Introduction. Lossless and lossy coding

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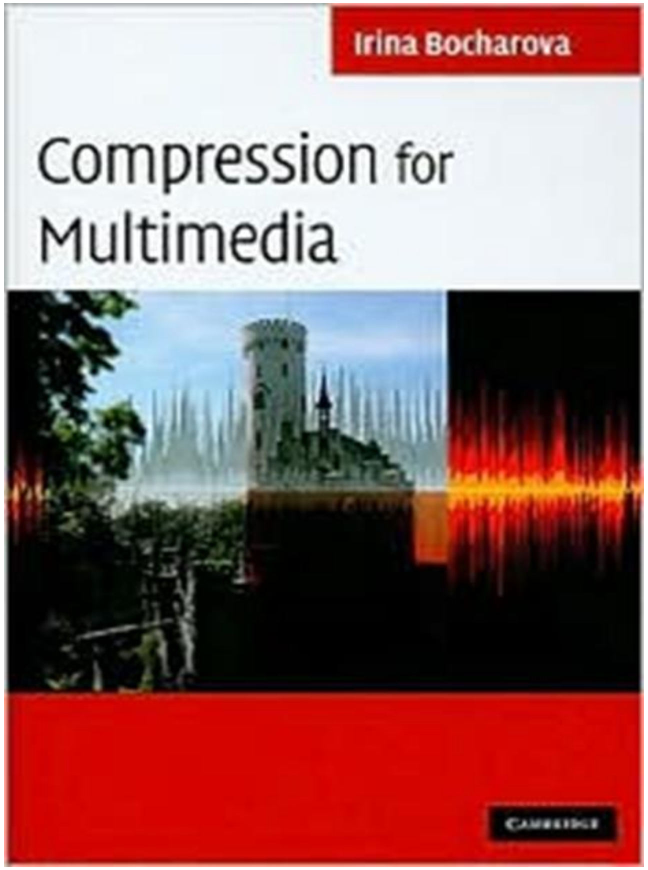
ACM International Collegiate Programming Contest Winners (Wikipedia)

Wins	Country	Institution	Most Recent
5	Russia	Saint Petersburg State University of Information Technologies, Mechanics and Optics	2013,2012, 2009, 2008, 2004
3	China	Shanghai Jiao Tong University	2010
3	United States	Stanford University	1991
2	Canada	University of Waterloo	1999
2	Poland	University of Warsaw	2007
2	Russia	Saint Petersburg State University	2001
2	United States	California Institute of Technology	1988
2	United States	Washington University in St. Louis	1980

ITMO


ICPC
2012
University
of Warsaw
Programming Contest
Europe Champions
2012
ACM-ICPC
Programming Contest
World Champion
2012



I. Bocharova *Compression for Multimedia*, Cambridge Univ. Press, 2010

- Multimedia data such as images, audio, video, speech are originally analog waveforms.

  The first problem is how to convert them into digital form to get multimedia data.
- The number of bits required to store multimedia data is large and this circumstance limits efficiency of the digital systems.

  The second problem is to compress multimedia data in order to transmit them faster and to store them more efficiently.

Examples

Without compression:

One picture obtained by 4 Mpixels digital camera requires 12 Mbytes of memory, that is, only 21 pictures can be stored in 256 Mbytes CF memory

(12 Mpixels camera, 16 Gbyte CF memory, 444 pictures).

Video sequence containing 30 frames (1sec movie) of size 480×720 takes $480\times720\times30\times3\cong31$Mbytes of memory or only 21sec of this movie can be recorded into a CD of capacity 650 Mbytes

(4.7 Gbyte DVD, 2.5 min., 15.9 Gbyte DVD, 8 min. of the movie).

Examples

Transmitting color image of size 352×288 pixels (each pixel takes 3 bytes and the image takes 2.4Mb) through the plain old telephone service (POTS) networks using modem with rate 33600 b/s requires approximately 1minute (72 s.).

(HDSL 10 Mb/s, 0.24 s, 4.0 frame/s.)

Transmitting 1 second of speech sampled at 8 kHz with 16 bits per sample using the same modem (33600 b/s) requires approximately 4 seconds.

