different functions.

## 4.13

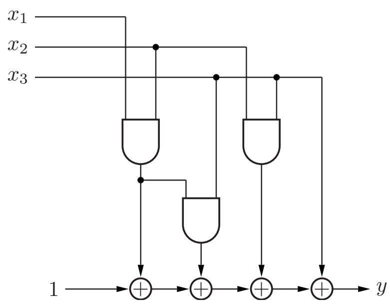
From the figure we get

$$\begin{aligned} y &= x_1'x_3' \vee x_1'x_2 \vee x_2'x_3' \vee x_2x_3 \\ &= x_1'x_3' \vee x_2'x_3' \vee x_2x_3 \end{aligned}$$

where we in the second equality used that $x_1'x_2$ is the conseneus term of $x_1'x_3'$ and $x_2x_3$. Since we should use AND-gates and Mod2-adders it should be rewtten in RSE:

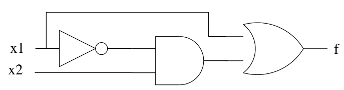
$$\begin{aligned} y &= x_1'x_3' \oplus x_2'x_3' \oplus x_2x_3 \oplus x_1'x_3'(x_2'x_3' \oplus x_2x_3) \\ &= (1 \oplus x_1)(1 \oplus x_3) \oplus (1 \oplus x_2)(1 \oplus x_3) \oplus x_2x_3 \oplus (1 \oplus x_1)(1 \oplus x_2)(1 \oplus x_3) \\ &= 1 \oplus x_3 \oplus x_1x_2 \oplus x_2x_3 \oplus x_1x_2x_3 \end{aligned}$$

Realization:



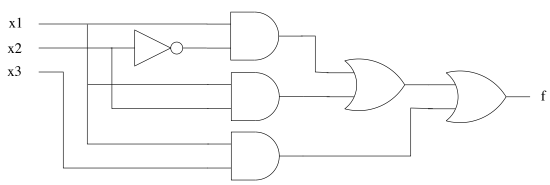

## 4.17

a)



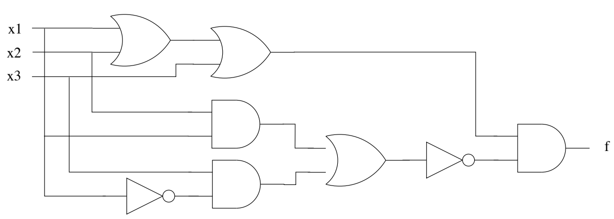

Size = 3
Depth = 3

b)



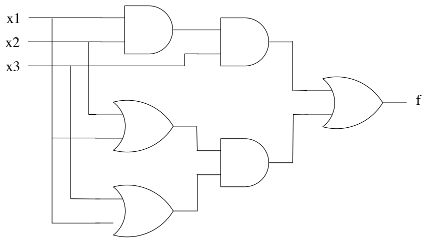

Size = 6
Depth = 4

c)




Size = 8
Depth = 5

d)




Size $= 6$
Depth $= 3$

## 4.18

a) 14 = 01110

14 = 001110

b) -14 = -01110 = 10001+1 = 10010

-14 = -001110 = 110001+1 = 110010

c) -6 = -00110 = 11001+1 = 11010

-6 = -000110 = 111001+1 = 111010

d) -7 = -00111 = 11000+1 = 11001

-7 = -000111 = 111000+1 = 111001

e) -12 = -01100 = 10011+1 = 10100

-12 = -001100 = 110011+1 = 110100

f) -1 = -00001 = 11110+1 = 11111

-1 = -000001 = 111110+1 = 111111

## 4.19

a) Instead of solving the subtraction we solve the addition of the two-complement, i.e., $101011 + 100101 + 1$.

$$\begin{array}{r} {\scriptstyle 1\ \ 1111}1 \\ 101011 \\ +100101 \\ \hline 010001 \end{array}$$

$ov = 1 \oplus 0 = 1$. $(-21 - 26 = -47 \notin [-32, 31]$.)

b) Solve 11101101+10110000+1.

$$\begin{array}{r} {\scriptstyle 111\qquad 1}1 \\ 11101101 \\ +10110000 \\ \hline 10011110 \end{array}$$

$ov = 1 \oplus 1 = 0$. $(-19 - 79 = -98 \in [-128, 127]$.)

b) Solve 11001+11001+1.

$$\begin{array}{r} {\scriptstyle 11\quad 1}1 \\ 11001 \\ +11001 \\ \hline 10011 \end{array}$$

$ov = 1 \oplus 1 = 0$. $(-7 - 6 = -13 \in [-16, 15]$.)

## 4.20

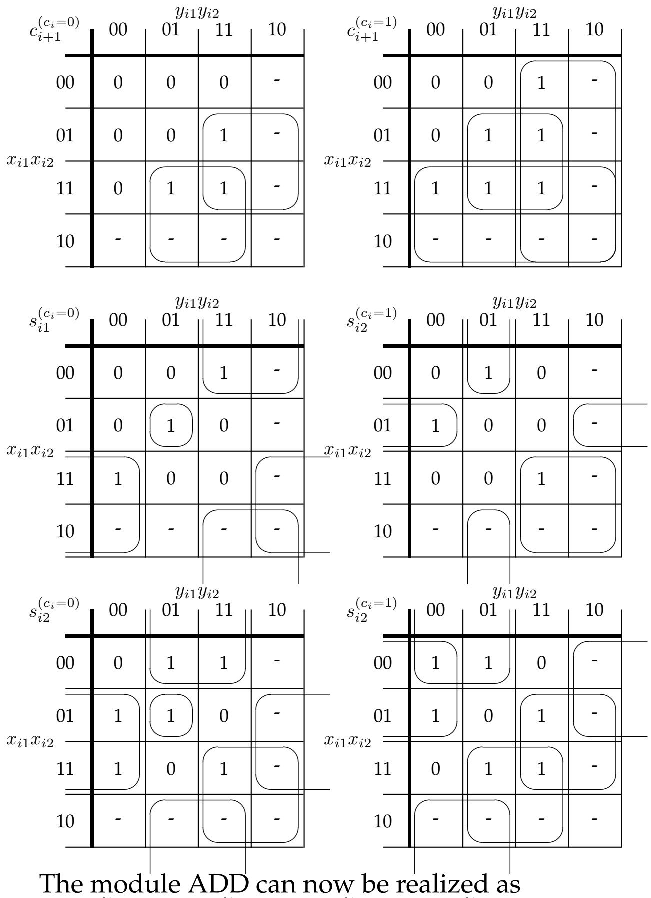
Overflow occurs when the addition of two positive numbers is negative, or when the addition of two negative numbers is positive, i.e., when $(x_{n-1}, y_{n-1}, z_{n-1}) \in \{(0,0,1),(1,1,0)\}$. Therefore $ov = x'_{n-1}y'_{n-1}z_{n-1} \vee x_{n-1}y_{n-1}z'_{n-1}$.

## 4.22

The problem can be solved in the same way as in the binary case, i.e., with one module (Full Adder) for each level. Therefore, we start by constructing a full adder (FA) for the trinary, or ternary, case. We have two input digits and onr carry from the previous level. Since $2+2+1=5=12_3$ the carry is either 0 or 1 and can be represented with only one bit. Below we have lited the functional table that will be realized.

| $(x+y+c=s)_3$ | $x_{i1}x_{i2}$ | $y_{i1}y_{i2}$ | $c_i$ | $c_{i+1}$ | $s_{i1}s_{i2}$ | $(x+y+c=s)_3$ | $x_{i1}x_{i2}$ | $y_{i1}y_{i2}$ | $c_i$ | $c_{i+1}$ | $s_{i1}s_{i2}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $0+0+0=0$ | 0 0 | 0 0 | 0 | 0 | 0 0 | $0+0+1=1$ | 0 0 | 0 0 | 1 | 0 | 0 1 |
| $0+1+0=1$ | 0 0 | 0 1 | 0 | 0 | 0 1 | $0+1+1=2$ | 0 0 | 0 1 | 1 | 0 | 1 1 |
| $0+2+0=2$ | 0 0 | 1 1 | 0 | 0 | 1 1 | $0+2+1=10$ | 0 0 | 1 1 | 1 | 1 | 0 0 |
| $1+0+0=1$ | 0 1 | 0 0 | 0 | 0 | 0 1 | $1+0+1=2$ | 0 1 | 0 0 | 1 | 0 | 1 1 |
| $1+1+0=2$ | 0 1 | 0 1 | 0 | 0 | 1 1 | $1+1+1=10$ | 0 1 | 0 1 | 1 | 1 | 0 0 |
| $1+2+0=10$ | 0 1 | 1 1 | 0 | 1 | 0 0 | $1+2+1=11$ | 0 1 | 1 1 | 1 | 1 | 0 1 |
| $2+0+0=2$ | 1 1 | 0 0 | 0 | 0 | 1 1 | $2+0+1=10$ | 1 1 | 0 0 | 1 | 1 | 0 0 |
| $2+1+0=10$ | 1 1 | 0 1 | 0 | 1 | 0 0 | $2+1+1=11$ | 1 1 | 0 1 | 1 | 1 | 0 1 |
| $2+2+0=11$ | 1 1 | 1 1 | 0 | 1 | 0 1 | $2+2+1=12$ | 1 1 | 1 1 | 1 | 1 | 1 1 |

We have three functions of five variables. Draw the Karnaugh maps to find expressions:

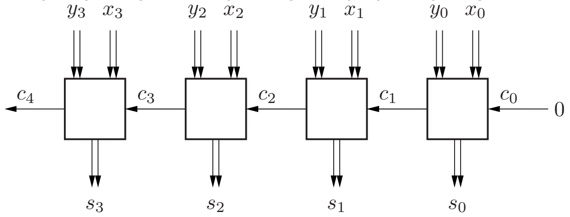


$$c_{i-1} = x_{i2}y_{i1} \vee x_{i1}y_{i2} \vee x_{i2}y_{i2}c_i \vee x_{i1}c_i \vee y_{i1}c_i$$

$$s_{i1} = x_{i1}y'_{i2}c'_i \vee x'_{i2}y_{i1}c'_i \vee x'_{i1}x_{i2}y'_{i1}y_{i2}c'_i \\ \vee x_{i1}y_{i1}c_i \vee x'_{i1}x_{i2}y'_{i2}c_i \vee x'_{i2}y'_{i1}y_{i2}c_i$$

$$s_{i2} = x_{i2}y'_{i2}c'_i \vee x'_{i2}y_{i2}c'_i \vee x_{i1}y_{i1} \vee x'_{i1}x_{i2}y'_{i1}y_{i2}c'_i \\ \vee x'_{i1}y'_{i2}c_i \vee x'_{i2}y'_{i1}c_i \vee x_{i1}y_{i2}c_i \vee x_{i2}y_{i1}c_i$$

The module ADD can now be realized as



## 5.1

(b) Applying the rules $xp \vee x'q = xp \vee x'q \vee pq$ (consensus) and $p \vee px = p$ (absorption) successively yields:

$$f_1() = \underset{A}{x_0'x_1'x_2'} \vee \underset{B}{x_0'x_1'x_2} \vee \underset{C}{x_0'x_1x_2} \vee \underset{D}{x_0x_1'x_2'} \vee \underset{E}{x_0x_1'x_2} \vee \underset{F}{x_0x_1x_2}.$$

Add consensus of A through F, then consensus of B through F, and so on.

$$\begin{aligned} &= \underset{A}{x_0'x_1'x_2'} \vee \underset{B}{x_0'x_1'x_2} \vee \underset{C}{x_0'x_1x_2} \vee \underset{D}{x_0x_1'x_2'} \vee \underset{E}{x_0x_1'x_2} \vee \underset{F}{x_0x_1x_2} \\ &\vee \underset{G=\text{cons}(A,B)}{x_0'x_1'} \vee \underset{H=\text{cons}(A,D)}{x_1'x_2'} \vee \underset{I=\text{cons}(B,C)}{x_0'x_2} \vee \underset{J=\text{cons}(B,E)}{x_1'x_2} \vee \underset{K=\text{cons}(C,F)}{x_1x_2} \vee \underset{L=\text{cons}(D,E)}{x_0x_1'} \vee \underset{M=\text{cons}(E,F)}{x_0x_2}. \end{aligned}$$

After adding the consensus, we now apply the absorption rule. G covers A and B, H covers D, I covers C, J covers E, and K covers F. Add consensus of G through M, then consensus of H through M, and so on.

$$= \underset{G}{x_0'x_1'} \vee \underset{H}{x_1'x_2'} \vee \underset{I}{x_0'x_2} \vee \underset{J}{x_1'x_2} \vee \underset{K}{x_1x_2} \vee \underset{L}{x_0x_1'} \vee \underset{M}{x_0x_2} \vee \underset{N=\text{cons}(G,L)}{x_1'} \vee \underset{O=\text{cons}(I,M)}{x_2}.$$

N covers G, H, J, and L, and O covers I, K, and M (and J). Further applications of the consensus rule do not add any new terms, indicating that the algorithm has terminated. The complete list of prime implicants is

$$= x_1' \vee x_2.$$

One could also have opted for first adding consensus of A and later terms, only, then applying absorption, and iteratively proceeding in this way. Like this:

$$\begin{aligned} f_1() &= \underset{A}{x_0'x_1'x_2'} \vee \underset{B}{x_0'x_1'x_2} \vee \underset{C}{x_0'x_1x_2} \vee \underset{D}{x_0x_1'x_2'} \vee \underset{E}{x_0x_1'x_2} \vee \underset{F}{x_0x_1x_2} \\ &= \underset{A}{x_0'x_1'x_2'} \vee \underset{B}{x_0'x_1'x_2} \vee \underset{C}{x_0'x_1x_2} \vee \underset{D}{x_0x_1'x_2'} \vee \underset{E}{x_0x_1'x_2} \vee \underset{F}{x_0x_1x_2} \vee \underset{G=\text{cons}(A,B)}{x_0'x_1'} \vee \underset{H=\text{cons}(A,D)}{x_1'x_2'} \\ &= \underset{C}{x_0'x_1x_2} \vee \underset{E}{x_0x_1'x_2} \vee \underset{F}{x_0x_1x_2} \vee \underset{G}{x_0'x_1'} \vee \underset{H}{x_1'x_2'} \vee \underset{I=\text{cons}(C,F)}{x_1x_2} \vee \underset{J=\text{cons}(C,G)}{x_0'x_2} \\ &= \underset{E}{x_0x_1'x_2} \vee \underset{G}{x_0'x_1'} \vee \underset{H}{x_1'x_2'} \vee \underset{I}{x_1x_2} \vee \underset{J}{x_0'x_2} \vee \underset{K=\text{cons}(E,G)}{x_1'x_2} \vee \underset{L=\text{cons}(E,H)}{x_0x_1'} \vee \underset{M=\text{cons}(E,I)}{x_0x_2} \\ &= \underset{G}{x_0'x_1'} \vee \underset{H}{x_1'x_2'} \vee \underset{I}{x_1x_2} \vee \underset{J}{x_0'x_2} \vee \underset{K}{x_1'x_2} \vee \underset{L}{x_0x_1'} \vee \underset{M}{x_0x_2} \vee \underset{N=\text{cons}(G,L)}{x_1'} \\ &= \underset{I}{x_1x_2} \vee \underset{J}{x_0'x_2} \vee \underset{M}{x_0x_2} \vee \underset{N}{x_1'} \vee \underset{O=\text{cons}(I,N)}{x_2} \\ &= x_1' \vee x_2. \end{aligned}$$

Thus, we see that iterative consensus can be run in several ways so long as, at each step, the onset $f_1^{-1}(1)$ is preserved. Likewise, one can start from any disjunctive form.

In the conjunctive form we can use the duals $(x \vee p)(x' \vee q) = (x \vee p)(x' \vee q)(p \vee q)$ (consensus) and $p(p \vee x) = p$ (absorption) and obtain:

$$\begin{aligned} f_2() &= \underset{A}{(x_0 \vee x_1 \vee x_2)}\underset{B}{(x_0 \vee x_1' \vee x_2)}\underset{C}{(x_0 \vee x_1' \vee x_2')}\underset{D}{(x_0' \vee x_1 \vee x_2)}\underset{E}{(x_0' \vee x_1' \vee x_2)}\underset{F}{(x_0' \vee x_1' \vee x_2')} \\ &= \underset{A}{(x_0 \vee x_1 \vee x_2)}\underset{B}{(x_0 \vee x_1' \vee x_2)}\underset{C}{(x_0 \vee x_1' \vee x_2')}\underset{D}{(x_0' \vee x_1 \vee x_2)}\underset{E}{(x_0' \vee x_1' \vee x_2)}\underset{F}{(x_0' \vee x_1' \vee x_2')} \\ &\quad \underset{G=\text{cons}(A,B)}{(x_0 \vee x_2)} \underset{H=\text{cons}(A,D)}{(x_1 \vee x_2)} \underset{I=\text{cons}(B,C)}{(x_0 \vee x_1')} \underset{J=\text{cons}(B,E)}{(x_1' \vee x_2)} \underset{K=\text{cons}(C,F)}{(x_1' \vee x_2')} \underset{L=\text{cons}(D,E)}{(x_0' \vee x_2)} \underset{M=\text{cons}(E,F)}{(x_0' \vee x_1')} \\ &= \underset{G}{(x_0 \vee x_2)}\,\underset{H}{(x_1 \vee x_2)}\,\underset{I}{(x_0 \vee x_1')}\,\underset{J}{(x_1' \vee x_2)}\,\underset{K}{(x_1' \vee x_2')}\,\underset{L}{(x_0' \vee x_2)}\,\underset{M}{(x_0' \vee x_1')}\,\underset{N=\text{cons}(G,L)}{(x_2)}\,\underset{O=\text{cons}(I,M)}{(x_1')} \\ &= x_1'x_2. \end{aligned}$$

## 5.2

(b) Let $f_{1D}$ and $f_{2D}$ denote the functions that describe the don't care sets.
First, we minimize $f_1()$ and $f_{1D}()$ separately:

$$\begin{aligned} f_1() &= x_0'x_1'x_2' \vee x_0'x_1x_2' \vee x_0'x_1x_2 = x_0'x_2' \vee x_0'x_1 \\ f_{1D}() &= x_0x_1'x_2 \vee x_0x_1x_2' \vee x_0x_1x_2 = x_0x_2 \vee x_0x_1 \end{aligned}$$

From $f_{1D}$ we only need to take those terms that help to simplify $f_1$. In this case this is only the second term $x_0x_1$. We obtain:

$$f_1() = x_0'x_2' \vee x_0'x_1 \vee x_0x_1 = x_0'x_2' \vee x_1$$

In the same way:

$$\begin{aligned} f_2() &= (x_0' \vee x_1 \vee x_2)(x_0' \vee x_1 \vee x_2')(x_0' \vee x_1' \vee x_2) = (x_0' \vee x_1)(x_0' \vee x_2) \\ f_{2D}() &= (x_0 \vee x_1 \vee x_2)(x_0 \vee x_1' \vee x_2)(x_0 \vee x_1' \vee x_2') = (x_0 \vee x_2)(x_0 \vee x_1') \end{aligned}$$

Here we take only the term $(x_0 \vee x_2)$ from $f_{2D}$ and obtain:

$$f_2() = (x_0' \vee x_1)(x_0' \vee x_2)(x_0 \vee x_2) = (x_0' \vee x_1)x_2$$

## 5.3

(a)

$$f() = \underset{A}{x_1x_2x_3'} \vee \underset{B}{x_1'x_2x_3} \vee \underset{C}{x_1x_2'x_3'} \vee \underset{D}{x_1x_3x_4}$$

