Filter banks

Separately, the lowpass and highpass filters are not invertible.

T_0 removes the highest frequency $\alpha = 1/2$ and T_1 removes the lowest frequency $\alpha = 0$.

Together these filters separate the signal into low-frequency and high-frequency subsequences and $T_1\mathbf{x}$ is the complement of $T_0\mathbf{x}$.

It is said that these filters form a filter bank.

It can be shown that there exists an inverse filter bank or in other words the transform which splits signal into low-frequency and high-frequency components is invertible.

Decimation

The problem connected with filter banks is :

The signal length has doubled since each component has the length equal to the length of the input sequence.

The solution is to downsample (or decimate).

Decimation means removing all odd-numbered components of the filtered sequence:

$$(\downarrow 2)\begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ \dots \end{pmatrix} = \begin{pmatrix} y(0) \\ y(2) \\ y(4) \\ y(6) \\ \dots \end{pmatrix}$$

Decimation

Decimation is not invertible. The odd-numbered components are lost.

Normally the even-numbered components of both (low-frequency and high-frequency) decimated signal parts will be needed to recover all components of the original vector.

For band-limited signals we can recover the odd-numbered components from the even-numbered components.

Two steps, filtering and decimation can be done with new matrices L and B.

Filter banks

$$L = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad B = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & \cdots \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Removing half of the rows leaves the matrices with a double-shift. To compensate losing half of the components we multiply the surviving components by $\sqrt{2}$.

Filter banks

Filtering extends the length of the output signal compared to the length of the input signal. If the input vector $\mathbf{x}$ has size N then the output vector $\mathbf{y}$ has size $N+l-1$,where l denotes the length of the impulse response.

Thus the corresponding matrices T_0, T_1 are matrices of size $N \times (N+l-1).$ To avoid such extension we use the so-called circular extension of the input signal. In this case the convolution

$$y(n) = \sum_{k=0}^{n} h(k)x(n-k) = \sum_{k=0}^{n} h(n-k)x(k)$$

can be considered as a circular convolution and T_0, T_1 become of size $N \times N$. We will consider matrices corresponding to the circular convolution.

Filter banks

The rectangular L and B fit into a square matrix which represents the whole analysis filter bank:

$$\begin{bmatrix} L \\ B \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & \ldots & & & \\ & & 1 & 1 & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ -1 & 1 & \ldots & & & .. \\ & & -1 & 1 & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

It executes the lowpass channel and the highpass channel (both decimated). The combined square matrix is invertible.

Filter banks

The inverse is the transpose:

$$\begin{bmatrix} L \\ B \end{bmatrix}^{-1} = \begin{bmatrix} L^T & B^T \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & . & -1 & & . \\ 1 & & . & 1 & & . \\ & 1 & . & & -1 & . \\ & 1 & . & & 1 & . \\ & & . & & & . \\ & & . & & & . \end{pmatrix}$$

The second matrix $\begin{bmatrix} L^T & B^T \end{bmatrix}$ is the synthesis filter bank.

This is an orthogonal filter bank, because inverse=transpose.

Filter banks

It is evident that: $$\begin{bmatrix} L^T & B^T \end{bmatrix} \begin{bmatrix} L \\ B \end{bmatrix} = L^T L + B^T B = I.$$

The synthesis bank is the transpose of the analysis bank.

When one follows the other we have perfect reconstruction.

The analysis bank had two steps: filtering and downsampling. The synthesis bank also can be organized to have two steps: upsampling and filtering.

The first step is to reconstruct full-length vectors. The odd-numbered components are returned as zeros by upsampling. Upsampling is denoted by $(\uparrow 2)$.

Filter banks

Applied to a half-length vector $\mathbf{y}$ it inserts zeros:

$$\tilde{\mathbf{y}} = (\uparrow 2)\begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ \cdots \\ \cdots \end{pmatrix} = \begin{pmatrix} y(0) \\ 0 \\ y(1) \\ 0 \\ y(2) \\ 0 \\ \cdots \end{pmatrix}$$

The second step in the synthesis bank is filtering. The lowpass filtering is equivalent to multiplying the extended vector by the matrix :

$$S_0 = \sqrt{2}\begin{pmatrix} \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ 0 & 0 & \frac{1}{2} & \cdots \\ \cdots & \cdots & 0 & \cdots \end{pmatrix}$$