New tendencies in multimedia compression

- Digital multimedia broadcasting (DMB)

(digital TV in mobile phones and laptops)

- Developing different portable devices such as: iphones, smartphones, iPADs, portable media players (ipods) intended for loading multimedia content.
- HDR imaging
- Recompression of multimedia content represented in “old’’ digital standards.
- Compression for sensor networks

Multimedia systems

Lossless compression systems

- Huffman coding
- Arithmetic coding
- Coding based on Ziv-Lempel algorithms etc.

Lossy compression systems

- Coding of the DFT coefficients
- Coding of DCT coefficients
- Wavelet filtering
- Differential coding

About the course

- Introduction
- Parts 2 and 3 give the basic theoretical aspects of lossy source coding
- Parts 4,5, and 6 describe commonly used coding techniques
- Parts 7,8,9, and 10 are devoted to modern standards for speech, image, video and audio compression, respectively.

Analog and digital signals

- A function $f(x)$ is *continuous at a point* $x = a$

if $\boxed{\lim_{x \to a} f(x) = f(a).}$

- A function is *continuous* if it is continuous at every point in its domain.
- A set of elements is a *discrete* set if it contains a *finite or countable* number of elements (elements of countable set can be enumerated).
- In the real world *analog signals* are *continuous* functions of *continuous* arguments.

Analog and digital signals

Discrete signals can be discrete over a set of function values and (or) over a set of argument values.

If an analog time signal is *sampled* we call this set of numbers a *discrete time or sampled* signal.

If the sample values are constrained to belong to a *discrete* set the signal becomes *digital.*

Analog to digital conversion

Images, speech, audio and video are originally analog signals.

To convert them into digital signals we should perform the following two operations:

- First the signal has to be sampled (time axis must be discretized or quantized).
- The second operation is to transform the sample values in such a manner that each resulting number belongs to a discrete set (alphabet). This operation is called quantization.

Analog to digital conversion

Let $x(t)$ be a continuous time function.

Sampling is taking samples of $x(t)$ at time instants $t = nT_s$ for all integer n, where T_s is sampling period, $f_s = 1/T_s$ is called sampling frequency.

Instead of $x(t)$ we obtain $x(nT_s)$

1. The question is: does sampling introduce distortion of the original continuous function?
2. How does the distortion, if any, depend on the value of T_s?

Analog to digital conversion

The forward and the inverse Fourier transform of $x(t)$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad x(t) = \int_{-\infty}^{\infty} X(f) e^{j\omega t} df$$

If $x(t)$ is a periodic function with period p it has the Fourier series expansion

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt\omega_p) + b_k \sin(kt\omega_p)$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{jkt\omega_p}, \quad c_k = \frac{1}{p} \int_{-p/2}^{p/2} x(t) \exp(-jkt\omega_p) dt,$$

$$\omega_p = 2\pi / p.$$

Analog to digital conversion

It is said that the Fourier series expansion for a periodical function with period p decomposes this function into a sum of harmonical functions with frequencies $k\omega_p$, $k = 1,2,\ldots$

The Fourier transform for nonperiodical function represents it as a sum of infinite number of harmonical functions with frequencies which differ in infinitesimal quantities.

A nonperiodical function of length T has the Fourier series expansion for its periodical continuation with period T.

Sampling theorem

If $x(t)$ *is a signal whose Fourier transform is identically zero* $X(f)=0$ *for* $|f|>f_H$ *then* $x(t)$ *is completely determined by its samples taken every* $1/(2f_H)$ *seconds*.

Proof

Let $\hat{X}(f)$ be a periodical continuation $X(f)$ with period $2f_H$

$$\hat{X}(f)=\sum_{k=-\infty}^{\infty} a_k e^{j2\pi fk/(2f_H)}, \quad a_k=\frac{1}{2f_H}\int_{-f_H}^{f_H}\hat{X}(f)e^{-j2\pi fk/(2f_H)}df$$

Sampling theorem

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df = \int_{-f_H}^{f_H} \hat{X}(f) e^{j2\pi ft} df$$

Consider samples of $x(t)$ in the discrete points $t = k/(2f_H)$

$$x\left(\frac{k}{2f_H}\right) = \int_{-f_H}^{f_H} \hat{X}(f) e^{j2\pi fk/(2f_H)} df$$

Comparing with formula for coefficient a_k

we obtain $$a_k = \frac{1}{2f_H} x\left(\frac{-k}{2f_H}\right)$$

Sampling theorem

If $x(t)$ is known at points $-2/(2f_H),\ -1/(2f_H),\ 0,$ $1/(2f_H),\ 2/(2f_H)\ldots$ then the coefficients a_k are determined. They in turn determine $\hat{X}(f)$ and thereby $X(f)$.