Add consensus of $A$ and later terms:

$$= \underset{A}{x_1x_2x_3'} \vee \underset{B}{x_1'x_2x_3} \vee \underset{C}{x_1x_2'x_3'} \vee \underset{D}{x_1x_3x_4} \vee \underset{E=C(A,C)}{x_1x_3'} \vee \underset{F=C(A,D)}{x_1x_2x_4}$$

$E$ covers $A$ and $C$. Add consensus of $B$ and later terms:

$$= \underset{B}{x_1'x_2x_3} \vee \underset{D}{x_1x_3x_4} \vee \underset{E}{x_1x_3'} \vee \underset{F}{x_1x_2x_4} \vee \underset{G=C(B,D)=C(B,F)}{x_2x_3x_4}$$

Add consensus of $D$ and later terms:

$$= \underset{B}{x_1'x_2x_3} \vee \underset{D}{x_1x_3x_4} \vee \underset{E}{x_1x_3'} \vee \underset{F}{x_1x_2x_4} \vee \underset{G}{x_2x_3x_4} \vee \underset{H=C(D,E)}{x_1x_4}$$

$H$ covers $D$ and $F$. Add consensus of $E$ and later terms:

$$= \underset{B}{x_1'x_2x_3} \vee \underset{E}{x_1x_3'} \vee \underset{G}{x_2x_3x_4} \vee \underset{H}{x_1x_4} \vee \underset{I=C(E,G)}{x_1x_2x_4}$$

$I$ is covered by $H$.
No further consensus terms can be built and hence all prime implicants of the function are:

$$x_1'x_2x_3, x_1x_3', x_2x_3x_4, x_1x_4.$$

## 5.4

**Disjunctive form**

| $f_1$ ($x_0$ \ $x_1x_2$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 0 |
| 1 | 1 | 1 | 1 | 0 |

$$f_1(x_0, x_1, x_2) = x_1' \vee x_2$$

| $f_2$ ($x_0$ \ $x_1x_2$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 |

$$f_2(x_0, x_1, x_2) = x_1'x_2$$

**Conjunctive form**

| $f_1$ ($x_0$ \ $x_1x_2$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 0 |
| 1 | 1 | 1 | 1 | 0 |

$$f_1(x_0, x_1, x_2) = (x_1x_2')' = x_1' \vee x_2$$

| $f_2$ ($x_0$ \ $x_1x_2$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 |

$$f_2(x_0, x_1, x_2) = (x_1 \vee x_2')' = x_1'x_2$$

## 5.5

**Disjunctive form**

| $f_1$ \ $x_1x_2$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| $x_0$ = 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | - | - | - |

$f_1(x_0, x_1, x_2) = x_0'x_2' \vee x_1$

| $f_2$ \ $x_1x_2$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| $x_0$ = 0 | - | 1 | - | - |
| 1 | 0 | 0 | 1 | 0 |

$f_2(x_0, x_1, x_2) = x_0' \vee x_1x_2$

**Conjunctive form**

| $f_1$ \ $x_1x_2$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| $x_0$ = 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | - | - | - |

$f_1(x_0, x_1, x_2) = (x_0 \vee x_1'x_2)' = x_0'(x_1 \vee x_2')$

| $f_2$ \ $x_1x_2$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| $x_0$ = 0 | - | 1 | - | - |
| 1 | 0 | 0 | 1 | 0 |

$f_2(x_0, x_1, x_2) = (x_2' \vee x_0x_1')' = x_2(x_0' \vee x_1)$

## 5.6

a)

| $f$: $x_1x_2$ \ $x_3x_4$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | 1 | 1 | 1 |
| 01 | 0 | 1 | 0 | 1 |
| 11 | 0 | 0 | 1 | 0 |
| 10 | 1 | 0 | 0 | 1 |

**Prime implicants:**
$PI = \{x_1'x_2', x_2'x_4', x_1'x_3'x_4, x_1'x_3x_4', x_1x_2x_3x_4\}$
All prime implicants are essential.

**Minimal function:**
$f = x_1'x_2' \vee x_2'x_4' \vee x_1'x_3'x_4 \vee x_1'x_3x_4' \vee x_1x_2x_3x_4$.

b)

| $f$: $x_1x_2$ \ $x_3x_4$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 1 | 1 |
| 01 | 0 | 1 | 1 | 1 |
| 11 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 |

**Prime implicants:**
$PI = \{x_1, x_3, x_4\}$
All prime implicants are essential.

**Minimal function:**
$f = x_1 \vee x_3 \vee x_4$.

c)

| $f$: $x_1x_2$ \ $x_3x_4$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | 1 | 0 | - |
| 01 | 1 | - | 1 | 0 |
| 11 | - | 1 | - | 0 |
| 10 | - | 1 | 0 | 1 |

**Prime implicants:**
$PI = \{x_2x_4, x_2'x_4', x_3'\}$
All prime implicants are essential.

**Minimal function:**
$f = x_2x_4 \vee x_2'x_4' \vee x_3'$.

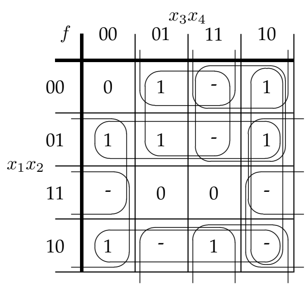
d)

| $f$: $x_1x_2$ \ $x_3x_4$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 0 | 1 |
| 01 | 0 | 1 | 0 | 0 |
| 11 | 1 | 0 | 1 | 0 |
| 10 | 0 | 1 | 1 | 0 |

**Prime implicants:**
$PI = \{\underline{x_1x_2x_3'x_4'}, \underline{x_1'x_2'x_3x_4'}, \underline{x_1'x_3'x_4}, x_2'x_3'x_4, x_1x_2'x_4, \underline{x_1x_3x_4}\}$
The essential prime implicants are underlined.

**Minimal function:**
$f = x_1x_2x_3'x_4' \vee x_1'x_2'x_3x_4' \vee x_1'x_3'x_4 \vee x_1x_3x_4 \vee x_2'x_3'x_4$
or
$f = x_1x_2x_3'x_4' \vee x_1'x_2'x_3x_4' \vee x_1'x_3'x_4 \vee x_1x_3x_4 \vee x_1x_2'x_4$.

e)




**Prime implicants:**
$PI = \{x_1'x_2, x_1x_2', x_2'x_4, x_1'x_4, x_2'x_3, x_1'x_3, x_2x_4', x_1x_4', x_3x_4'\}$
None of the prime implicants are essential.

**Minimal function:**
There are several minimal functions. For example
$f = x_1'x_2 \vee x_1x_2' \vee x_2'x_4 \vee x_2'x_3$.

**5.8**

(a) Minimal disjunctive form:

$$f_{\text{MDF}} = x_0'x_2' \vee x_2'x_3 \vee x_1x_2$$

Minimal conjunctive form:

$$f_{\text{MCF}} = (x_0' \vee x_2 \vee x_3)(x_1 \vee x_2')$$

(b) Minimal disjunctive form:

$$f_{\text{MDF}} = x_0'x_1 \vee x_0'x_2 \vee x_0x_3$$

Minimal conjunctive form:

$$f_{\text{MCF}} = (x_0' \vee x_3)(x_0 \vee x_1 \vee x_2)$$

(c) Minimal disjunctive form:

$$f_{\text{MDF}} = x_0'x_3' \vee x_0'x_1'x_2$$

Minimal conjunctive form:

$$f_{\text{MCF}} = x_0'(x_2 \vee x_3')(x_1' \vee x_3')$$

Notice that the functions are not necessarily unique.

## 5.9

a)

$$1 \wedge - = - \qquad 1 \vee - = 1 \qquad 1 \oplus - = -$$
$$0 \wedge - = 0 \qquad 0 \vee - = - \qquad 0 \oplus - = -$$

b) From part a and the following Karnaugh-maps we see that the function $f_1 \oplus f_2$ is given by $(f_1 \oplus f_2)^{-1}(1) = \{0, 8, 12\}$ and $(f_1 \oplus f_2)^{-1}(0) = \{3, 4, 5, 6, 9, 10\}$.

| $f_1$ ($x_1x_2$ \ $x_3x_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | - | 1 | - |
| 01 | 0 | 1 | 1 | 1 |
| 11 | 0 | - | - | - |
| 10 | 1 | 0 | - | 0 |

| $f_2$ ($x_1x_2$ \ $x_3x_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | - | 1 | 0 |
| 01 | 0 | 1 | - | 1 |
| 11 | 1 | - | - | 0 |
| 10 | 0 | 0 | 1 | 0 |

| $f_1 \oplus f_2$ ($x_1x_2$ \ $x_3x_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | - | 0 | - |
| 01 | 0 | 0 | - | 0 |
| 11 | 1 | - | - | - |
| 10 | 1 | 0 | - | 0 |

Corresponding minimal functions can be found in for example

$$f_1 \oplus f_2 = x_1x_2 \vee x_2'x_3'x_4',$$

or

$$f_1 \oplus f_2 = x_1x_3'x_4' \vee x_2'x_3'x_4'$$

(This is not a complete list. There are other minimal functions.)

## 5.10

a) $PI = \{\underline{x_1'x_2x_3}, \underline{x_1x_2'x_3}, \underline{x_1x_2x_3'}, x_1x_2x_4, x_2x_3x_4, x_1x_3x_4\}$
The essential prime implicants are underlined.

b) $PI = \{x_1x_2x_3', x_2'x_4, x_3'x_4\}$
All prime implicants are essential.

## 5.11

a) Yes. The four minterms and $x_1'x_2q_2$, $x_1'x_2q_1$, and $x_2q_1q_2'$.

b) Yes.

c) Yes, since

$$q_2^+ = x_1'x_2q_2 \vee x_2q_1q_2' = x_1'x_2q_2 \oplus x_2q_1q_2' \oplus \underbrace{x_1'x_2q_1q_2q_2'}_{=0}$$
$$= (1 \oplus x_1)x_2q_2 \oplus x_2q_1(1 \oplus q_2) = x_2q_2 \oplus x_1x_2q_2 \oplus x_2q_1 \oplus x_2q_1q_2$$

d) Yes. Consider the implicants

$$A = x_1'q_2'$$
$$B = x_2'q_2'$$
$$C = x_2q_1q_2'$$
$$D = x_1'x_2q_1$$
$$E = x_1'x_2q_2$$

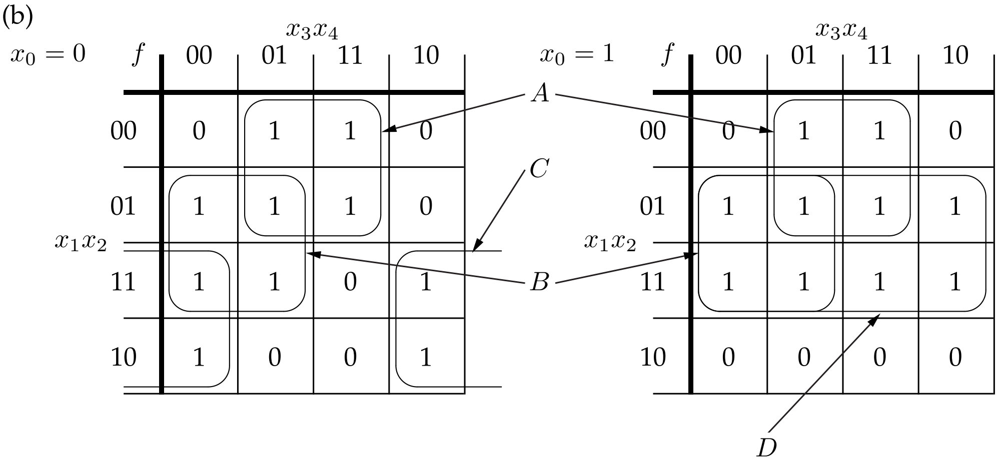
Then

$$q_1^+ = A \vee B \vee C \vee D$$
$$q_2^+ = C \vee E$$

Notice, that $C$ is not a prime implicant in $q_1^+$.

## 5.13

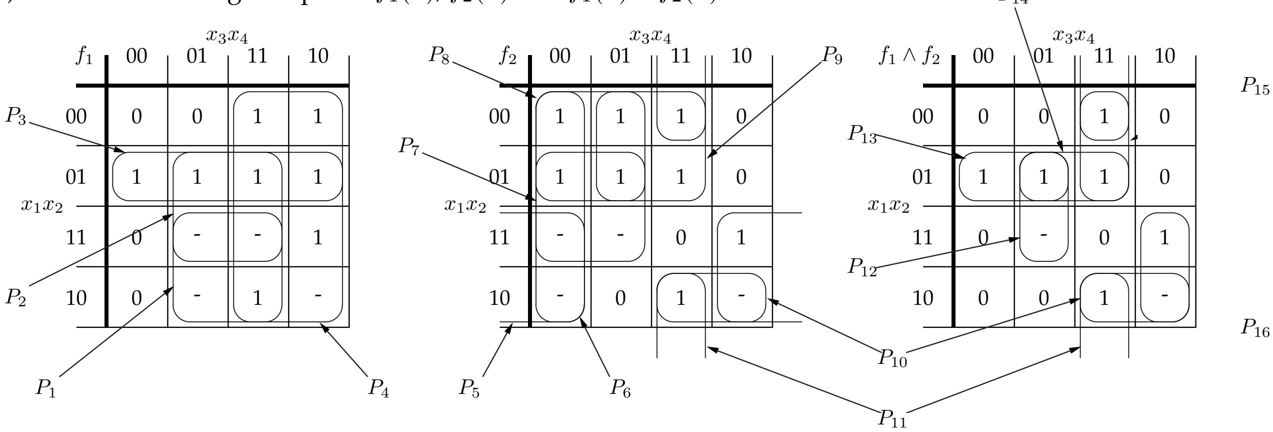
(b)




$$f(x_1, x_2, x_3, x_4, x_5) = A \vee B \vee C \vee D = x_1' x_4 \vee x_2 x_3' \vee x_0' x_1 x_4' \vee x_0 x_2$$

## 5.14

(e) Draw the Karnaugh maps for $f_1(x)$, $f_2(x)$ and $f_1(x) \wedge f_2(x)$




The P-table:

| | $f_1$ | | | | | | | | $f_2$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 | 7 | 11 | 14 | 0 | 1 | 3 | 4 | 5 | 7 | 11 | 14 |
| $P_1$ | | | | | | | x | | | | | | | | | |
| $P_2$ | | | | x | | x | | | | | | | | | | |
| $P_3$ | | | x | x | x | x | | | | | | | | | | |
| $P_4$ | x | x | | | x | x | x | x | | | | | | | | |
| $P_5$ | | | | | | | | | | | | | | | | x |
| $P_6$ | | | | | | | | | x | | | x | | | | |
| $P_7$ | | | | | | | | | | | | x | x | | | |
| $P_8$ | | | | | | | | | x | x | | x | x | | | |
| $P_9$ | | | | | | | | | | x | x | | x | x | | |
| $P_{10}$ | | | | | | | x | | | | | | | | x | |
| $P_{11}$ | | x | | | | | x | | | | x | | | | x | |
| $P_{12}$ | | | | x | | | | | | | | | x | | | |
| $P_{13}$ | | | x | x | | | | | | | | x | x | | | |
| $P_{14}$ | | | | x | | x | | | | | | | x | x | | |
| $P_{15}$ | | x | | | | x | | | | | x | | | x | | |
| $P_{16}$ | | | | | | | | x | | | | | | | | x |

Here we see that $P_4$ is essential in $f_1$. A reduced table becomes

| | $f_1$ | | $f_2$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 4 | 5 | 0 | 1 | 3 | 4 | 5 | 7 | 11 | 14 |
| $P_{13} \supset P_2$ | | x | | | | | | | | |
| $P_{13} \supset P_3$ | x | x | | | | | | | | |
| $P_5$ | | | | | | | | | | x |
| $P_8 \supset P_6$ | | | x | | | x | | | | |
| $P_{13} \supset P_7$ | | | | | | x | x | | | |
| $P_8$ | | | x | x | | x | x | | | |
| $P_9$ | | | | x | x | | x | x | | |
| $P_{11} \supset P_{10}$ | | | | | | | | | x | |
| $P_{11}$ | | | | | x | | | | x | |
| $P_{13} \supset P_{12}$ | | x | | | | | x | | | |
| $P_{13}$ | x | x | | | | x | x | | | |
| $P_{14}$ | | x | | | | | x | x | | |
| $P_9 \supset P_{15}$ | | | | | x | | | x | | |
| $P_5 \supset P_{16}$ | | | | | | | | | | x |

$P_5$ dominates $P_{16}$. $P_{13}$ dominates $P_2$, $P_3$, $P_7$ and $P_{12}$. $P_8$ dominates $P_6$. $P_{11}$ dominates $P_{10}$. $P_9$ dominates $P_{15}$. Hence,

| | $f_1$ | | $f_2$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 4 | 5 | 0 | 1 | 3 | 4 | 5 | 7 | 11 | 14 |
| $P_5$ | | | | | | | | | | x |
| $P_8$ | | | x | x | | x | x | | | |
| $P_9$ | | | | x | x | | x | x | | |
| $P_{11}$ | | | | | x | | | | x | |
| $P_{13}$ | x | x | | | | x | x | | | |
| $P_{14}$ | | x | | | | | x | x | | |

$P_{13}$ is essential in $f_1$. Since, this implicant now is for free we can also use it in $f_2$ to cover 4 and 5. The implicants $P_5$, $P_8$, and $P_{11}$ are essential in $f_2$. $P_8$ also covers 4 and 5 so we can actually choose if we like to use $P_{13}$ in the minimal cover of $f_2$ or not. What is left of the table is now

| | $f_2$ |
|---|---|
| | 7 |
| $P_9$ | x |
| $P_{14}$ | x |

We can choose if we like $P_9$ or $P_{14}$ to cover 7. Hence, a minmal cover of $f_1$ and $f_2$ is

$$f_1 = P_4 \vee P_{13} = x_3 \vee x_1' x_2 x_3'$$
$$f_2 = P_5 \vee P_8 \vee P_9 \vee P_{11} = x_1 x_4' \vee x_1' x_3' \vee x_1' x_4 \vee x_2' x_3 x_4$$

Notice that $P_{13}$ can be added to the cover of $f_2$ and it will still be a minimal cover of the functions.

## 5.15

Let us first consider the function $f_1(\quad)$. Then we have

$$f_1^{-1}(1) = \{000, 001, 011, 100, 101, 111\}$$

and

$$f_1^{-1}(0) = \{010, 110\}\,.$$

In the table below we calculate for all minterms their blocking functions, span functions, and core-span functions.

| $i$ | $m_i$ | $B^{m_i}(\boldsymbol{x})$ | $(B^{m_i}(\boldsymbol{x}))'$ | $cs^{m_i}(\boldsymbol{x})$ |
|---|---|---|---|---|
| 0 | $x_0'x_1'x_2'$ | $x_1 \vee x_0x_1 = x_1$ | $x_1'$ | $x_1'$ |
| 1 | $x_0'x_1'x_2$ | $x_1x_2' \vee x_0x_1x_2' = x_1x_2'$ | $(x_1x_2')' = x_1' \vee x_2$ | $x_1'x_2$ |
| 3 | $x_0'x_1x_2$ | $x_2' \vee x_0x_2' = x_2'$ | $x_2$ | $x_2$ |
| 4 | $x_0x_1'x_2'$ | $x_0'x_1 \vee x_1 = x_1$ | $x_1'$ | $x_1'$ |
| 5 | $x_0x_1'x_2$ | $x_0'x_1x_2' \vee x_1x_2'$ | $x_1' \vee x_2$ | $x_1'x_2$ |
| 7 | $x_0x_1x_2$ | $x_0'x_2' \vee x_2' = x_2'$ | $x_2$ | $x_2$ |

We now have to check which of the core-span functions are essential signature cubes.

1. We first check if $x_1'$ is essential. We use Def. 5.7 (page 187 in the *Signature cubes* hand-out). Since $x_1' = cs^{m_0}(\boldsymbol{x})$ and there exists no minterm $m_i$ for which $x_1' < cs^{m_i}(\boldsymbol{x})$, it follows that $x_1'$ is essential.
2. We now check if $x_1'x_2$ is essential. Again, we use Def. 5.7 and conclude that $x_1'x_2$ is *not* essential since $x_1'x_2 < cs^{m_0}(\boldsymbol{x}) = x_1'$.
3. We check if $x_2$ is essential. Clearly, $x_2 = cs^{m_3}(\boldsymbol{x})$ and there exists no minterm $m_i$ for which $x_2 < cs^{m_i}(\boldsymbol{x})$. Thus, from Def. 5.7 it follows that $x_2$ is essential.

There are no other distinct core-span functions. Thus, $x_1'$ and $x_2$ are the only two essential signature cubes. Since they coincide with the prime implicants in their span functions, the prime-table is trivial and can be omitted. We conclude that

$$f_1(\boldsymbol{x}) = x_1' \vee x_2 \,.$$

Let us consider now the second function

$$f_2^{-1}(1) = \{001, 101\}$$
$$f_2^{-1}(0) = \{000, 010, 011, 100, 110, 111\} \,.$$

In the table below we calculate for all minterms their blocking functions, span functions, and core-span functions.

| $i$ | $m_i$ | $B^{m_i}(\boldsymbol{x})$ | $(B^{m_i}(\boldsymbol{x}))'$ | $cs^{m_i}(\boldsymbol{x})$ |
|---|---|---|---|---|
| 1 | $x_0'x_1'x_2$ | $x_2' \vee x_1x_2' \vee x_1 \vee x_0x_2' \vee x_0x_1x_2' \vee x_0x_1$ $= x_1 \vee x_2'$ | $(x_1 \vee x_2')' = x_1'x_2$ | $x_1'x_2$ |
| 5 | $x_0x_1'x_2$ | $x_0'x_2' \vee x_0'x_1x_2' \vee x_0'x_1 \vee x_2' \vee x_1x_2' \vee x_1$ $= x_1 \vee x_2'$ | $x_1'x_2$ | $x_1'x_2$ |

If follows directly from Def. 5.7 that $x_1'x_2$ is an essential signature cube. Thus,

$$f_2(\boldsymbol{x}) = x_1'x_2 \,.$$

## 5.19

To minimize the size of the AND-matrix and the OR-matrix is the same as minimizing the number of minterms. Let $(z_1z_0)_2 = (x_1x_0)_2 \oplus_4 (y_1y_0)_2$. Then

| $x_1x_0$ | $y_1y_0$ | $z_1z_0$ |
|---|---|---|
| 00 | 00 | 00 |
| 00 | 01 | 01 |
| 00 | 10 | 10 |
| 00 | 11 | 11 |
| 01 | 00 | 01 |
| 01 | 01 | 10 |
| 01 | 10 | 11 |
| 01 | 11 | 00 |
| 10 | 00 | 10 |
| 10 | 01 | 11 |
| 10 | 10 | 00 |
| 10 | 11 | 01 |
| 11 | 00 | 11 |
| 11 | 01 | 00 |
| 11 | 10 | 01 |
| 11 | 11 | 10 |

$z_1$:

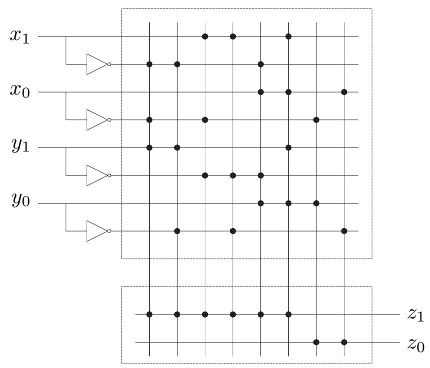
| $x_1x_0$ \ $y_1y_0$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 0 | 1 | 1 |
| 01 | 0 | 1 | 0 | 1 |
| 11 | 1 | 0 | 1 | 0 |
| 10 | 1 | 1 | 0 | 0 |

$z_0$:

| $x_1x_0$ \ $y_1y_0$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 1 | 0 |
| 01 | 1 | 0 | 0 | 1 |
| 11 | 1 | 0 | 0 | 1 |
| 10 | 0 | 1 | 1 | 0 |

$$z_1 = x_1'x_0'y_1 \vee x_1'y_1y_0' \vee x_1x_0'y_1' \vee x_1y_1'y_0' \vee x_1'x_0y_1'y_0 \vee x_1x_0y_1y_0$$
$$z_0 = x_0'y_0 \vee x_0y_0'$$

An implementation with PLA:




## 5.21

We start to write a table for all the functions

| $x_1x_2x_3$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_b = f_1 \oplus f_2$ | $f_e = (f_4 \wedge f_5) \oplus (f_5 \wedge f_6)$ |
|---|---|---|---|---|---|---|---|---|
| 0 0 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| 0 0 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 |
| 0 1 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 1 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| 1 0 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| 1 0 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
| 1 1 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| 1 1 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 |

a) Yes, $f_1$ is linear.

$$\begin{aligned} f_1 &= x_1 \oplus x_2 \oplus x_2'x_3 \oplus x_1' \oplus x_2x_3 \oplus 1 \\ &= x_1 \oplus x_2 \oplus (1 \oplus x_2)x_3 \oplus (1 \oplus x_1) \oplus x_2x_3 \oplus 1 \\ &= x_2 \oplus x_3 \end{aligned}$$

b) The minimal forms are minimal disjunctive form, minimal conjunctive form, and RSE. Here we choose the disjunctive form. Write the function $f_b$ above in a Karnaugh map,

| $f_b$ \ $x_1$ / $x_2x_3$ | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 |

and we get $f_1 \oplus f_2 = x_1'x_2' \vee x_3$.

c) $f_3$ is given in conjunctive normal form and we see that $f_3^{-1}(0) = \{001, 111\}$. In DNF we get

$$\begin{aligned} f_3 &= m_{000} \vee m_{010} \vee m_{011} \vee m_{100} \vee m_{101} \vee m_{110} \\ &= x_1'x_2'x_3' \vee x_1'x_2x_3' \vee x_1'x_2x_3 \vee x_1x_2'x_3' \vee x_1x_2'x_3 \vee x_1x_2x_3' \end{aligned}$$

d)

$$\begin{aligned} f_4 &= m_{001} \vee m_{100} \vee m_{101} \vee m_{111} = m_{001} \oplus m_{100} \oplus m_{101} \oplus m_{111} \\ &= x_1'x_2'x_3 \oplus x_1x_2'x_3' \oplus x_1x_2'x_3 \oplus x_1x_2x_3 \\ &= (1 \oplus x_1)(1 \oplus x_2)x_3 \oplus x_1(1 \oplus x_2)(1 \oplus x_3) \oplus x_1(1 \oplus x_2)x_3 \oplus x_1x_2x_3 \\ &= x_1 \oplus x_3 \oplus x_1x_2 \oplus x_1x_3 \oplus x_2x_3. \end{aligned}$$

e) From the table above we get the following Karnaugh map

| $f_e$ | $x_2x_3$ = 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| $x_1$ = 0 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 |

Hence, $f_e = x_3(x_1' \vee x_2')$.

## 5.23

For all combinations of inputs $x_1$, $x_2$, $x_3$ we first calculate values $f_1(x_1, x_2, x_3)$, $f_2(x_1, x_2, x_3)$, and $f_3(x_1, x_2, x_3)$. Then we calculate $f(x_1, x_2, x_3) = g(f_1, f_2, f_3)$. For example, input $x_1x_2x_3 = 010$ gives $f_1f_2f_3 = 011$, and therefore $f(010) = g(011) = 1$

| $x_1x_2x_3$ | $f_1f_2f_3$ | $g$ | $f$ |
|---|---|---|---|
| 0 0 0 | 0 - 1 | - | 1 |
| 0 0 1 | - 1 0 | 1 | 0 |
| 0 1 0 | 0 1 1 | 0 | 1 |
| 0 1 1 | 0 1 0 | 1 | 0 |
| 1 0 0 | 1 1 1 | 1 | - |
| 1 0 1 | - 1 0 | 1 | 0 |
| 1 1 0 | 1 0 1 | 0 | 1 |
| 1 1 1 | 1 0 - | - | 1 |

Now we have a choice. Either we realize the function $f$ which is 1 for the input combination $1, 0, 0$ or we realize the function $f$ which is 0 for $1, 0, 0$. Obviously we have to choose the one which gives minimum number of AND gates in the realization.

The first variant is

$$f^{(1)} = x_1'x_2'x_3' \vee x_1'x_2x_3' \vee x_1x_2'x_3' \vee x_1x_2x_3' \vee x_1x_2x_3 = 1 \oplus x_3 \oplus x_1x_2x_3.$$

The second variant is

$$f^{(2)} = x_1'x_2'x_3' \vee x_1'x_2x_3' \vee x_1x_2x_3' \vee x_1x_2x_3 = 1 \oplus x_1 \oplus x_3 \oplus x_1x_2 \oplus x_1x_3.$$

We choose $f = f^{(1)} = 1 \oplus x_3 \oplus x_1x_2x_3$ for realization.

## 5.24

This problem can be solved by a graph with 13 sates where the state transitions follows

$$s_0 \to s_1 \to s_2 \to \cdots \to s_{12} \to s_0 \to \cdots$$

If the length of the shift register is $L$ each state is a part of the sequence starting at that position. Since the period is 13 long that is also the maximum number of state variables needed. That gives the following table

| $s$ | Period |
|---|---|
| $s_0$ | 1101010011011 |
| $s_1$ | 1010100110111 |
| $s_2$ | 0101001101111 |
| $s_3$ | 1010011011110 |
| $s_4$ | 0100110111101 |
| $s_5$ | 1001101111010 |
| $s_6$ | 0011011110101 |
| $s_7$ | 0110111101010 |
| $s_8$ | 1101111010100 |
| $s_9$ | 1011110101001 |
| $s_{10}$ | 0111101010011 |
| $s_{11}$ | 1111010100110 |
| $s_{12}$ | 1110101001101 |

Now, we try to use as few $D$-elements as possible, i.e., we start from left and try to use as little as possible from the sequence. We need at least four bits, but if we only use those we get $s_0 = s_8$ and $s_1 = s_3$. With five bits we get a unique state assignment as follows

| $s$ | $q_0q_1q_2q_3q_4$ | $u$ |
|---|---|---|
| $s_0$ | 1 1 0 1 0 | 1 |
| $s_1$ | 1 0 1 0 1 | 0 |
| $s_2$ | 0 1 0 1 0 | 0 |
| $s_3$ | 1 0 1 0 0 | 1 |
| $s_4$ | 0 1 0 0 1 | 1 |
| $s_5$ | 1 0 0 1 1 | 0 |
| $s_6$ | 0 0 1 1 0 | 1 |
| $s_7$ | 0 1 1 0 1 | 1 |
| $s_8$ | 1 1 0 1 1 | 1 |
| $s_9$ | 1 0 1 1 1 | 1 |
| $s_{10}$ | 0 1 1 1 1 | 0 |
| $s_{11}$ | 1 1 1 1 0 | 1 |
| $s_{12}$ | 1 1 1 0 1 | 0 |

where the output is the input for the shift register, i.e., the last bit of the next state.

The prime implicants can be found by the Karnaugh map

| $f_{q_0=0}$ ($q_1q_2$ \ $q_3q_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | - | - | - | - |
| 01 | - | - | - | 1 |
| 11 | - | 1 | 0 | - |
| 10 | - | 1 | - | 0 |

| $f_{q_0=1}$ ($q_1q_2$ \ $q_3q_4$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | - | - | 0 | - |
| 01 | 1 | 0 | 1 | - |
| 11 | - | 0 | - | 1 |
| 10 | - | - | 1 | 1 |

The Prime table below help us to find a minimal solution

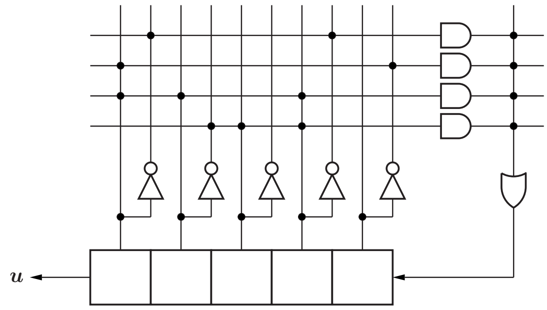
| | 6 | 9 | 13 | 20 | 23 | 26 | 27 | 30 |
|---|---|---|---|---|---|---|---|---|
| $A = q_0'q_3'$ | | x | x | | | | | |
| $B = q_0'q_1'$ | x | | | | | | | |
| $C = q_0q_4'$ | | | | x | | x | | x |
| $D = q_0q_2q_3$ | | | | | x | | | x |
| $E = q_0q_1q_3$ | | | | | | x | x | x |
| $F = q_0q_1q_2'$ | | | | | | x | x | |
| $G = q_1'q_2q_3$ | x | | | | x | | | |
| $H = q_1q_2'q_4$ | | | x | | | | x | |

We directly see that $A$ and $C$ are essential. Implicant $B$ is dominated by $G$ and should be removed. Implicant $F$ is dominated by $E$ and should be removed. We have the table

| | 6 | 23 | 27 |
|---|---|---|---|
| $D = q_0q_2q_3$ | | x | |
| $E = q_0q_1q_3$ | | | x |
| $G = q_1'q_2q_3$ | x | x | |
| $H = q_1q_2'q_4$ | | | x |

Now, $G$ is (secondary) essential. To cover minterm 27 we can use either $E$ or $H$. Hence, a minimal function is

$$u = A \vee C \vee E \vee G = q_0'q_3' \vee q_0q_4' \vee q_0q_1q_3 \vee q_1'q_2q_3$$


u

## 6.1

(a)

$$\begin{array}{lcl} P1 & : & \{s_0\}, \{s_1, s_5\}, \{s_2\}, \{s_3, s_4\} \\ P2 & : & \{s_0\}, \{s_1, s_5\}, \{s_2\}, \{s_3, s_4\} \quad = \quad P1 \end{array}$$

Table for the reduced machine:

| | $i_0$ | $i_1$ |
|---|---|---|
| $s_0$ | $s_2/z_1$ | $s_{34}/z_1$ |
| $s_{15}$ | $s_{15}/z_0$ | $s_2/z_1$ |
| $s_2$ | $s_2/z_1$ | $s_0/z_0$ |
| $s_{34}$ | $s_{34}/z_0$ | $s_2/z_0$ |

With the coding $i_0 = 0$, $i_1 = 1$, $z_0 = 0$, $z_1 = 1$ and

| | $s_0$ | $s_{15}$ | $s_2$ | $s_{34}$ |
|---|---|---|---|---|
| $q_0q_1$ | 00 | 01 | 10 | 11 |

we get the realization

$$\begin{aligned} q_0^+ &= q_1'i' \vee q_0'i \vee q_0q_1 \\ q_1^+ &= q_0'q_1'i \vee q_1i' \\ z &= q_0'i \vee q_1'i' \end{aligned}$$

(b)

$$P1 \ : \ \{s_0\}, \{s_1\}, \{s_2\}, \{s_3\}$$

Since the states are maximally partitioned already by the output signal, the machine is minimal. With the natural binary encoding of signals and states we get

$$\begin{aligned} q_0^+ &= q_1i \vee q_0q_1'i' \\ q_1^+ &= q_0'i \vee q_1i' \vee q_1'i \\ z &= q_0'q_1'i \vee q_1i' \vee q_0q_1 \end{aligned}$$

(c)

$$\begin{array}{lcl} P1 & : & \{s_0, s_4, s_6\}, \{s_1, s_5\}, \{s_2\}, \{s_3\} \\ P2 & : & \{s_0, s_4\}, \{s_1, s_5\}, \{s_2\}, \{s_3\}, \{s_6\} \\ P3 & : & \{s_0\}, \{s_1, s_5\}, \{s_2\}, \{s_3\}, \{s_4\}, \{s_6\} \\ P4 & : & \{s_0\}, \{s_1, s_5\}, \{s_2\}, \{s_3\}, \{s_4\}, \{s_6\} \quad = \quad P3 \end{array}$$

With the coding $i_0 = 0$, $i_1 = 1$, $z_0 = 0$, $z_1 = 1$ and

| | $s_0$ | $s_{15}$ | $s_2$ | $s_3$ | $s_4$ | $s_6$ |
|---|---|---|---|---|---|---|
| $q_0q_1q_2$ | 000 | 001 | 010 | 011 | 100 | 101 |

We get the realization

$$\begin{aligned} q_0^+ &= q_0q_2' \vee q_1i \vee q_1q_2 \\ q_1^+ &= q_0'q_1'q_2i \vee q_1q_2'i' \vee q_0q_2i' \\ q_2^+ &= q_0'q_1'q_2 \vee q_1i' \vee q_0i \\ z &= q_1'q_2'i \vee q_1i' \vee q_1q_2 \vee q_0i \end{aligned}$$

## 6.2

(a) It follows from the figure that

$$\begin{aligned} q_{11}^{+} &= x_1 \\ q_{12}^{+} &= q_{11} \\ q_{21}^{+} &= x_2 \\ u &= q_{11} \oplus q_{12} \oplus x_1 \oplus q_{21} \oplus x_2 \oplus x_3. \end{aligned}$$

Let us define

$$v = q_{11} \oplus q_{12} \oplus x_1 \oplus q_{21} \oplus x_2.$$

Then, the output $u$ can be written as

$$u = v \oplus x_3.$$

The problem of finding a minimal realization of $u$ can now be restated as minimizing a realization of $v$.