On the other hand $X(f)$ determines $x(t)$ for all values of t.

It means that there exists a unique time function which does not contain frequencies higher than f_H and passes through the given sampling points spaced $1/(2f_H)$ seconds.

Sampling theorem

$$x(t) = \int_{-f_H}^{f_H} X(f) e^{j2\pi f t} df = \int_{-f_H}^{f_H} \sum_k a_k e^{j2\pi f\left(\frac{k}{2f_H}+t\right)} df = \sum_k a_k \int_{-f_H}^{f_H} e^{j2\pi f\left(\frac{k}{2f_H}+t\right)} df$$

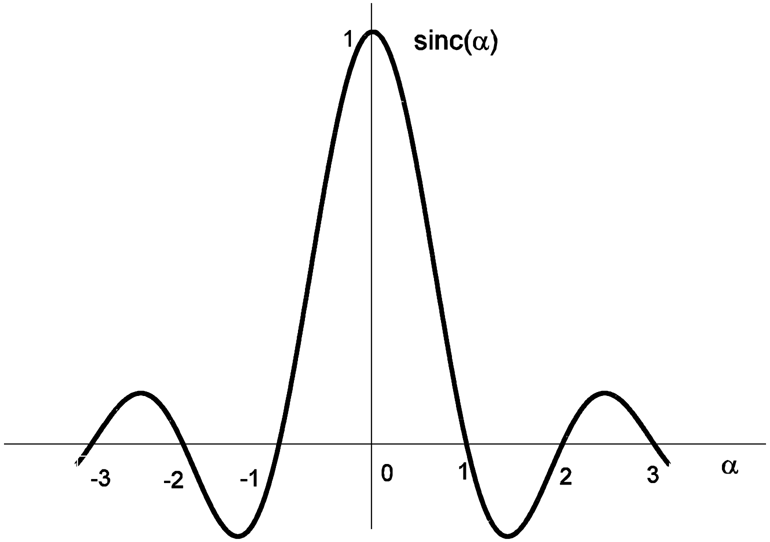
$$= 2\sum_k a_k \int_0^{f_H} \cos 2\pi f\left(\frac{k}{2f_H}+t\right) df = 2\sum_k a_k \left.\frac{\sin 2\pi f\left(\frac{k}{2f_H}+t\right)}{2\pi\left(\frac{k}{2f_H}+t\right)}\right|_0^{f_H}$$

$$x(t) = 2f_H \sum_{k=-\infty}^{\infty} a_k \operatorname{sinc}\left(2f\left(\frac{k}{2f_H}+t\right)\right)\Bigg|_0^{f_H}, \quad \operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$x(t) = \sum_{i=-\infty}^{\infty} x\left(\frac{i}{2f_H}\right) \operatorname{sinc}(2f_H t - i), \; i = -k$$