We use an octal notation for the state 3-tuple $(q_{11}q_{12}q_{21})$, i.e., $S_0 \equiv (000)$, $S_1 \equiv (001)$, $S_2 \equiv (010)$, etc.,. and obtain the following table for *next state/$v$*:

| Present state | Input 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| $S_0$ | $S_0/0$ | $S_1/1$ | $S_4/1$ | $S_5/0$ |
| $S_1$ | $S_0/1$ | $S_1/0$ | $S_4/0$ | $S_5/1$ |
| $S_2$ | $S_0/1$ | $S_1/0$ | $S_4/0$ | $S_5/1$ |
| $S_3$ | $S_0/0$ | $S_1/1$ | $S_4/1$ | $S_5/0$ |
| $S_4$ | $S_2/1$ | $S_3/0$ | $S_6/0$ | $S_7/1$ |
| $S_5$ | $S_2/0$ | $S_3/1$ | $S_6/1$ | $S_7/0$ |
| $S_6$ | $S_2/0$ | $S_3/1$ | $S_6/1$ | $S_7/0$ |
| $S_7$ | $S_2/1$ | $S_3/0$ | $S_6/0$ | $S_7/1$ |

From the table given above we get

$P1 : \{S_0, S_3, S_5, S_6\}, \{S_1, S_2, S_4, S_7\}$

and

$P2 : \{S_0, S_3\}, \{S_1, S_2\}, \{S_4, S_7\}, \{S_5, S_6\}.$

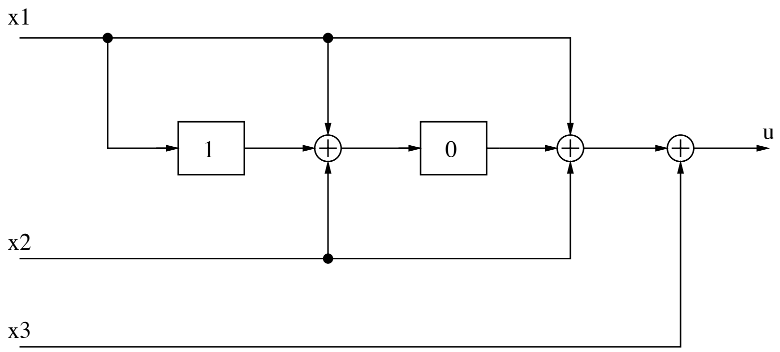
Since $P2 = P3$ the minimal realization has four states, which we can represent by $q_1q_2$. We now obtain the following (minimized) table for *next state* $q_1^+q_2^+/v$:

| Present state $(q_0q_1)$ | Input $x_1x_2$ 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| $\{S_0, S_3\} \equiv 00$ | 00/0 | 01/1 | 11/1 | 10/0 |
| $\{S_1, S_2\} \equiv 01$ | 00/1 | 01/0 | 11/0 | 10/1 |
| $\{S_4, S_7\} \equiv 11$ | 01/1 | 00/0 | 10/0 | 11/1 |
| $\{S_5, S_6\} \equiv 10$ | 01/0 | 00/1 | 10/1 | 11/0 |

Using Karnaugh maps we obtain

$$\begin{aligned} q_0^{+} &= x1 \\ q_1^{+} &= q_0x_1'x_2' \vee q_0'x_1'x_2 \vee q_0'x_1x_2' \vee q_0x_1x_2 \\ &= q_0 \oplus x_1 \oplus x_2 \\ v &= q_1x_1'x_2' \vee q_1'x_1'x_2 \vee q_1'x_1x_2' \vee q_1x_1x_2 \\ &= q_1 \oplus x_1 \oplus x_2 \end{aligned}$$

The minimal realization is




(b)

| $s$ | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0/0 | 0/1 | 2/1 | 2/0 | 8/1 | 8/0 | A/0 | A/1 |
| 1 | 0/1 | 0/0 | 2/0 | 2/1 | 8/0 | 8/1 | A/1 | A/0 |
| 2 | 1/0 | 1/1 | 3/1 | 3/0 | 9/1 | 9/0 | B/0 | B/1 |
| 3 | 1/1 | 1/0 | 3/0 | 3/1 | 9/0 | 9/1 | B/1 | B/0 |
| 4 | 0/1 | 0/0 | 2/0 | 2/1 | 8/0 | 8/1 | A/1 | A/0 |
| 5 | 0/0 | 0/1 | 2/1 | 2/0 | 8/1 | 8/0 | A/0 | A/1 |
| 6 | 1/1 | 1/0 | 3/0 | 3/1 | 9/0 | 9/1 | B/1 | B/0 |
| 7 | 1/0 | 1/1 | 3/1 | 3/0 | 9/1 | 9/0 | B/0 | B/1 |
| 8 | 4/1 | 4/0 | 6/0 | 6/1 | C/0 | C/1 | E/1 | E/0 |
| 9 | 4/0 | 4/1 | 6/1 | 6/0 | C/1 | C/0 | E/0 | E/1 |
| A | 5/1 | 5/0 | 7/0 | 7/1 | D/0 | D/1 | F/1 | F/0 |
| B | 5/0 | 5/1 | 7/1 | 7/0 | D/1 | D/0 | F/0 | F/1 |
| C | 4/0 | 4/1 | 6/1 | 6/0 | C/1 | C/0 | E/0 | E/1 |
| D | 4/1 | 4/0 | 6/0 | 6/1 | C/0 | C/1 | E/1 | E/0 |
| E | 5/0 | 5/1 | 7/1 | 7/0 | D/1 | D/0 | F/0 | F/1 |
| F | 5/1 | 5/0 | 7/0 | 7/1 | D/0 | D/1 | F/1 | F/0 |

(Column heads are values of $x_1x_2x_3$.)

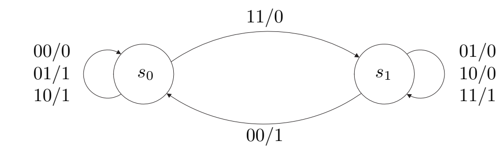
$$\begin{aligned}
P1 &: \{0,2,5,7,9,B,C,E\},\{1,3,4,6,8,A,D,F\}\\
P2 &: \{0,5,B,E\},\{2,7,9,C\},\{1,4,A,F\},\{3,6,8,D\}\\
P3 &: \{0,5,B,E\},\{2,7,9,C\},\{1,4,A,F\},\{3,6,8,D\} \quad = \quad P2
\end{aligned}$$

so the minimal realization has four states and is given by

$$\begin{aligned}
q_0^+ &= x_1'x_2 \vee x_1x_2' = x_1 \oplus x_2\\
q_1^+ &= q_0'x_1 \vee q_0x_1' = q_0 \oplus x_1\\
u &= (q_1'x_1'x_2 \vee q_1'x_1x_2' \vee q_1x_1'x_2' \vee q_1x_1x_2) \oplus x_3 = q_1 \oplus x_1 \oplus x_2 \oplus x_3
\end{aligned}$$

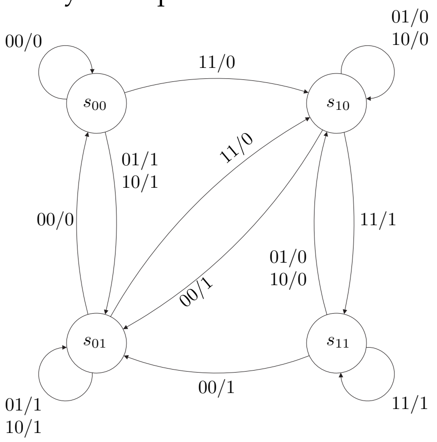
## 6.3

The graph for the Mealy-machine:

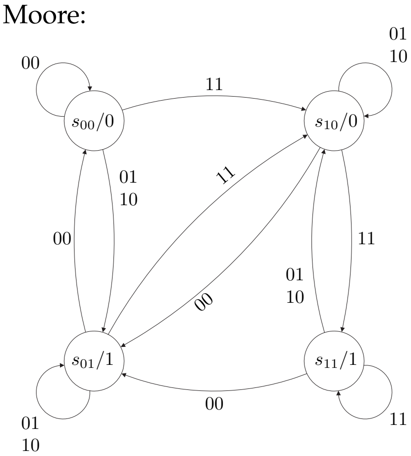



Split the states so all entering edges have the same output. Then, move the outputs into the state to get the graph for the Moore-machine:

Mealy with split states:




Moore:

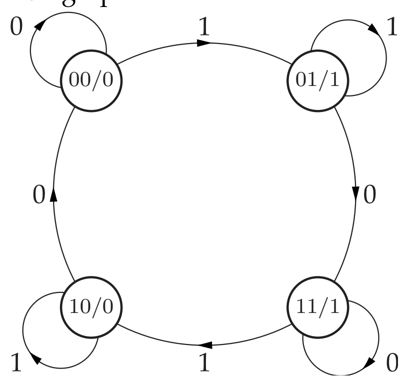



## 6.4

(1) Number the delay elements from below. The equations for the state-transitions and the output is then given by

$$
\begin{aligned}
q_o^+ &= q_0 x \vee q_1 x' \\
q_1^+ &= q_1 x' \vee q_o' x \\
q_2^+ &= q_1 x' \vee q_o' x \vee q_0' q_1 = q_1 x' \vee q_o' x = q_1^+ \Rightarrow z = q_1
\end{aligned}
$$

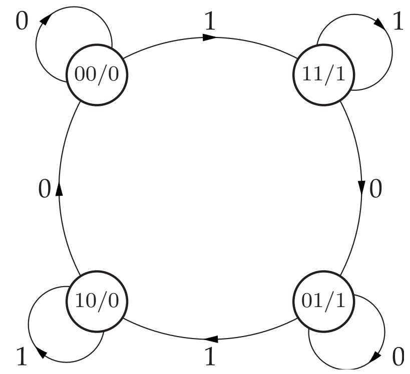
Hence, the $D$-element on the output implies that the sequential circuit is of Moore-type. From the equations we construct the state-transition graph:




(2) Number the delay elements from the left hand side. The equations for the state-transitions and the output is then given by

$$
\begin{aligned}
q_o^+ &= x \\
q_1^+ &= q_1 x' \vee q_0 q_1 \vee q_0' q_1' x \\
z &= q_1
\end{aligned}
$$

Which gives:




The two graphs differ only in the state assignments. Since this does not affect the behaviour of the construction we conclude that the two circuits give the same output when given the same input. Hence, $\boldsymbol{u} = \mathbf{0}$ for all inputs.

**6.5**

The counter is specified by the following table.

| | $i_0$ | $i_1$ |
|---|---|---|
| $s_0$ | $s_1$ | $s_5$ |
| $s_1$ | $s_2$ | $s_0$ |
| $s_2$ | $s_3$ | $s_1$ |
| $s_3$ | $s_4$ | $s_2$ |
| $s_4$ | $s_5$ | $s_3$ |
| $s_5$ | $s_0$ | $s_4$ |

We will use the coding $i_0 = 0$, $i_1 = 1$ for the input.

(a) The natural binary coding yields the minimal realization

$$\begin{aligned} q_0^+ &= q_0'q_1'q_2'i \vee q_1q_2i' \vee q_0q_2'i' \vee q_0q_2i \\ q_1^+ &= q_0'q_1'q_2i' \vee q_1q_2'i' \vee q_1q_2i \vee q_0q_2'i \\ q_2^+ &= q_2' \end{aligned}$$

with 2 OR gates and 8 AND gates.

(b)

$$\begin{aligned} &\overline{s_0, s_1} \longrightarrow \overline{s_0, s_1, s_2, s_5} \longrightarrow \overline{s_0, s_1, s_2, s_3, s_4, s_5} \quad \text{(not useful)} \\ &\overline{s_0, s_2} \longrightarrow \overline{s_0, s_2}; \overline{s_1, s_3, s_5} \longrightarrow \overline{s_0, s_2, s_4}; \overline{s_1, s_3, s_5} \\ &\overline{s_0, s_3} \longrightarrow \overline{s_0, s_3}; \overline{s_1, s_4}; \overline{s_2, s_5} \end{aligned}$$

Using the last two partitions, we obtain the state assignment

| | $s_0$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|---|
| $q_0q_1q_2$ | 000 | 111 | 001 | 100 | 011 | 101 |

where $q_0$ discriminates between $\overline{s_0, s_2, s_4}$ and $\overline{s_1, s_3, s_5}$, and $q_1q_2$ between $\overline{s_0, s_3}$, $\overline{s_1, s_4}$, and $\overline{s_2, s_5}$. Then the minimal realization is

$$\begin{aligned} q_0^+ &= q_0' \\ q_1^+ &= q_2'i' \vee q_1'q_2i \\ q_2^+ &= q_2' \vee q_1'i \vee q_1i' \end{aligned}$$

(2 OR gates and 4 AND gates).

(c) The Gray code

| | $s_0$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|---|
| $q_0q_1q_2$ | 000 | 001 | 011 | 010 | 110 | 100 |

(notice that there are several Gray codes to choose from) yields

$$\begin{aligned} q_0^+ &= q_1 q_2' i' \vee q_1' q_2' i \\ q_1^+ &= q_2 i' \vee q_0' q_1 q_2' \vee q_0 i \\ q_2^+ &= q_0' q_1' i' \vee q_0' q_1 i \end{aligned}$$

with 3 OR gates and 7 AND gates.

## 6.6

There are two state partitions to use for the reduced dependancy encoding:

$\overline{s_0 s_1} \rightarrow \overline{s_0 s_1}; \overline{s_3 s_5} \rightarrow \overline{s_0 s_1 s_2}; \overline{s_3 s_4 s_5}$

$\overline{s_0 s_5} \rightarrow \overline{s_0 s_5}; \overline{s_2 s_3} \rightarrow \overline{s_0 s_5}; \overline{s_2 s_3}; \overline{s_1 s_4}$

This gives for example the following state assignment and result:

| state | $q_0 q_1 q_2$ |
|---|---|
| $s_0$ | 000 |
| $s_1$ | 001 |
| $s_2$ | 010 |
| $s_3$ | 110 |
| $s_4$ | 101 |
| $s_5$ | 100 |

$\Rightarrow$

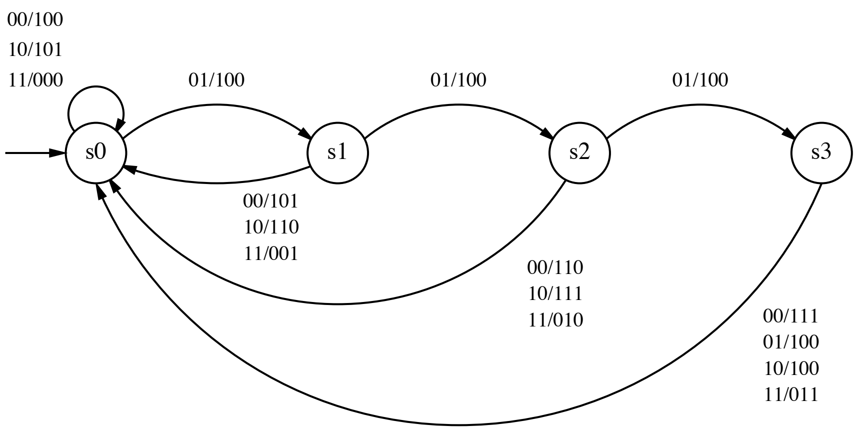
| $q_0 q_1 q_2$ | $x$ | $q_0^+ q_1^+ q_2^+$ | $u$ |
|---|---|---|---|
| 000 | 0 | 110 | 0 |
| 000 | 1 | 010 | 0 |
| 001 | 0 | 100 | 0 |
| 001 | 1 | 010 | 0 |
| 010 | 0 | 101 | 0 |
| 010 | 1 | 001 | 0 |
| 011 | 0 | - - - | - |
| 011 | 1 | - - - | - |
| 100 | 0 | 010 | 0 |
| 100 | 1 | 110 | 0 |
| 101 | 0 | 000 | 0 |
| 101 | 1 | 110 | 0 |
| 110 | 0 | 001 | 1 |
| 110 | 1 | 101 | 1 |
| 111 | 0 | - - - | - |
| 111 | 1 | - - - | - |

$\Rightarrow$

$$\begin{aligned} u &= q_0 q_1 \\ q_0^+ &= q_0' x' \vee q_0 x \\ q_1^+ &= q_1' x \vee q_1' q_2' \\ q_2^+ &= q_1 \end{aligned}$$

## 6.7

The machine can be described by the following minimal graph:




Let $i_1, i_0$ denote the input variables and $q_1, q_0$ the state variables. With the state assignment

| $s$ | $q_1 q_0$ |
|---|---|
| $s_0$ | 00 |
| $s_1$ | 01 |
| $s_2$ | 10 |
| $s_3$ | 11 |

we obtain the following table:

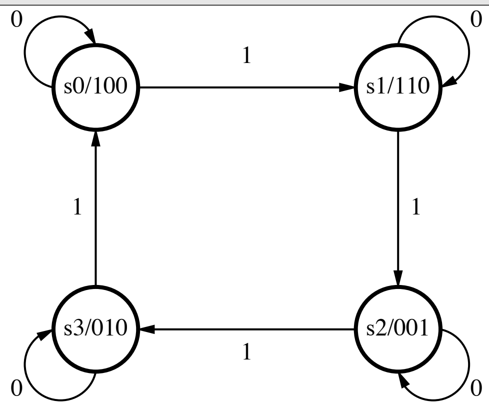
| $i_1 i_0 q_1 q_0$ | $u_2 u_1 u_0 q_1^+ q_0^+$ |
|---|---|
| 0 0 0 0 | 1 0 0 0 0 |
| 0 0 0 1 | 1 0 1 0 0 |
| 0 0 1 0 | 1 1 0 0 0 |
| 0 0 1 1 | 1 1 1 0 0 |
| 0 1 0 0 | 1 0 0 0 1 |
| 0 1 0 1 | 1 0 0 1 0 |
| 0 1 1 0 | 1 0 0 1 1 |
| 0 1 1 1 | 1 0 0 0 0 |
| 1 0 0 0 | 1 0 1 0 0 |
| 1 0 0 1 | 1 1 0 0 0 |
| 1 0 1 0 | 1 1 1 0 0 |
| 1 0 1 1 | 1 0 0 0 0 |
| 1 1 0 0 | 0 0 0 0 0 |
| 1 1 0 1 | 0 0 1 0 0 |
| 1 1 1 0 | 0 1 0 0 0 |
| 1 1 1 1 | 0 1 1 0 0 |

One possible realization of the outputs $u_2, u_1, u_0$, in disjunctive form, is

$$\begin{aligned}
u_2 &= i_1' \vee i_0' \\
u_1 &= i_0' i_1' q_1 \vee i_0 i_1 q_1 \vee i_1 q_1 q_0' \vee i_1 i_0' q_1' q_0 \\
u_0 &= i_1' i_0' q_0 \vee i_1 i_0 q_0 \vee i_1 i_0' q_0' \\
q_1^+ &= i_1' i_0 q_1' q_0 \vee i_1' i_0 q_1 q_0' \\
q_2^+ &= i_1' i_0 q_0'
\end{aligned}$$

## 6.8

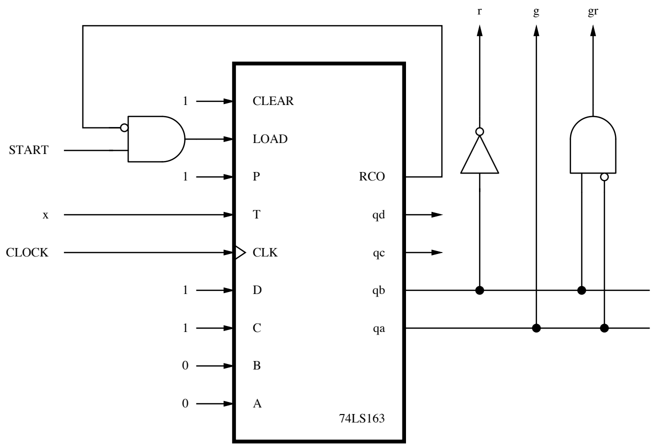
The traffic light can be described by the state-transition graph to the right (Moore machine). The three output bits correspond to the lamps for red, yellow and green light, respectively ($r$, $g$ and $gr$). The input $x$ is 1 when the lamps change.



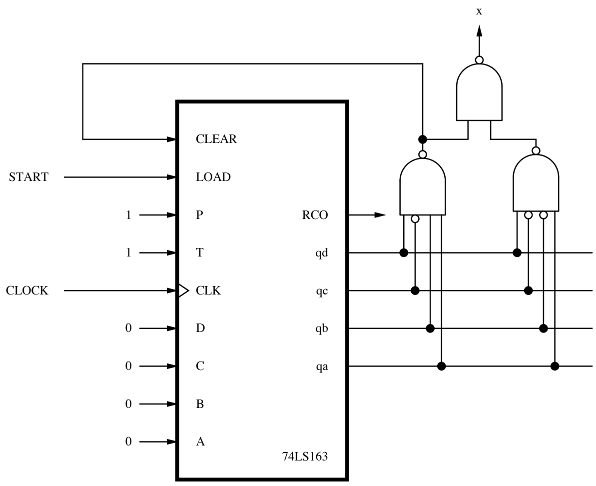

If we use natural state assignment, the outputs can be calculated as

$$r = q_1' \qquad g = q_0 \qquad gr = q_1 \wedge q_0'$$

We choose to realize the graph with a modulo 4 counter (other realizations are possible, as well):



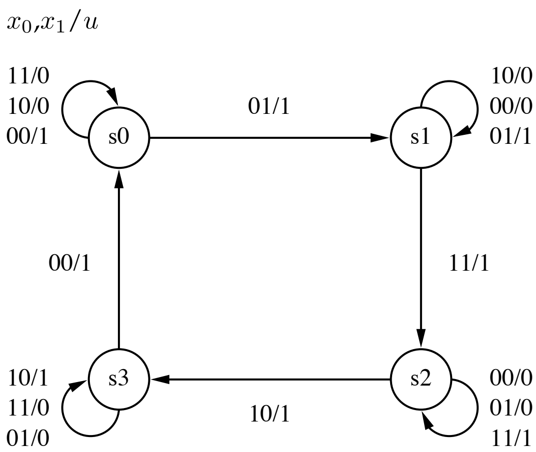

It remains to generate the input signal $x$. Again, we use a counter, now modulo 12. It's output, $x$, is 1 if the counter's state equals 9 (1001 in binary notation) or 11 (1011 in binary notation):




The signal that we call $\overline{\text{START}}$ can be used to put the traffic light into the starting state.