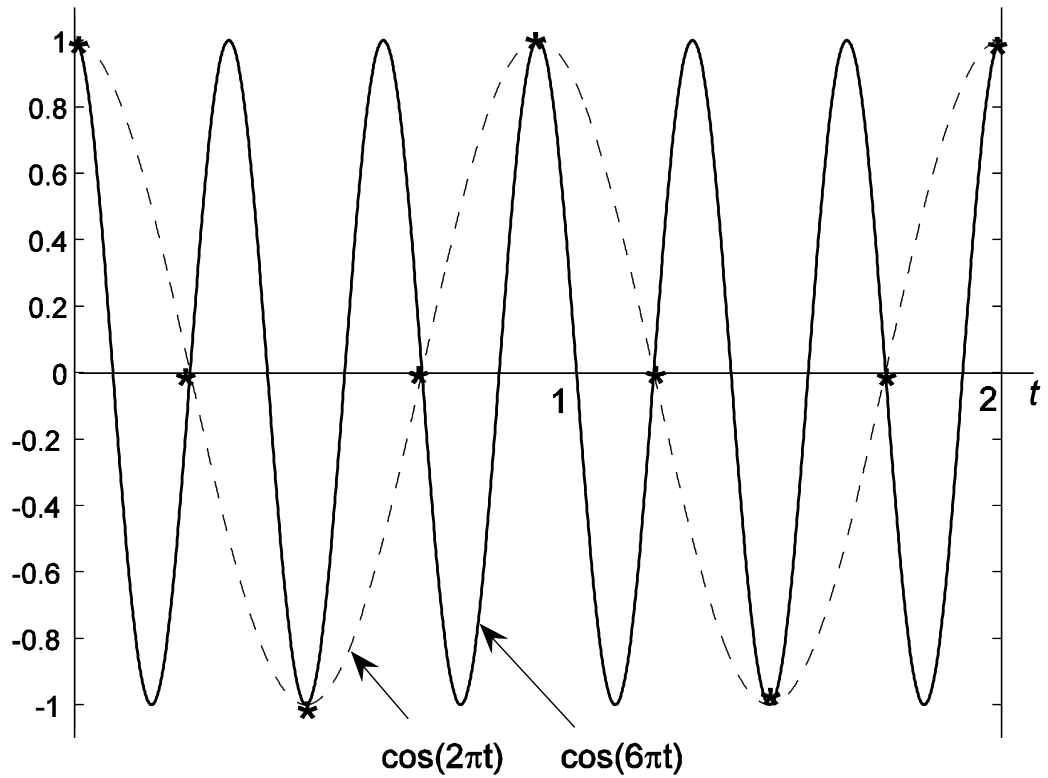
Sampling theorem

The time function $x(t)$ can be represented as a sum of elementary functions in the form $\operatorname{sinc}(x)$ centered in the sampling points. In order to reconstruct this function it is necessary to generate a train of impulses which have form $\operatorname{sinc}(x)$ and are proportional to samples, and to summarize them.



Aliasing

The error caused by sampling too slowly is called aliasing. The 3Hz signal sampled at 4Hz is disguising itself as a 1Hz signal.



Aliasing

The sampling yields $x_s(t) = \sum_{k=-\infty}^{k=\infty} x(kT_s)\delta(t-kT_s).$

Since $\delta(t-kT_s)$ is equal to 0 everywhere except at the point kT_s

$$x_s(t) = x(t)\sum_{k=-\infty}^{\infty}\delta(t-kT_s) \qquad (1.1)$$

$$\sum_{k=-\infty}^{\infty}\delta(t-kT_s) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi tn/T_s}, \; c_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2}\delta(t)e^{-j2\pi tn/T_s}dt = \frac{1}{T_s} \quad (1.2)$$

Substituting (1.2) into (1.1)

$$x_s(t) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} x(t)e^{j2\pi tn/T_s}.$$

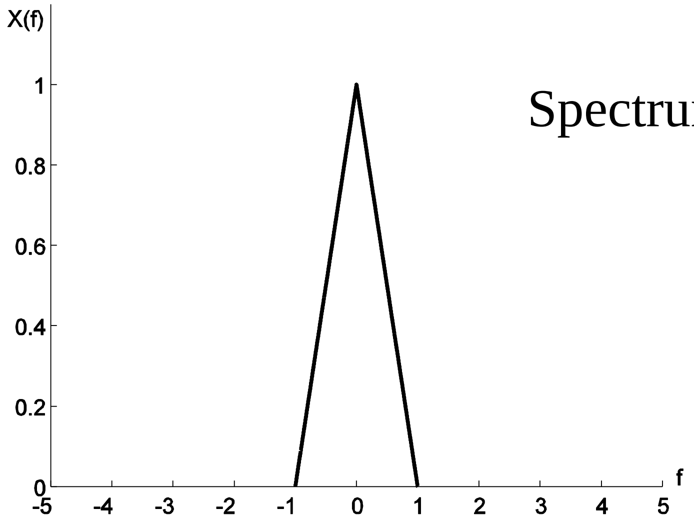
Multiplication by $\exp(j2\pi\alpha t)$ corresponds to the shift of the Fourier transform by α

$$X_s(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X\left(f-\frac{n}{T_s}\right)$$

Aliasing

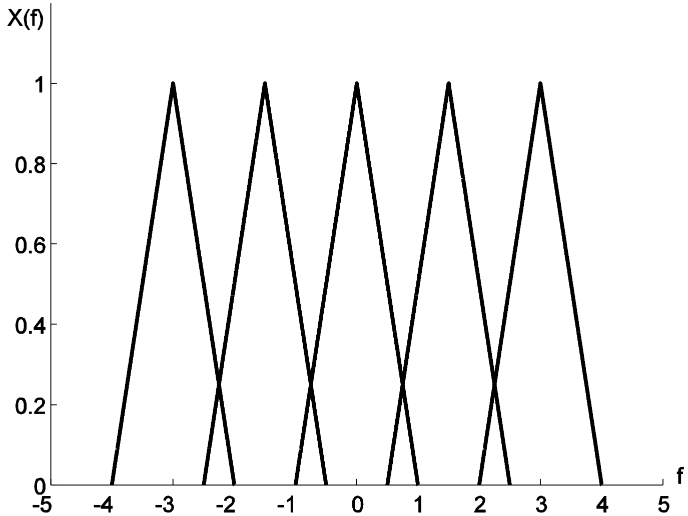
- If the sampling of the signal is too slow the copies of its spectrum overlap which leads to an aliasing error.
- To reconstruct the signal from the sequence of its samples it is sufficient to filter this sequence by the ideal low-pass filter with cut-off frequency equal to $f_H = 1/(2T_s)$.

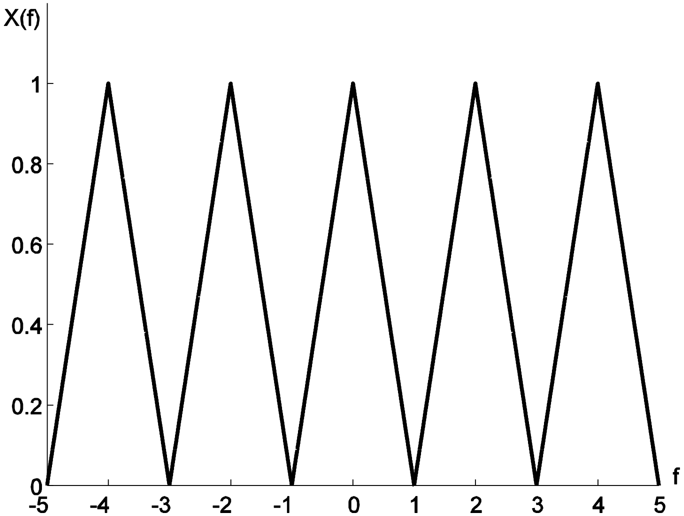
Aliasing



Spectrum of the original signal

Spectra of the sampled signal





Aliasing

The Fourier transform of the reconstructed signal

$$X(f) = X_s(f)T_s H_L(f), \quad H_L(f) = \begin{cases} 1, & \text{if } |f| \le f_H \\ 0, & \text{otherwise} \end{cases}$$

The impulse response of the ideal low-pass filter

$$h_L(t) = \int_{-\infty}^{\infty} H_L(f) e^{j2\pi ft} df = 2\int_0^{f_H} \cos(2\pi ft)\, df = \frac{1}{T_s}\operatorname{sinc}(t/T_s).$$

The reconstructed signal is sum of weighted and shifted copies of impulse response

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT_s)\operatorname{sinc}\left(\frac{t-kT_s}{T_s}\right).$$

Aliasing

- The sampling theorem requires that samples be taken for all time instants in the infinite interval, and that every sample is used to reconstruct the value of the original function at any particular time.
- In a real system, the signal is observed during a limited time. The Fourier transform of any time limited signal is nonzero outside any finite frequency band.

How to apply sampling theorem to such signals?

Aliasing

Usually we restrict our signals to frequencies inside a certain finite band, $[-f_H, f_H]$.

We choose f_H to satisfy the condition

$$\frac{\int_{-f_H}^{f_H} |X(f)|^2 df}{\int_{-\infty}^{\infty} |X(f)|^2 df} = \alpha ,$$

where α is approximately 0.90–0.99.

We call f_H the effective bandwidth with respect to α.

Aliasing

- Sampling of time-limited signals always introduces an aliasing error due to overlapping of the copies of the signal spectrum.
- To reduce this error we pre-filter time-limited signals.
- We choose sampling period $T_s << 1/(2 f_H)$sec.

Aliasing effect demo

	Speech	Music
Original files 32 kHz		
Downsampling to 16 kHz with prefiltering		
Downsampling to 16 kHz without prefiltering		
Downsampling to 8 kHz with prefiltering		
Downsampling to 8 kHz without prefiltering		