## 6.11

Consider the graph that we obtained as alternative solution to Problem 2.13:




We observe that this graph already is asynchronously realizable (cf. Definition 6.8). In order to avoid race we can, for example, use the state assignment:

| | $s_0$ | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|---|
| $q_0q_1$ | 00 | 01 | 11 | 10 |

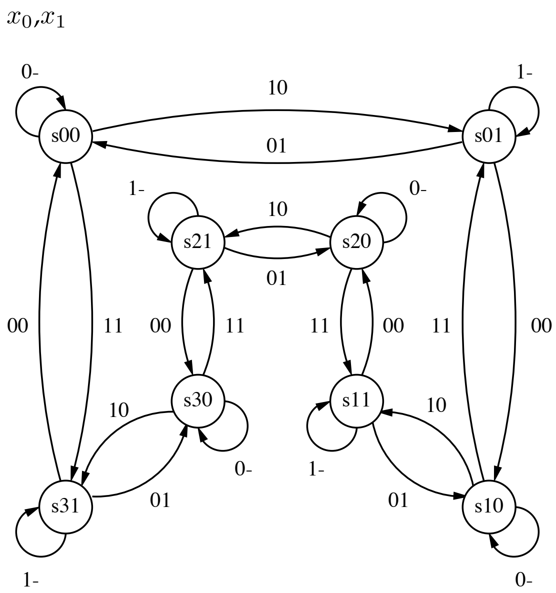
.

For a hazard free realization we include *all* prime implicants, which, for example, can be obtained from Karnaugh diagrams. This results in:

$$
\begin{aligned}
q_0^+ &= q_0q_1 \vee q_0x_1 \vee q_0x_0 \vee q_1x_0x_1 \\
q_1^+ &= q_0'q_1 \vee q_1x_0' \vee q_1x_1 \vee q_0'x_0'x_1 \\
u &= q_0'q_1'x_0' \vee q_0'x_0'x_1 \vee q_0'q_1x_1 \vee q_1x_0x_1 \vee q_0q_1x_0 \vee q_0x_0x_1' \vee q_1'x_0'x_1' \vee q_0q_1'x_1' \quad .
\end{aligned}
$$

## 6.12

For a synchronous realization of a modulo 4 counter we can immediately obtain a state-transition graph with four states. This graph is, however, not asynchronously realizable. For an asynchronous counter we have to introduce additional states:




One race-free state assignment is:

| | $s_{00}$ | $s_{01}$ | $s_{10}$ | $s_{11}$ | $s_{20}$ | $s_{21}$ | $s_{30}$ | $s_{31}$ |
|---|---|---|---|---|---|---|---|---|
| $q_0q_1q_2$ | 000 | 001 | 011 | 111 | 101 | 100 | 110 | 010 |

The outputs can directly be derived from the state assignment:

| $q_0q_1q_2$ | $u_0u_1$ |
|---|---|
| 00– | 00 |
| –11 | 01 |
| 10– | 10 |
| –10 | 11 |

$$
\Rightarrow \qquad
\begin{aligned}
u_0 &= q_0q_1' \vee q_1q_2' \vee q_0q_2' \\
u_1 &= q_1
\end{aligned}
$$

The complete prime implicant (hazard-free) form of the next-state variables becomes:

$$
\begin{aligned}
q_0^+ &= q_0q_1' \vee q_1q_2'x_0'x_1 \vee q_1q_2x_0x_1' \vee q_0x_0'x_1' \vee q_0x_0x_1 \vee q_0q_2'x_1 \vee q_0q_2x_1' \vee q_0q_2'x_0' \vee q_0q_2x_0 \\
q_1^+ &= q_0'q_2'x_0x_1 \vee q_0'q_2x_0'x_1' \vee q_0q_2'x_0'x_1' \vee q_0q_2x_0x_1 \vee q_0'q_1q_2'x_1 \vee q_0'q_1q_2x_1' \vee q_0q_1q_2'x_1' \vee q_0q_1q_2x_1 \\
&\quad \vee q_0'q_1q_2'x_0 \vee q_0'q_1q_2x_0' \vee q_0q_1q_2'x_0' \vee q_0q_1q_2x_0 \vee q_1x_0'x_1 \vee q_1x_0x_1' \\
q_2^+ &= q_0'q_1'x_0x_1' \vee q_0q_1'x_0'x_1 \vee q_2x_0'x_1' \vee q_2x_0x_1 \vee q_0'q_2x_1' \vee q_0q_2x_1 \vee q_0'q_2x_0 \vee q_0q_2x_0' \vee q_1q_2 \quad .
\end{aligned}
$$

## 6.14

(a) No, Mealy.

(b) Yes.

(c) No, $10 \rightarrow 01$.

(d) Yes, $\boldsymbol{x} = 101$

## 6.16

The graph is given in the solution of Problem 2.6. In the following tabulars we write the graph and the state assignment

| | 0 | 1 |
|---|---|---|
| $s_0$ | $s_1/0$ | $s_4/0$ |
| $s_1$ | $s_1/0$ | $s_2/0$ |
| $s_2$ | $s_2/0$ | $s_3/1$ |
| $s_3$ | $s_3/0$ | $s_3/0$ |
| $s_4$ | $s_2/0$ | $s_3/0$ |

| | $q_0q_1q_2$ |
|---|---|
| $s_0$ | 0 0 0 |
| $s_1$ | 0 0 1 |
| $s_2$ | 0 1 1 |
| $s_3$ | 1 1 1 |
| $s_4$ | 0 1 0 |

We get the following Karnaugh maps

| $q_0^+$ ($q_0q_1$ \ $q_2x$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 0 | 0 | 0 |
| 01 | 0 | 1 | 1 | 0 |
| 11 | - | - | 1 | 1 |
| 10 | - | - | - | - |

| $q_1^+$ ($q_0q_1$ \ $q_2x$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | 1 | 0 |
| 01 | 1 | 1 | 1 | 1 |
| 11 | - | - | 1 | 1 |
| 10 | - | - | - | - |

| $q_2^+$ ($q_0q_1$ \ $q_2x$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 1 | 0 | 1 | 1 |
| 01 | 1 | 1 | 1 | 1 |
| 11 | - | - | 1 | 1 |
| 10 | - | - | - | - |

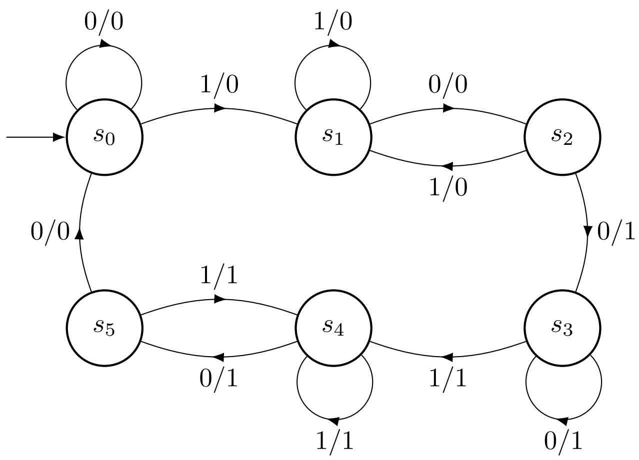
| $u$ ($q_0q_1$ \ $q_2x$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 0 | 0 | 0 |
| 01 | 0 | 0 | 1 | 0 |
| 11 | - | - | 0 | 0 |
| 10 | - | - | - | - |

and the functions

$$
\begin{aligned}
q_0^+ &= q_1x \vee q_0 \\
q_1^+ &= q_1 \vee x \\
q_2^+ &= q_1 \vee q_2 \vee x' \\
u &= q_0'q_1q_2x
\end{aligned}
$$

## 6.17

We first construct a graph that fulfills the requirements.



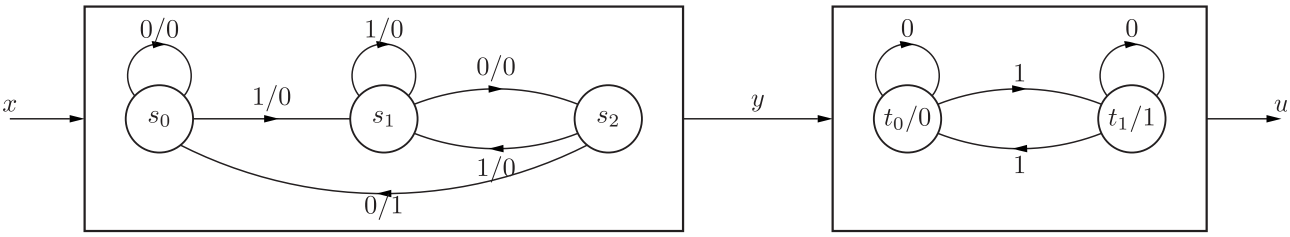

With the state assignment $s_0 = 000$, $s_1 = 001$, $s_2 = 011$, $s_3 = 111$, $s_4 = 101$, and $s_5 = 100$ we get the following functional table. Notice that the output is equivalent to $q_1$.

| $q_1q_2q_3x$ | $q_1^+q_2^+q_3^+$ | $q_1q_2q_3x$ | $q_1^+q_2^+q_3^+$ |
|---|---|---|---|
| 000 0 | 000 | 100 0 | 000 |
| 000 1 | 001 | 100 1 | 101 |
| 001 0 | 011 | 101 0 | 100 |
| 001 1 | 001 | 101 1 | 101 |
| 011 0 | 111 | 111 0 | 111 |
| 011 1 | 001 | 111 1 | 101 |

This can be realized with for example

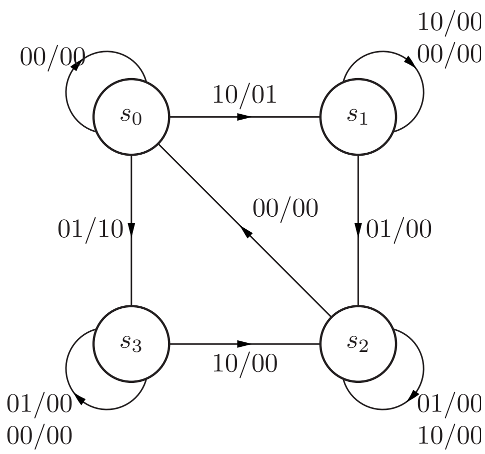
$$
\begin{aligned}
q_1^+ &= q_2x' \vee q_1x \vee q_1q_3\\
q_2^+ &= q_2x' \vee q_1'q_3x'\\
q_3^+ &= q_2x' \vee q_1'q_3x' \vee x
\end{aligned}
$$

**Remark:** This problem can also be solved by combining two machines, one that detects the pattern 100 and one modulo-2 counter as follows. Notice that we have first used a Mealy machine that affects the output asynchronously, and then a Moore machine so that the timing is correct.




**6.18**

The graph below can be used to solve the problem.




With the state assignment

| $s$ | $s_0$ | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|---|
| $q_0q_1$ | 00 | 01 | 11 | 10 |

we get the following table

| $q_0q_1w_1w_2$ | $q_0^+q_1^+u_lu_r$ |
|---|---|
| 0 0 0 0 | 0 0 0 0 |
| 0 0 0 1 | 1 0 1 0 |
| 0 0 1 0 | 0 1 0 1 |
| 0 0 1 1 | - - - - |
| 0 1 0 0 | 0 1 0 0 |
| 0 1 0 1 | 1 1 0 0 |
| 0 1 1 0 | 0 1 0 0 |
| 0 1 1 1 | - - - - |
| 1 0 0 0 | 1 0 0 0 |
| 1 0 0 1 | 1 0 0 0 |
| 1 0 1 0 | 1 1 0 0 |
| 1 0 1 1 | - - - - |
| 1 1 0 0 | 0 0 0 0 |
| 1 1 0 1 | 1 1 0 0 |
| 1 1 1 0 | 1 1 0 0 |
| 1 1 1 1 | - - - - |

This can, for example, be realized with

$$\begin{aligned}
q_0^+ &= w_2 \vee q_0w_1 \vee q_0q_1' \\
q_1^+ &= w_1 \vee q_1w_2 \vee q_0'q_1 \\
u_l &= q_0'q_1'w_2 \\
u_r &= q_0'q_1'w_1
\end{aligned}$$

The realization is omitted.

## 6.19

Construct a combinational circuit $K$ with three inputs and two outputs. One of the inputs is the $i$th input $x_i$ and the other two inputs, $(a_1a_0)$, represent the modulo three sum of the inputs $x_1, x_2, \ldots, x_{i-1}$. The otputs, $(b_1b_0)$, represent the modulo three sum of $(a_1a_0)_2$ and $x_i$. The table for the circuit is

| $a_1a_0$ | $x_i$ | $b_1b_0$ |
|---|---|---|
| 0 0 | 0 | 0 0 |
| 0 0 | 1 | 0 1 |
| 0 1 | 0 | 0 1 |
| 0 1 | 1 | 1 0 |
| 1 0 | 0 | 1 0 |
| 1 0 | 1 | 0 0 |
| 1 1 | 0 | - - |
| 1 1 | 1 | - - |

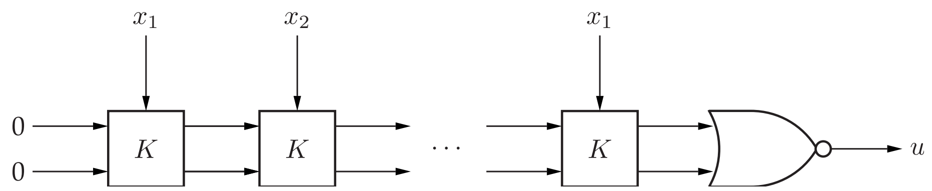
which can be realized as

$$\begin{aligned}
b_1 &= a_0x_i \vee a_1x_i' \\
b_0 &= a_1'a_0'x_i \vee a_0x_i'
\end{aligned}$$

The realization is omitted.

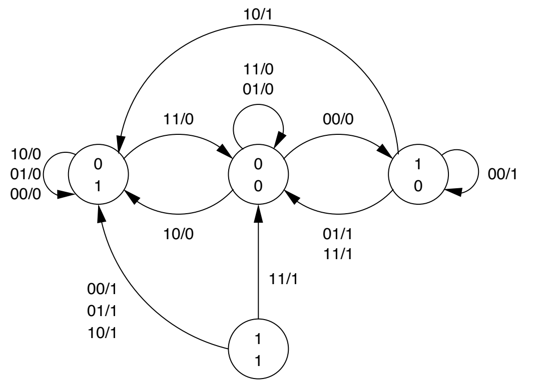
(This is the same combinational circuit we would hae used to construct a sequential modulo three counter.)

Use $N$ such circuits as follows for the solution.

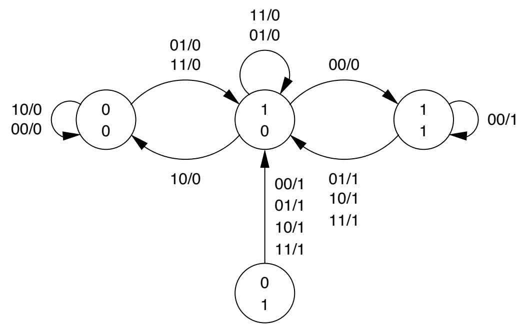



## 6.20

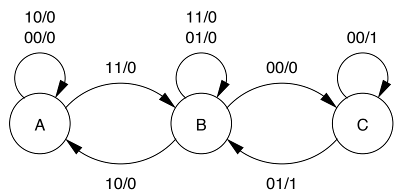
a) We first draw the graph of circuit 1:




Let us now draw the graph of circuit 2:




If we compare the two graphs (and ignore the state coding), we see that circuit 1 does not exactly realize graph 2, and circuit 2 does not exactly realize graph 1. Since we know, however, that both circuits must realize the same graph, it can only be that the original graph did not fully specify the behavior of the circuit. That is, some input combinations are considered to be impossible. Hence, when realizing the graph, we treat these input combinations as "don't care" terms. Obviously, for the realizations 1 and 2, the "don't care" terms were interpreted differently. If we compare graph 1 and graph 2, and draw parts that are in common, we can obtain the following graph.




Notice that this graph is not unique. For instance, we could have included the fourth state as well.

b) The realization can, for instance, be used to detect when a lion leaves its cage. (State A corresponds to the lion being in the cage, and the output 1 indicates danger.)

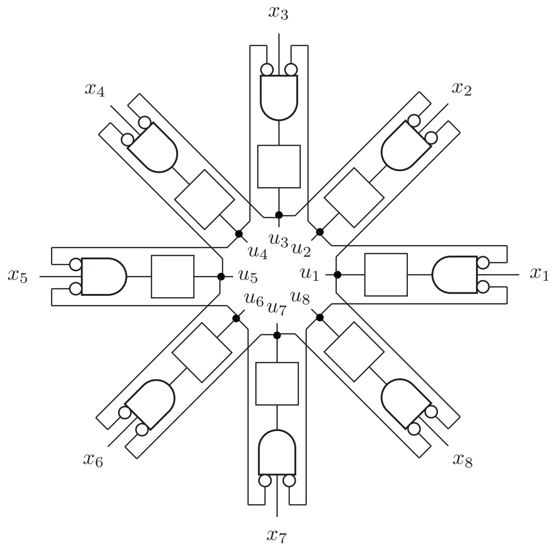
## 6.22

This can be implemented as a Moore machine where the output and the state is the same thing. Define one state variable for each witch and let it be 1 if she eats and 0 if she does not eat. Here we will use a synchronous state machine with a high frequence clock signal so that only one witch can decide to start or stop to eat during one clock cycle.

We can now concentrate on one witch, say witch $i$. Then the corresponding state variable $q_i$ updates as follows

| $q_{i-1}q_iq_{i+1}$ | $x_i$ | $q_i^+$ | Comment | $q_{i-1}q_iq_{i+1}$ | $x_i$ | $q_i^+$ | Comment |
|---|---|---|---|---|---|---|---|
| 0 0 0 | 0 | 0 | Thinking | 1 0 0 | 0 | 0 | Thinking |
| 0 0 0 | 1 | 1 | Start eating | 1 0 0 | 1 | 0 | Wait for witch $i-1$ |
| 0 0 1 | 0 | 0 | Thinking | 1 0 1 | 0 | 0 | Thinking |
| 0 0 1 | 1 | 0 | Wait for witch $i+1$ | 1 0 1 | 1 | 0 | Wait for witches $i-1$ and $i+1$ |
| 0 1 0 | 0 | 0 | Stop eating | 1 1 0 | 0 | - | Cannot happen |
| 0 1 0 | 1 | 1 | Continue to eat | 1 1 0 | 1 | - | Cannot happen |
| 0 1 1 | 0 | - | Cannot happen | 1 1 1 | 0 | - | Cannot happen |
| 0 1 1 | 1 | - | Cannot happen | 1 1 1 | 1 | - | Cannot happen |

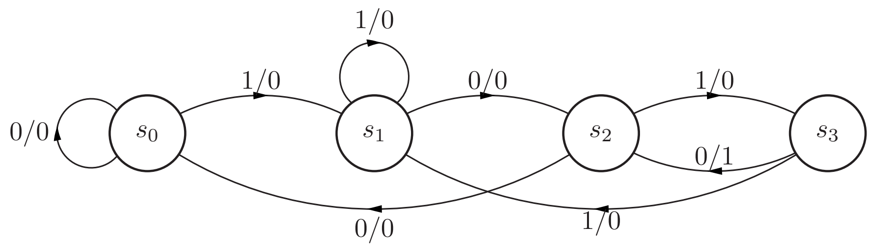
This can be realized with $q_i^+ = q'_{i-1}q'_{i+1}x_i$, i.e., eat if neighbours are not eating. The realization becomes




Since the functions only use one implicant each there are no risks for hazard and the delay elements can be dropped and we get an asyncronous sequential circuit. Also notice that the same type of realization can be used for an arbitrary number of witches.

## 6.23

The following (minimal) graph desribes the state machine:




With the state assignment $s_0 = 00$, $s_1 = 01$, $s_2 = 10$, and $s_3 = 11$ (NBCD) and minimization by Karnaugh maps we get the following functions:

$$u = q_0 q_1 x'$$
$$q_0^+ = q_1 x' \vee q_0 q_1' x$$
$$q_1^+ = x$$

The figure is omitted in this solution.

## 6.24

If the two machines are equivalent, we see that the first machine (M1) is non-minimal, since the second machine (M2) has at most four states. Hence, we try to minimize M1 and see what we get.

State transition table:

| $Q^+/u$ | 0 | 1 |
|---|---|---|
| $s_1$ | $s_5/1$ | $s_4/0$ |
| $s_2$ | $s_6/1$ | $s_3/0$ |
| $s_3$ | $s_4/1$ | $s_1/1$ |
| $s_4$ | $s_3/1$ | $s_2/1$ |
| $s_5$ | $s_3/1$ | $s_6/1$ |
| $s_6$ | $s_4/1$ | $s_5/1$ |

State minimization:

$$P_1 = \overline{s_1 s_2}; \overline{s_3 s_4 s_5 s_6}$$
$$P_2 = \overline{s_1 s_2}; \overline{s_3 s_4}; \overline{s_5 s_6}$$
$$P_3 = P_2$$

A minimal graph is given by:

| $Q^+/u$ | 0 | 1 |
|---|---|---|
| $s_{12}$ | $s_{56}/1$ | $s_{34}/0$ |
| $s_{34}$ | $s_{34}/1$ | $s_{12}/1$ |
| $s_{56}$ | $s_{34}/1$ | $s_{56}/1$ |

Now, we draw the graph for M2. Write expressions for the functions and draw the corresponding state transition table (one way is to write the functions in Karnaugh maps and then rewrite it as a normal table). Here we denote the state variable in the upper D-elelment $q_1$ and the lower $q_2$.

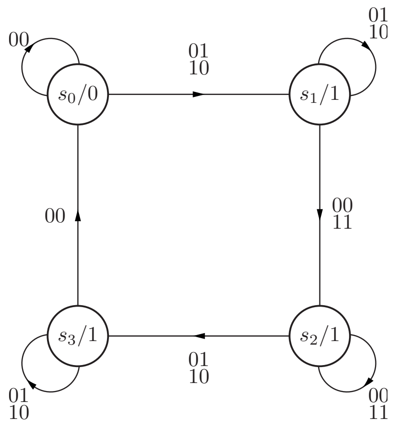
$$
\begin{aligned}
u &= q_1 \vee x' \\
q_1^+ &= q_2 \vee x' \vee q_1' \\
q_2^+ &= ((q_1' \vee x)' \vee (q_2 \vee x')')' = (q_1' \vee x)(q_2 \vee x')
\end{aligned}
$$

| $Q^+/u$ | 0 | 1 |
|---|---|---|
| 00 | 11/1 | 10/0 |
| 01 | 11/1 | 11/0 |
| 11 | 10/1 | 11/1 |
| 10 | 10/1 | 00/1 |

With the mapping $s_{12} = 00$, $s_{34} = 10$, and $s_{56} = 11$ (the state 01 is unreachable from the others) the two tables are equal, and therefore the machines equivalent (assuming M2 does not start in the state 01).

## 6.25

First draw a graph and decide on a state assignment (we assume that the trains can be more that 2km):




| $\mathcal{S}$ | $q_0 q_1$ | $L$ |
|---|---|---|
| $s_0$ | 0 0 | 0 |
| $s_1$ | 0 1 | 1 |
| $s_2$ | 1 1 | 1 |
| $s_3$ | 1 0 | 1 |

Use Karnaugh maps to realize the functions:

| $q_0^+$ ($q_0q_1$ \ $G_1G_2$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 0 | - | 0 |
| 01 | 1 | 0 | 1 | 0 |
| 11 | 1 | 1 | 1 | 1 |
| 10 | 0 | 1 | - | 1 |

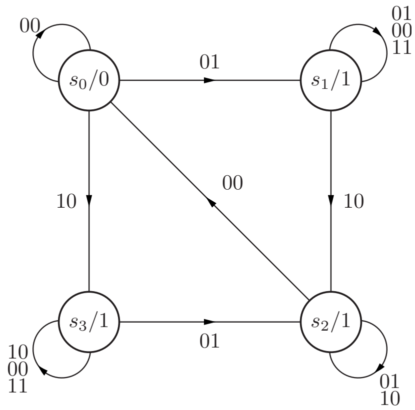
| $q_1^+$ ($q_0q_1$ \ $G_1G_2$) | 00 | 01 | 11 | 10 |
|---|---|---|---|---|
| 00 | 0 | 1 | - | 1 |
| 01 | 1 | 1 | 1 | 1 |
| 11 | 1 | 0 | 1 | 0 |
| 10 | 0 | 0 | - | 0 |

| $L$ ($q_0$ \ $q_1$) | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

$$
\begin{aligned}
q_0^+ &= q_0 G_2 \vee q_0 G_1 \vee \underline{q_1 G_1' G_2'} \vee \underline{G_1 G_2} \\
q_0^+ &= q_0' G_2 \vee q_o' G_1 \vee \underline{q_1 G_1' G_2'} \vee \underline{G_1 G_2} \\
L &= q_0 \vee q_1
\end{aligned}
$$

The figure is omitted.

**Remark:** An alternative graph can also be used:


00
01
$s_0/0$
01
00
11
$s_1/1$
10
00
10
$s_3/1$
01
$s_2/1$
10
00
11
01
10

## 7.1

First express the function in disjunctive normal form (DNF). Each minterm is 1 for only one combination of the variables, and there are not two minterms that are one for the same combination. Therefore, applying the definition of OR (3.152) on minterms we get

$$m_a \vee m_b = m_a \oplus m_b \oplus m_a m_b = m_a \oplus m_b, \quad a \neq b.$$

Hence,

$$\begin{aligned} f(x_1, x_2, x_3) &\overset{DNF}{=} x_1' x_2' x_3 \vee x_1' x_2 x_3 \vee x_1 x_2' x_3' \vee x_1 x_2 x_3 \\ &\overset{RSE}{=} (x_1 \oplus 1)(x_2 \oplus 1)x_3 \oplus (x_1 \oplus 1)x_2 x_3 \oplus x_1(x_2 \oplus 1)(x_3 \oplus 1) \oplus x_1 x_2 x_3 \\ &= x_1 \oplus x_3 \oplus x_1 x_2. \end{aligned}$$

The function is not linear.

## 7.2

Similar to 7.1 we get

$$\begin{aligned} f(x_1, x_2, x_3) &\overset{DNF}{=} x_1' x_2' x_3 \vee x_1' x_2 x_3' \vee x_1 x_2' x_3 \vee x_1 x_2 x_3' \\ &\overset{RSE}{=} (x_1 \oplus 1)(x_2 \oplus 1)x_3 \oplus (x_1 \oplus 1)x_2(x_3 \oplus 1) \oplus x_1(x_2 \oplus 1)x_3 \oplus x_1 x_2(x_3 \oplus 1) \\ &= x_2 \oplus x_3. \end{aligned}$$

The function is linear.

## 7.3

The following table specifies the conversion from NBCD to Gray (unit distance) code.

| $x_1x_2x_3x_4$ | $abcd$ | $x_1x_2x_3x_4$ | $abcd$ |
|---|---|---|---|
| 0000 | 0000 | 1000 | 1100 |
| 0001 | 0001 | 1001 | 1101 |
| 0010 | 0011 | 1010 | 1111 |
| 0011 | 0010 | 1011 | 1110 |
| 0100 | 0110 | 1100 | 1010 |
| 0101 | 0111 | 1101 | 1011 |
| 0110 | 0101 | 1110 | 1001 |
| 0111 | 0100 | 1111 | 1000 |

The realisation is given by,

$$\begin{cases} a = x_1 \\ b = x_1 \oplus x_2 \\ c = x_2 \oplus x_3 \\ d = x_3 \oplus x_4 \end{cases}$$

## 7.4

Write $g(k)$ in the following way,

$$g(k) = (1 \oplus x_1) \oplus (1 \oplus x_2) \oplus \cdots \oplus (1 \oplus x_k) = f(k) \oplus \underbrace{1 \oplus 1 \oplus \cdots \oplus 1}_{k-terms}.$$

Thus,

$$g(k) = \begin{cases} f(k) & , k \text{ even} \\ f(k) \oplus 1 = f'(k) & , k \text{ odd;} \end{cases}$$

**7.5**

See Example 7.2.

**7.6**

(a) Consider a sequential circuit given in Problem 6.2 (a). Assume we enumerate delay modules from the left to the right and from the top to the bottom. The next state variables and the output can be expressed by the following system of linear equations

$$\begin{aligned} q_1^+ &= x_1 \qquad u = x_1 \oplus q_1 \oplus q_2 \oplus x_2 \oplus q_3 \oplus x_3 \\ q_2^+ &= q_1 \\ q_3^+ &= x_2 \end{aligned}$$

An equivalent representation is

$$\begin{aligned} \boldsymbol{q}^+ &= A\boldsymbol{q} + B\boldsymbol{x} \\ \boldsymbol{u} &= C\boldsymbol{q} + H\boldsymbol{x} \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

First we compute the diagnostic matrix $K$:

$$\begin{aligned} CA &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \\ CA^2 &= \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Hence

$$K = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It can be observed that $rank(K) = 2$ since a system of the two first rows of $K$ is linearly independent but the system of all three rows is linearly dependent. We form a matrix $T$ from the first two linearly independent rows of $K$

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

One of the right inverses of $T$ is

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$$

The reduced form $\mathcal{L}_{\min} = (A_{\min}, B_{\min}, C_{\min}, H_{\min})$ of the linear circuit is now determined:

$$A_{\min} = TAR = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$B_{\min} = TB = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$C_{\min} = CR = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$H_{\min} = H = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

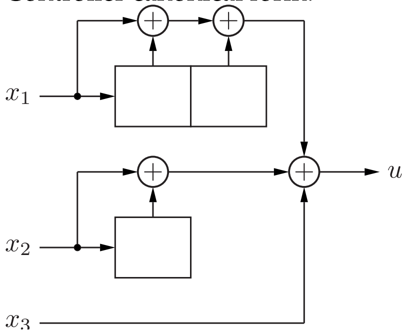
It results in the equation system

$$\begin{aligned} q_1^+ &= q_2 \oplus x_1 \oplus x_2 \qquad u = q_1 \oplus x_1 \oplus x_2 \oplus x_3 \\ q_2^+ &= x_1 \end{aligned}$$

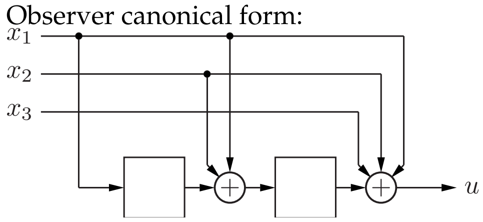
**Realizations in CCF and OCF:**
First derive the transfer function $G(D)$

$$\begin{aligned} G(D) &= \begin{pmatrix}1 & 0\end{pmatrix} \left( \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \oplus \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} D \right)^{-1} \begin{pmatrix}1 & 1 & 0\\1 & 0 & 0\end{pmatrix} D \oplus \begin{pmatrix}1 & 1 & 1\end{pmatrix} \\ &= \begin{pmatrix}1 \oplus D \oplus D^2 & 1 \oplus D & 1\end{pmatrix} \end{aligned}$$

Controller canonical form:




Observer canonical form:




(b) Consider a sequential circuit given in Problem 6.2 (b). Assume we enumerate delay modules from the left to the right and from the top to the bottom. The next state variables and the output can be expressed by the following system of linear equations

$$\begin{aligned} q_1^+ &= x_1 \qquad u = x_1 \oplus q_1 \oplus q_2 \oplus x_2 \oplus q_4 \oplus x_3 \\ q_2^+ &= q_1 \\ q_3^+ &= x_2 \\ q_4^+ &= q_3 \end{aligned}$$

An equivalent representation is

$$\begin{aligned} \boldsymbol{q}^+ &= A\boldsymbol{q} + B\boldsymbol{x} \\ \boldsymbol{u} &= C\boldsymbol{q} + H\boldsymbol{x} \end{aligned}$$

where

$$A = \begin{pmatrix}0 & 0 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\end{pmatrix} \qquad B = \begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix} \qquad C = \begin{pmatrix}1 & 1 & 0 & 1\end{pmatrix} \qquad H = \begin{pmatrix}1 & 1 & 1\end{pmatrix}$$

First we compute the diagnostic matrix $K$:

$$\begin{aligned} CA &= \begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix} \\ CA^2 &= \begin{pmatrix}0 & 0 & 0 & 0\end{pmatrix} \\ CA^3 &= \begin{pmatrix}0 & 0 & 0 & 0\end{pmatrix} \end{aligned}$$

Hence

$$K = \begin{pmatrix}1 & 1 & 0 & 1\\1 & 0 & 1 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}$$

It can be observed that $rank(K) = 2$. We form a matrix $T$ from the first two linearly independent rows of $K$

$$T = \begin{pmatrix}1 & 1 & 0 & 1\\1 & 0 & 1 & 0\end{pmatrix}$$

One of the right inverses of $T$ is

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The reduced form $\mathcal{L}_{\min} = (A_{\min}, B_{\min}, C_{\min}, H_{\min})$ of the linear circuit is now determined:

$$A_{\min} = TAR = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$B_{\min} = TB = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$C_{\min} = CR = \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$H_{\min} = H = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

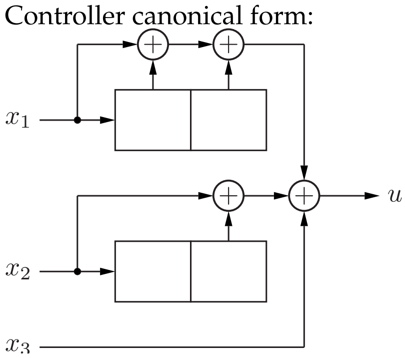
It results in the equation system

$$q_1^+ = q_2 \oplus x_1 \qquad u = q_1 \oplus x_1 \oplus x_2 \oplus x_3$$
$$q_2^+ = x_1 \oplus x_2$$

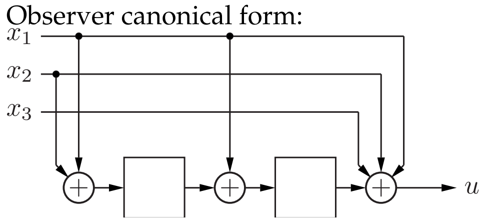
**Realizations in CCF and OCF:**
First derive the transfer function $G(D)$

$$G(D) = \begin{pmatrix} 1 & 0 \end{pmatrix} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} D \right)^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} D \oplus \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \oplus D \oplus D^2 & 1 \oplus D^2 & 1 \end{pmatrix}$$

Controller canonical form:




Observer canonical form:




## 7.7

We number the $D$-elements from left to right and from top and down. Then we get the following matrix representation:

$$\begin{pmatrix} q_0^+ \\ q_1^+ \\ q_2^+ \\ q_3^+ \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} x$$

$$u = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \oplus (0)x$$

Derive the diagnostic matrix

$$K = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

This has $\mathrm{rank} = 3$. Use the first 3 linearly independent rows to form

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

A right invers can be found in

$$R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

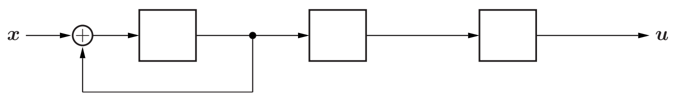
Hence, a minimal realization yields

$$\begin{aligned} A_{\min} &= TAR = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ B_{\min} &= TB = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ C_{\min} &= CR = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \\ H_{\min} &= H = 0 \end{aligned}$$

Hence,

$$\begin{aligned} q_0^+ &= q_1 \\ q_1^+ &= q_2 \\ q_2^+ &= q_2 \oplus x \\ u &= q_0 \end{aligned}$$

The realization:




## 7.8

Assume we enumerate delay modules from the left to the right. The next state variables and the output can be expressed by the following system of linear equations

$$\begin{aligned} q_1^+ &= x \oplus q_2 \oplus q_3 \oplus q_4 \qquad u = q_2 \oplus q_4 \\ q_2^+ &= q_1 \oplus q_3 \oplus q_4 \\ q_3^+ &= q_1 \oplus q_2 \oplus q_3 \\ q_4^+ &= q_1 \oplus q_3 \end{aligned}$$

An equivalent representation is

$$\begin{aligned} \boldsymbol{q}^+ &= A\boldsymbol{q} + B\boldsymbol{x} \\ \boldsymbol{u} &= C\boldsymbol{q} + H\boldsymbol{x} \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix} \qquad H = \begin{pmatrix} 0 \end{pmatrix}$$

First we compute the diagnostic matrix $K$:

$$\begin{aligned} CA &= \begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix} \\ CA^2 &= \begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix} \\ CA^3 &= \begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix} \end{aligned}$$

Hence

$$K = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Since $rank(K) = 4$ the circuit is already in minimal form.

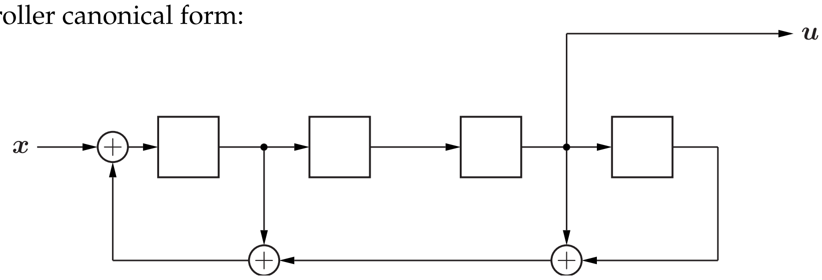
To get the transfer function matrix we derive

$$\begin{aligned} (I \oplus AD)^{-1} &= \begin{pmatrix} 1 & D & D & D \\ D & 1 & D & D \\ D & D & 1 \oplus D & 0 \\ D & 0 & D & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{1 \oplus D \oplus D^3 \oplus D^4} \begin{pmatrix} D \oplus 1 \oplus D^2 \oplus D^3 & D \oplus D^3 & D \oplus D^3 & D \oplus D^3 \\ D \oplus D^2 & D \oplus 1 & D \oplus D^3 & D \oplus D^3 \\ D^3 \oplus D^2 \oplus D & D^3 \oplus D^2 \oplus D & D^3 \oplus 1 & 0 \\ D & D^3 & D \oplus D^2 & D \oplus 1 \oplus D^2 \oplus D^3 \end{pmatrix} \end{aligned}$$

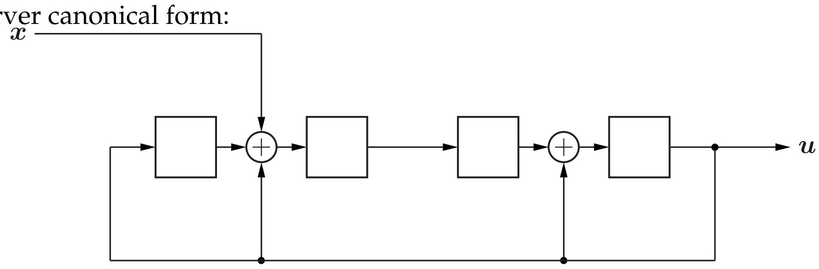
Hence,

$$G(D) = \left(C\left(I \oplus AD\right)^{-1} BD \oplus H\right)^T = \frac{D^3}{1 \oplus D \oplus D^3 \oplus D^4}$$

Realization in controller canonical form:




Realization in observer canonical form:



## 7.10

a) From the figure we get

$$\begin{pmatrix} q_1^+ \\ q_2^+ \end{pmatrix} = \begin{pmatrix} q_1 \oplus q_2 \oplus x_1 \\ q_1 \oplus x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} q_1 \oplus x_1 \\ q_2 \oplus x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Hence,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

b) Derive $G(D)$ from the marices above,

$$G(D) = C(I \oplus AD)^{-1} BD \oplus H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \oplus D & D \\ D & 1 \end{pmatrix}^{-1} \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The inverse of $(I \oplus AD)$ is

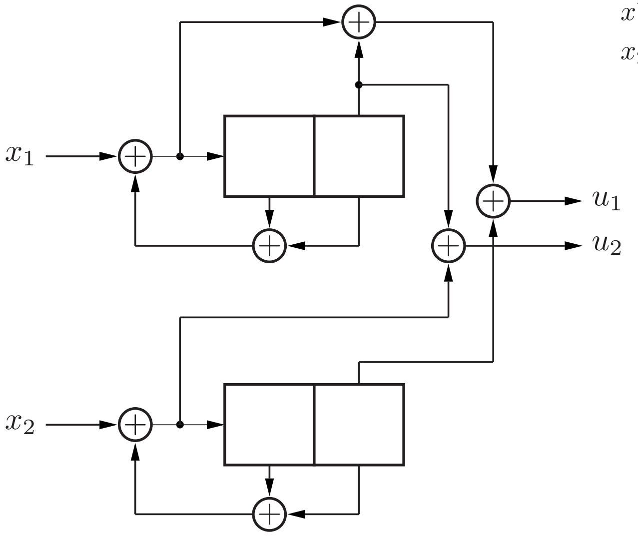
$$\begin{pmatrix} 1 \oplus D & D \\ D & 1 \end{pmatrix}^{-1} = \frac{1}{1 \oplus D \oplus D^2} \begin{pmatrix} 1 & D \\ D & 1 \oplus D \end{pmatrix}$$

Hence,

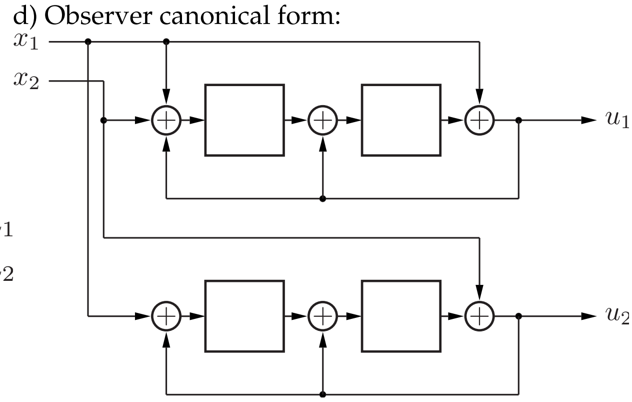
$$G(D) = \begin{pmatrix} \frac{1 \oplus D^2}{1 \oplus D \oplus D^2} & \frac{D^2}{1 \oplus D \oplus D^2} \\ \frac{D^2}{1 \oplus D \oplus D^2} & \frac{1}{1 \oplus D \oplus D^2} \end{pmatrix}$$

For an alternative solution see the exam from August 2003, Problem 5.

c) Controller canonical form:




d) Observer canonical form:




## 7.11

a) from the figure it is clear that the first output for the second (right) encoder equals the input, i.e., $v_1' = u'$. Therefore, we can directly get the input sequence as the sequence for $v_1'$. Hence,

$$\boldsymbol{u}' = 0111111100\ldots$$

b) The two encoders are both linear sequential circuits. To get some more knowledge about the behaviour we find the generator matrices for the two encoders. We start with the left encoder and write the $D$-transform for the outputs.

$$v_1(D) = u(D) \oplus u(D)D \oplus u(D)D^3 = u(D)\big(1 \oplus D \oplus D^3\big)$$
$$v_2(D) = u(D) \oplus u(D)D^2 \oplus u(D)D^3 = u(D)\big(1 \oplus D^2 \oplus D^3\big)$$

In matrix form this is equivalent to

$$v(D) = \big(v_1(D)\; v_2(D)\big) = u(D)G(D) \quad \text{where } G(D) = \big(1 \oplus D \oplus D^3 \quad 1 \oplus D^2 \oplus D^3\big)$$

Similarly, for the right encoder we get

$$v_1'(D) = u'(D)$$
$$v_2'(D) = u'(D) \oplus u'(D)D^2 \oplus u'(D)D^3 \oplus v_2'(D)D \oplus v_2'D^3$$

The second output can be rewritten as

$$v_2'(D)\big(1 \oplus D \oplus D^3\big) = u'(D)\big(1 \oplus D^2 \oplus D^3\big)$$

or, equivalently,

$$v_2'(D) = u'(D)\frac{1 \oplus D^2 \oplus D^3}{1 \oplus D \oplus D^3}$$

Hence, in matrix form we get

$$v'(D) = \big(v_1'(D)\; v_2'(D)\big) = u'(D)G'(D) \quad \text{where } G'(D) = \left(1 \quad \frac{1 \oplus D^2 \oplus D^3}{1 \oplus D \oplus D^3}\right)$$

If the left encoder produces the output $v(D)$ for the input $u(D)$, then

$$\begin{aligned} v(D) &= u(D)\big(1 \oplus D \oplus D^3 \quad 1 \oplus D^2 \oplus D^3\big) \\ &= u(D)\big(1 \oplus D \oplus D^3\big)\left(1 \quad \frac{1 \oplus D^2 \oplus D^3}{1 \oplus D \oplus D^3}\right) = u'(D)\left(1 \quad \frac{1 \oplus D^2 \oplus D^3}{1 \oplus D \oplus D^3}\right) \end{aligned}$$

Hence, the right encoder will generate the same output for the input

$$u'(D) = u(D)\big(1 \oplus D \oplus D^3\big)$$

Similarly, if the right encoder produces the output $v'(D)$ for the input $u'(D)$, then

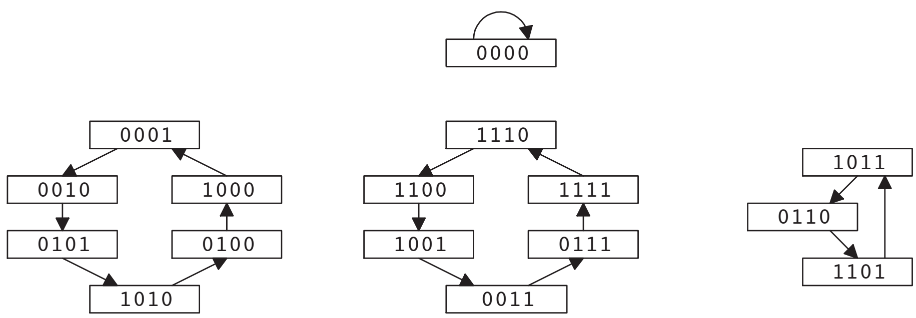
$$v'(D) = u'(D)\left(1 \quad \frac{1 \oplus D^2 \oplus D^3}{1 \oplus D \oplus D^3}\right) = u'(D)\frac{1}{1 \oplus D \oplus D^3}\big(1 \oplus D \oplus D^3 \quad 1 \oplus D^2 \oplus D^3\big)$$

Hence, the left encoder will generate the same output for the input

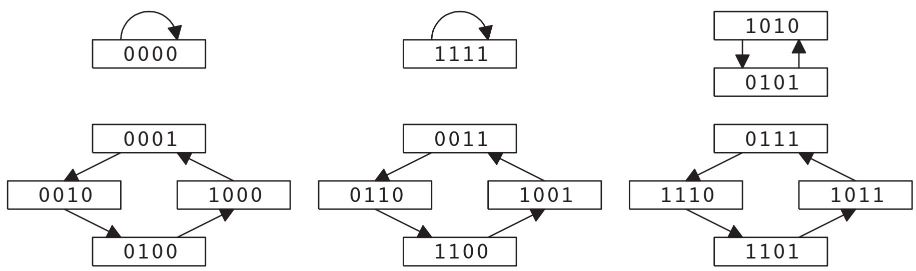
$$u(D) = \frac{u'(D)}{1 \oplus D \oplus D^3}$$

## 7.12

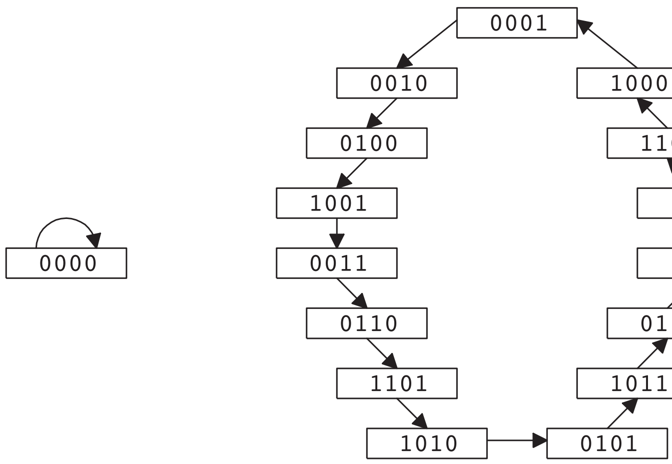
(a) The connection polynomial is $C(D) = 1 \oplus D^2 \oplus D^4$. The state-transition graphs are




(b) The connection polynomial is $C(D) = 1 \oplus D^4$. The corresponding state-transition graphs are



(c) The connection polynomial is $C(D) = 1 \oplus D^3 \oplus D^4$. The corresponding state-transition graphs are




## 7.13

(a)

$$s = [1001]^\infty \xrightarrow{\mathcal{D}} S(D) = \frac{1 \oplus D^3}{1 \oplus D^4} = \frac{(1 \oplus D \oplus D^2)(1 \oplus D)}{(1 \oplus D \oplus D^2 \oplus D^3)(1 \oplus D)}. = \frac{1 \oplus D \oplus D^2}{1 \oplus D \oplus D^2 \oplus D^3}.$$

(b)

$$s = [1010]^\infty \xrightarrow{\mathcal{D}} S(D) = \frac{1 \oplus D^2}{1 \oplus D^4} = \frac{1 \oplus D^2}{(1 \oplus D^2)^2} = \frac{1}{1 \oplus D^2} \quad (\Rightarrow s = [10]^\infty).$$

(c)

$$s = [0001]^\infty \xrightarrow{\mathcal{D}} S(D) = \frac{D^3}{1 \oplus D^4}.$$

(d) denote $t = 0$ with a dot ($\dot{}$). Then

$$\begin{aligned} s &= 10\dot{1}[01]^\infty = 10\dot{1}0101010101\ldots \\ &\xrightarrow{\mathcal{D}} S(D) = D^{-2} \oplus 1 \oplus D^2 \oplus D^4 \oplus D^6 \oplus \ldots \\ &= D^{-2}(1 \oplus D^2 \oplus D^4 \oplus D^6 \oplus \ldots) = D^{-2}\frac{1}{1 \oplus D^2}. \end{aligned}$$

## 7.14

(a) This problem can be solved in two ways. The first is by series expansion, in our case long division,

$$\begin{array}{r|l} & D^2 \oplus D^3 \quad \oplus D^5 \oplus D^6 \quad \oplus D^8 \oplus D^9 \oplus \cdots \xrightarrow{\mathcal{D}^{-1}} 00110110110\ldots = 00[110]^\infty = 0[011]^\infty \\ \hline 1 \oplus D \oplus D^2 & D^2 \\ & \underline{D^2 \oplus D^3 \oplus D^4} \\ & D^3 \oplus D^4 \\ & \underline{D^3 \oplus D^4 \oplus D^5} \\ & D^5 \\ & \underline{D^5 \oplus D^6 \oplus D^7} \\ & D^6 \oplus D^7 \\ & \underline{D^6 \oplus D^7 \oplus D^8} \\ & D^8 \end{array}$$

In the alternative solution we notice that $(1 \oplus D \oplus D^2)(1 \oplus D) = 1 \oplus D^3$. Then, $S(D)$ can be expressed as

$$S(D) = \frac{D^2}{1 \oplus D \oplus D^2} = \frac{D^2(1 \oplus D)}{1 \oplus D^3} = D^2\frac{1 \oplus D}{1 \oplus D^3} \xrightarrow{\mathcal{D}^{-1}} 00[110]^\infty.$$

(b) Again use $(1 \oplus D \oplus D^2)(1 \oplus D) = 1 \oplus D^3$ to rewrite $S(D)$,

$$S(D) = D\frac{1 \oplus D}{1 \oplus D \oplus D^2} = D\frac{(1 \oplus D)(1 \oplus D)}{1 \oplus D^3} = D\frac{1 \oplus D^2}{1 \oplus D^3} \xrightarrow{\mathcal{D}^{-1}} \boldsymbol{s} = 0[101]^\infty.$$

(c) Again use the $\mathcal{D}$-transform and rewrite $S(D)$,

$$\begin{aligned} S(D) &= \frac{1 \oplus D^2}{1 \oplus D \oplus D^2} = \frac{(1 \oplus D)(1 \oplus D^2)}{1 \oplus D^3} = \frac{1 \oplus D \oplus D^2 \oplus D^3}{1 \oplus D^3} \\ &= \frac{1 \oplus D^3}{1 \oplus D^3} \oplus \frac{D \oplus D^2}{1 \oplus D^3} = 1 \oplus D\frac{1 \oplus D}{1 \oplus D^3} \xrightarrow{\mathcal{D}^{-1}} 1[110]^\infty. \end{aligned}$$

(d) Similarly,

$$\begin{aligned} S(D) &= \frac{D}{1 \oplus D} \oplus \frac{1}{1 \oplus D \oplus D^2} = \frac{D(1 \oplus D \oplus D^2)}{1 \oplus D^3} \oplus \frac{1 \oplus D}{1 \oplus D^3} \\ &= \frac{1 \oplus D^2 \oplus D^3}{1 \oplus D^3} = 1 \oplus \frac{D^2}{1 \oplus D^3} = 1 \oplus D\frac{D}{1 \oplus D^3} \xrightarrow{\mathcal{D}^{-1}} \boldsymbol{s} = 1[010]^\infty. \end{aligned}$$

## 7.15

a) Expand $\frac{1}{C(D)} = \frac{1}{1 \oplus D^3 \oplus D^4}$ with long division until the rest is $R(D) = D^p$, then the period is $T = p$.

$$\begin{array}{r|l}
 & 1\oplus D^3\oplus D^4\oplus D^6 \quad \oplus D^8\oplus D^9\oplus D^{10}\oplus D^{11} \\
\hline
1\oplus D^3\oplus D^4 & 1 \\
 & 1\oplus D^3\oplus D^4 \\ \hline
 & D^3\oplus D^4 \\
 & D^3 \quad \oplus D^6\oplus D^7 \\ \hline
 & D^4\oplus D^6\oplus D^7 \\
 & D^4 \quad \oplus D^7\oplus D^8 \\ \hline
 & D^6 \quad \oplus D^8 \\
 & D^6 \quad \oplus D^9\oplus D^{10} \\ \hline
 & D^8\oplus D^9\oplus D^{10} \\
 & D^8 \quad \oplus D^{11}\oplus D^{12} \\ \hline
 & D^9\oplus D^{10}\oplus D^{11}\oplus D^{12} \\
 & D^9 \quad \oplus D^{12}\oplus D^{13} \\ \hline
 & D^{10}\oplus D^{11} \quad \oplus D^{13} \\
 & D^{10} \quad \oplus D^{13}\oplus D^{14} \\ \hline
 & D^{11} \quad \oplus D^{14} \\
 & D^{11} \quad \oplus D^{14}\oplus D^{15} \\ \hline
 & D^{15}
\end{array}$$

Hence, $T = 15$.

b) Expand $\frac{1}{1\oplus D^3\oplus D^6}$.

$$\begin{array}{r|l}
 & 1\oplus D^3 \\
\hline
1\oplus D^3\oplus D^6 & 1 \\
 & 1\oplus D^3\oplus D^6 \\ \hline
 & D^3\oplus D^6 \\
 & D^3\oplus D^6\oplus D^9 \\ \hline
 & D^9
\end{array}$$

Hence, the period is $T = 9$.

## 7.16

First use the $\mathcal{D}$-transform to get $S(D)$

$$\boldsymbol{s} = [011011101010111]^\infty \xrightarrow{\mathcal{D}} S(D) = \frac{D(1\oplus D\oplus D^3\oplus D^4\oplus D^5\oplus D^7\oplus D^9\oplus D^{11}\oplus D^{12}\oplus D^{13})}{1\oplus D^{15}}.$$

Calculate $\gcd(1\oplus D^{15}, 1\oplus D\oplus D^3\oplus D^4\oplus D^5\oplus D^7\oplus D^9\oplus D^{11}\oplus D^{12}\oplus D^{13})$ with Euclid's algorithm

$$\begin{aligned}
1\oplus D^{15} &= (1\oplus D\oplus D^3\oplus D^4\oplus D^5\oplus D^7\oplus D^9\oplus D^{11}\oplus D^{12}\oplus D^{13})(D\oplus D^2)\\
&\quad \oplus(1\oplus D\oplus D^3\oplus D^4\oplus D^7\oplus D^8\oplus D^9\oplus D^{10}\oplus D^{11}\oplus D^{12})\\
1\oplus D\oplus D^3\oplus D^4\oplus D^5\oplus D^7\oplus D^9\oplus D^{11}\oplus D^{12}\oplus D^{13} &= (1\oplus D\oplus D^3\oplus D^4\oplus D^7\oplus D^8\oplus D^9\oplus D^{10}\oplus D^{11}\oplus D^{12})D\\
&\quad \oplus(1\oplus D^2\oplus D^3\oplus D^7\oplus D^8\oplus D^{10})\\
1\oplus D\oplus D^3\oplus D^4\oplus D^7\oplus D^8\oplus D^9\oplus D^{10}\oplus D^{11}\oplus D^{12} &= (1\oplus D^2\oplus D^3\oplus D^7\oplus D^8\oplus D^{10})(D\oplus D^2)\\
&\quad \oplus(1\oplus D^2\oplus D^4\oplus D^5\oplus D^7\oplus D^9)\\
1\oplus D^2\oplus D^3\oplus D^7\oplus D^8\oplus D^{10} &= (1\oplus D^2\oplus D^4\oplus D^5\oplus D^7\oplus D^9)D\\
&\quad \oplus(1\oplus D\oplus D^2\oplus D^5\oplus D^6\oplus D^7)\\
1\oplus D^2\oplus D^4\oplus D^5\oplus D^7\oplus D^9 &= (1\oplus D\oplus D^2\oplus D^5\oplus D^6\oplus D^7)(1\oplus D\oplus D^2)\oplus 0\\
&\Rightarrow \text{gcd} = 1\oplus D\oplus D^2\oplus D^5\oplus D^6\oplus D^7.
\end{aligned}$$

Extracting this from $S(D)$ gives

$$\begin{aligned}
S(D) &= \frac{1\oplus D\oplus D^2\oplus D^5\oplus D^6\oplus D^7}{1\oplus D\oplus D^2\oplus D^5\oplus D^6\oplus D^7}\cdot\frac{D\oplus D^3\oplus D^7}{1\oplus D\oplus D^3\oplus D^4\oplus D^5\oplus D^7\oplus D^8}\\
&= \frac{D\oplus D^3\oplus D^7}{1\oplus D\oplus D^3\oplus D^4\oplus D^5\oplus D^7\oplus D^8}
\end{aligned}$$

Hence, the connection polynomial for the shortest shift register is

$$C(D) = 1\oplus D\oplus D^3\oplus D^4\oplus D^5\oplus D^7\oplus D^8.$$

## 7.17

We start with the upper LFSR. The connection polynomial is $C_1(D) = 1 \oplus D \oplus D^4$. Since the starting state is $s_0s_1s_2s_3 = 1000$ we get the numerator $P(D)$ from

$$\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix} = \begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

Hence, the sequence generated by the upper LFSR can be written as

$$S_1(D) = \frac{P_1(D)}{C_1(D)} = \frac{1\oplus D}{1\oplus D\oplus D^4}$$

Similarly, the connection polynomial for the lower LFSR is $C_2(D) = 1 \oplus D \oplus D^2 \oplus D^3$ and the numerator can be derived as

$$\begin{pmatrix}p_0\\p_1\\p_2\end{pmatrix} = \begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

Hence, the sequence from the lower LFSR can be written as

$$S_2(D) = \frac{P_2(D)}{C_2(D)} = \frac{1+D}{1\oplus D\oplus D^2\oplus D^3}$$

Combining these two sequences with a modulo two adder yields

$$\begin{aligned}
S(D) = S_1(D)\oplus S_2(D) &= \frac{1\oplus D}{1\oplus D\oplus D^4}\oplus\frac{1+D}{1\oplus D\oplus D^2\oplus D^3}\\
&= \frac{(1\oplus D)(1\oplus D\oplus D^2\oplus D^3)\oplus(1+D)(1\oplus D\oplus D^4)}{(1\oplus D\oplus D^4)(1\oplus D\oplus D^2\oplus D^3)}\\
&= \frac{D^2\oplus D^5}{1\oplus D^5\oplus D^6\oplus D^7}\\
&= \frac{D^2\oplus D^3\oplus D^4}{1\oplus D\oplus D^2\oplus D^3\oplus D^4\oplus D^6}
\end{aligned}$$

since $\gcd(D^2 \oplus D^5, 1 \oplus D^5 \oplus D^6 \oplus D^7) = 1 \oplus D$. This implies that the sequence $\boldsymbol{s}$ can be generated by an LFSR of length 6 and connection polynomial $C(D) = 1 \oplus D \oplus D^2 \oplus D^3 \oplus D^4 \oplus D^6$. The construction is not the cheapest one.

An alternative solution is to notice that the sequence from the lower LFSR can be written as

$$S_2(D) = \frac{P_2(D)}{C_2(D)} = \frac{1 + D}{1 \oplus D \oplus D^2 \oplus D^3} = \frac{1}{1 \oplus D^2}$$

That is, we can replace the lower LFSR with an LFSR of length 2 and connection polynoimial $C_2(D) = 1 \oplus D^2$.

**7.18**

Denote the upper sequence by $\boldsymbol{x}$ and the lower sequence by $\boldsymbol{y}$. Then the resulting sequence is $\boldsymbol{z} = \boldsymbol{x} \wedge \boldsymbol{y}$. The sequences are:

$$\begin{aligned}
\boldsymbol{x} = [110]^\infty &= 110110110110110110110\ldots \\
\boldsymbol{y} = [1011100]^\infty &= 101110010111001011100101\ldots \\
\boldsymbol{z} &= 100110010110000010100100\ldots = [100110010110000010100]^\infty
\end{aligned}$$

Hence,

$$\begin{aligned}
Z(D) &= \frac{1 \oplus D^3 \oplus D^4 \oplus D^7 \oplus D^9 \oplus D^{10} \oplus D^{16} \oplus D^{18}}{1 \oplus D^{21}} = \frac{P(D)}{1 \oplus D^{21}} \\
&= \frac{(1 \oplus D^2 \oplus D^3)G(D)}{(1 \oplus D^2 \oplus D^4 \oplus D^5 \oplus D^6)G(D)} \\
&= \frac{1 \oplus D^2 \oplus D^3}{1 \oplus D^2 \oplus D^4 \oplus D^5 \oplus D^6},
\end{aligned}$$

where

$$G(D) = \gcd(P(D), 1 \oplus D^{21}) = 1 \oplus D^2 \oplus D^5 \oplus D^8 \oplus D^9 \oplus D^{12} \oplus D^{14} \oplus D^{15}.$$

**7.19**

With long division of $\frac{1}{C(D)}$ we get the period $T = 15$, which is the longest possible period for a linear shift register of length $L = 4$. (There are $2^L = 16$ states and one of them is the zero state, which we cannot use)

a) false.

b) true.

c) false.

d) true.

Again use long division to get the series expansion of $S(D)$,

$$S(D) = \frac{P(D)}{C(D)} = 1 \oplus D^3 \oplus D^7 \oplus D^8 \oplus \cdots \xrightarrow{\mathcal{D}^{-1}} 100100011\ldots,$$

from which we conclude that the starting state for the LFSR is 1001.

e) false.

f) true.

The starting state for the observer canonical form of $C(D)$ is $P(D)$ in $S(D) = \frac{P(D)}{C(D)}$.

g) true.

h) false.

## 7.20

(a)

| $i$ | $s_i$ | $\delta$ | $T(D)$ | $C(D)$ | $L$ | $C_p(D)$ | $l$ |
|---|---|---|---|---|---|---|---|
| – | – | – | – | 1 | 0 | 1 | 1 |
| 0 | 0 | 0 | – | ″ | ″ | ″ | 2 |
| 1 | 0 | 0 | – | ″ | ″ | ″ | 3 |
| 2 | 1 | 1 | 1 | $1 \oplus D^3$ | 3 | ″ | 1 |
| 3 | 1 | 1 | $1 \oplus D^3$ | $1 \oplus D \oplus D^3$ | ″ | ″ | 2 |
| 4 | 0 | 1 | $1 \oplus D \oplus D^3$ | $1 \oplus D \oplus D^2 \oplus D^3$ | ″ | ″ | 3 |
| 5 | 0 | 0 | ″ | ″ | ″ | ″ | 4 |
| 6 | 0 | 1 | $1 \oplus D \oplus D^2 \oplus D^3$ | $1 \oplus D \oplus D^2 \oplus D^3 \oplus D^4$ | 4 | $1 \oplus D \oplus D^2 \oplus D^3$ | 1 |
| 7 | 1 | 0 | ″ | ″ | ″ | ″ | 2 |

(b)

| $i$ | $s_i$ | $\delta$ | $T(D)$ | $C(D)$ | $L$ | $C_p(D)$ | $l$ |
|---|---|---|---|---|---|---|---|
| – | – | – | – | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | $1 \oplus D$ | 1 | ″ | 1 |
| 1 | 1 | 0 | ″ | ″ | ″ | ″ | 2 |
| 2 | 1 | 0 | ″ | ″ | ″ | ″ | 3 |
| 3 | 0 | 1 | $1 \oplus D$ | $1 \oplus D \oplus D^3$ | 3 | $1 \oplus D$ | 1 |
| 4 | 0 | 1 | $1 \oplus D \oplus D^3$ | $1 \oplus D^2 \oplus D^3$ | ″ | ″ | 2 |
| 5 | 1 | 0 | ″ | ″ | ″ | ″ | 3 |
| 6 | 0 | 0 | ″ | ″ | ″ | ″ | 4 |
| 7 | 1 | 0 | ″ | ″ | ″ | ″ | 5 |
| 8 | 1 | 0 | ″ | ″ | ″ | ″ | 6 |
| 9 | 1 | 0 | ″ | ″ | ″ | ″ | 7 |

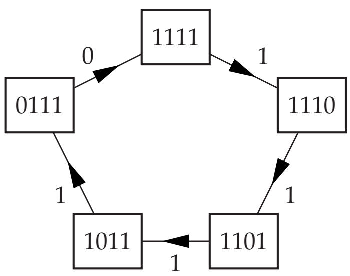
(c)

| $i$ | $s_i$ | $\delta$ | $T(D)$ | $C(D)$ | $L$ | $C_p(D)$ | $l$ |
|---|---|---|---|---|---|---|---|
| – | – | – | – | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | $1 \oplus D$ | 1 | ″ | 1 |
| 1 | 0 | 1 | $1 \oplus D$ | 1 | ″ | ″ | 2 |
| 2 | 1 | 1 | 1 | $1 \oplus D^2$ | 2 | ″ | 1 |
| 3 | 0 | 0 | ″ | ″ | ″ | ″ | 2 |
| 4 | 1 | 0 | ″ | ″ | ″ | ″ | 3 |
| 5 | 0 | 0 | ″ | ″ | ″ | ″ | 4 |
| 6 | 1 | 0 | ″ | ″ | ″ | ″ | 5 |
| 7 | 0 | 0 | ″ | ″ | ″ | ″ | 6 |
| 8 | 1 | 0 | ″ | ″ | ″ | ″ | 7 |
| 9 | 1 | 1 | $1 \oplus D^2$ | $1 \oplus D^2 \oplus D^7$ | 8 | $1 \oplus D^2$ | 1 |

## 7.25

The output sequence from the LSFR is $s = [11110]^\infty$. Thus $S(D) = (1 \oplus D \oplus D^2 \oplus D^3)/(1 \oplus D^5)$. First we can try to find a shorter LFSR by using either Euclid's algorithm for polynomials or the Berlekamp-Massey algorithm with stop criterion $i = N + L$. However, by computations we see that $\gcd(1 \oplus D \oplus D^2 \oplus D^3, 1 \oplus D^5) = 1 \oplus D$ and $(1 \oplus D^5)/(1 \oplus D) = 1 \oplus D \oplus D^2 \oplus D^3 \oplus D^4$ which is the coupling polynomial in the original LFSR. Thus no shorter LFSR exists.

From the starting state, we get the following (reachable) state transitions graph where the labels on the brances are the outputs



We now try to find a (non-linear) sequential circuit. Note that the following solution is not unique.

From the table we see that only five states are reachable from the starting state, so we need $\lceil \log_2 5 \rceil = 3$ state variables. For example we can use the following mapping

| $q_0q_1q_2$ | $q_0^+q_1^+q_2^+$ | $u$ |
|---|---|---|
| 100 | 101 | 1 |
| 101 | 111 | 1 |
| 111 | 110 | 1 |
| 110 | 000 | 1 |
| 000 | 100 | 0 |

That is, $q_0q_1q_2 = 100$ corresponds to the state 1111, $q_0q_1q_2 = 101$ to 1110 and so on. Then we have

$$q_0^+ = q_1' \vee q_2$$
$$q_1^+ = q_2$$
$$q_2^+ = q_0 \vee q_1'$$
$$u = q_0$$

The circuit layout is omitted.

**7.26**

a) The output for a given state is the left-most bit. A state with a 1 as output is the inverse of a state with a 0 as output. Therefore, we conclude that half of the states give output 1 and half of the states give output 0. A cycle in a maximal-length sequence run through all possible non-zero states. Hence, there are $2^{m-1}$ 1s and $2^{m-1} - 1$ 0s.

b) Since $(a \oplus b)^2 = a^2 \oplus b^2$ we can write the sequence as

$$S_2(D) = \frac{1}{(C(D))^2} = \left(\frac{1}{C(D)}\right)^2 = (S(D))^2 = (s_0 \oplus s_1 D \oplus s_2 D^2 \oplus \cdots)^2 = s_0 \oplus s_1 D^2 \oplus s_2 D^4 \oplus \cdots$$

or, equivalently,

$$\boldsymbol{s}_2 = s_0 0 s_1 0 s_2 0 \ldots$$

Hence, the period for $\boldsymbol{s}_2$ is $T_2 = 2T = 2^{m+1} - 2$. The number of ones is still $2^{m-1}$. Since there is one 0 inserted afeter every bit in $\boldsymbol{s}$ the number of 0s is $2^{m-1} - 1 + 2^m - 1 = 3 \cdot 2^{m-1} - 2$.

c) Since $C(D)$ is irreducible and that $\deg(C(D) > 2$ we have that $\gcd(C(D), 1 \oplus D) = 1$. By using the hint we get

$$S_3(D) = \frac{1}{C(D)(1 \oplus D)} = \frac{C(D)A(D) \oplus (1 \oplus D)B(D)}{C(D)(1 \oplus D)}$$

From $\deg(A(D)) < \deg(1 \oplus D)$ we get that $A(D) = 1$. Then we can rewrite the above expression as

$$S_3(D) = \frac{C(D) \oplus (1 \oplus D)B(D)}{C(D)(1 \oplus D)} = \frac{B(D)}{C(D)} \oplus \frac{1}{(1 \oplus D)}.$$

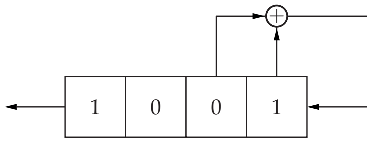
Since $\deg(B(D)) < \deg(C(D)$ and that $S(D)$ is a maximal-length sequence we know that $\frac{B(D)}{C(D)}$ is a shift of $S(D)$. We also know that $\frac{1}{(1 \oplus D)}$ is the all one sequence. Therefore, the sequence $S_3(D)$ must be the inverse of a shift of $S(D)$ and that will have the same period as $S(D)$, $T_3 = 2^m - 1$. The number of 1s and zeros is $2^{m-1} - 1$ and $2^{m-1}$, respectively.

**7.27**

a) Use the Berleycamp-Massey algorithm to find (one of) the shortest linear feedback shift register that gives the output $\boldsymbol{u} = 1001101$.

| $i$ | $s_i$ | $\delta$ | $T(D)$ | $C(D)$ | $L$ | $C_p(D)$ | $l$ |
|---|---|---|---|---|---|---|---|
| - | - | - | - | $1$ | 0 | $1$ | 1 |
| 0 | 1 | 1 | $1$ | $1 \oplus D$ | 1 | $1$ | 1 |
| 1 | 0 | 1 | $1 \oplus D$ | $1$ | | | 2 |
| 2 | 0 | 0 | | | | | 3 |
| 3 | 1 | 1 | $1$ | $1 \oplus D^3$ | 3 | | 1 |
| 4 | 1 | 1 | $1 \oplus D^3$ | $1 \oplus D \oplus D^3$ | | | 2 |
| 5 | 0 | 1 | $1 \oplus D \oplus D^3$ | $1 \oplus D \oplus D^2 \oplus D^3$ | | | 3 |
| 6 | 1 | 1 | $1 \oplus D \oplus D^2 \oplus D^3$ | $1 \oplus D \oplus D^2$ | 4 | $1 \oplus D \oplus D^2 \oplus D^3$ | 1 |
| 7 | Stop | | | | | | |

Hence, the shortest LFSR that generates the sequence $s$ ones is

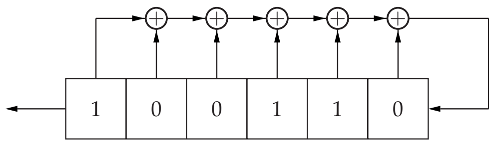



The starting state is marked in the figure.

b) To get a minimal LFSR that generates the sequence cyclically we continue the BM algorithm until $i = N + L$.

| $i$ | $s_i$ | $\delta$ | $T(D)$ | $C(D)$ | $L$ | $C_p(D)$ | $l$ |
|---|---|---|---|---|---|---|---|
| 6 | 1 | 1 | $1 \oplus D \oplus D^2 \oplus D^3$ | $1 \oplus D \oplus D^2$ | 4 | $1 \oplus D \oplus D^2 \oplus D^3$ | 1 |
| 7 | 1 | 0 | | | | | 2 |
| 8 | 0 | 0 | | | | | 3 |
| 9 | 0 | 1 | $1 \oplus D \oplus D^2$ | $1 \oplus D \oplus D^2 \oplus D^3 \oplus D^4 \oplus D^5 \oplus D^6$ | 6 | $1 \oplus D \oplus D^2$ | 1 |
| 10 | 1 | 0 | | | | | 2 |
| 11 | 1 | 0 | | | | | 3 |
| 12 | 0 | 0 | | | | | 4 |
| 13 | Stop | | | | | | |

This gives that the following LFSR is of minimal length




c) We can use the result in b) to derive the gcd. First we notice that

$$S(D) = \frac{1 \oplus D^3 \oplus D^4 \oplus D^6}{1 \oplus D^7} = \frac{P(D)}{1 \oplus D \oplus D^2 \oplus D^3 \oplus D^4 \oplus D^5 \oplus D^6}$$

where $\gcd(P(D), 1 \oplus D \oplus D^2 \oplus \cdots \oplus D^6) = 1$. Since

$$(1 \oplus D \oplus D^2 \oplus \cdots \oplus D^6)(1 \oplus D) = 1 \oplus D^7$$

we have that $\gcd(1 \oplus D^7, 1 \oplus D^3 \oplus D^4 \oplus D^6) = 1 \oplus D$.

## 7.28

Use the Berlecamp-Massy algorithm to find the linear complexity:

| $i$ | $s_i$ | $\delta$ | $T(D)$ | $C(D)$ | $L$ | $C_p(D)$ | $l$ |
|---|---|---|---|---|---|---|---|
| – | – | – | – | $1$ | 0 | $1$ | 1 |
| 0 | 0 | 0 | | | | $1$ | 2 |
| 1 | 1 | 1 | $1$ | $1 + D^2$ | 2 | | 1 |
| 2 | 1 | 1 | $1 + D^2$ | $1 + D + D^2$ | | | 2 |
| 3 | 0 | 0 | | | | | 3 |
| 4 | 0 | 1 | $1 + D + D^2$ | $1 + D + D^2 + D^3$ | 3 | $1 + D + D^2$ | 1 |
| 5 | 1 | 0 | | | | | 2 |
| 6 | 0 | 1 | $1 + D + D^2 + D^3$ | $1 + D + D^4$ | 4 | $1 + D + D^2 + D^3$ | 1 |
| 7 | 0 | 0 | | | | | 2 |

Since the last $L$ is $4$ the linear complexity is $L_l(\boldsymbol{s}) = 4$. To get the cyclic complexity we continue with the algorithm to see if the minimum length of an LFSR that generates the sequence cyclically also is $4$. To get the cyclic recursion we must continue until $i = N + L$.

| $i$ | $s_i$ | $\delta$ | $T(D)$ | $C(D)$ | $L$ | $C_p(D)$ | $l$ |
|---|---|---|---|---|---|---|---|
| 7 | 0 | 0 | - | $1 + D + D^4$ | 4 | $1 + D + D^2 + D^3$ | 2 |
| 8 | 0 | 0 | | | | | 3 |
| 9 | 1 | 0 | | | | | 4 |
| 10 | 1 | 0 | | | | | 5 |
| 11 | 0 | 1 | $1 + D + D^4$ | $1 + D + D^4 + D^5 + D^6 + D^7 + D^8$ | 8 | $1 + D + D^4$ | 1 |

In step 11 the complexity grows to 8, and since it will not decrease, and the length of $\boldsymbol{s}$ is 8, we conclude that this will also be the result. Hence, the two complexities are not equal.

(If we would have continued the algorithm until $i = 16$ we get the feedback polynomial

$$C(D) = 1 + D^2 + D^3 + D^4 + D^5 + D^7 + D^8$$

and $L = 8$. Since we also have the obvious polynomial $C(D) = 1 + D^8$ we see an example that there can be several LFSRs of the same length that solves the problem.)

**7.29**

See solution to Problem 5.24.