Filter banks

The highpass filtering is equivalent to multiplying the extended vector by matrix

$$S_1 \doteq \sqrt{2}\begin{pmatrix} -\frac{1}{2} & 0 & 0 & \dots \\ \frac{1}{2} & -\frac{1}{2} & 0 & \dots \\ 0 & \frac{1}{2} & -\frac{1}{2} & \dots \\ 0 & 0 & \frac{1}{2} & \dots \\ \dots & \dots & 0 & \dots \end{pmatrix}.$$

The synthesis lowpass filter is: $y(n) = \frac{1}{\sqrt{2}}\left(x(n) + x(n-1)\right)$

The synthesis highpass filter is: $y(n) = \frac{1}{\sqrt{2}}\left(x(n-1) - x(n)\right).$

This is the Haar filter bank.

Example

Let $x(n) = x(0),\ x(1), x(2), x(3), x(4), x(5)$ be input sequence of the the Haar filter bank.

Output of the lowpass filter is:

$$y_0(n) = x(0), x(0)+x(1), x(1)+x(2), x(2)+x(3), x(3)+x(4), x(4)+x(5), x(5)$$

Output of the highpass filter is:

$$y_1(n) = x(0), x(1)-x(0), x(2)-x(1), x(3)-x(2), x(4)-x(3), x(5)-x(4), -x(5)$$

Decimated output of the lowpass filter is:

$$v_0(n) = x(0),\ x(1)+x(2),\ x(3)+x(4),\ x(5)$$

Decimated output of the highpass filter is:

$$v_1(n) = x(0),\ x(2)-x(1),\ x(4)-x(3), -x(5)$$

Example

The extended low-frequency part is:

$$\tilde{y}_0(n) = x(0), 0, x(1)+x(2), 0, x(3)+x(4), 0, x(5)$$

The extended high-frequency part is:

$$\tilde{y}_1(n) = x(0),\ 0, x(2)-x(1), 0,\ x(4) - x(3), 0, -x(5)$$

Lowpass filtered $\tilde{y}_0$ has the form

$$w_0(n) = x(0), x(0), x(1)+x(2), x(1)+x(2), x(3)+x(4), x(3)+x(4)$$

Highpass filtered $\tilde{y}_1$ has the form

$$w_1(n) = -x(0), x(0), x(1)-x(2), x(2)-x(1), x(3)-x(4),\ x(4) - x(3)$$

Summing up $w_0(n)$ and $w_1(n)$ after normalization we get

$$\tilde{x}(n) = 0, x(0), x(1), x(2), x(3), x(4) = x(n-1).$$

Wavelet filtering

The most important feature of the wavelet filter banks is their hierarchical structure.

The recursive nature of wavelets leads to a tree structure of the corresponding filter bank.

The input sequence is decomposed into the so-called reference (low-frequency) subsequences with diminishing resolutions and related with them the so-called detail (high-frequency) subsequences.

At each level of decomposition the wavelet filtering is invertible, that is, the reference signal of this level together with the corresponding detail signal provide perfect reconstruction of the reference signal of the next level (with higher resolution)

Wavelet filtering

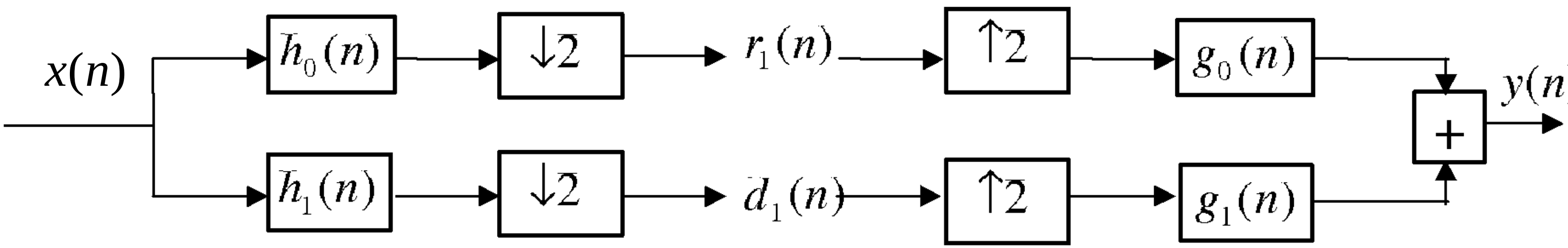


Fig.8.1 One level of wavelet decomposition followed by reconstruction

Wavelet filtering

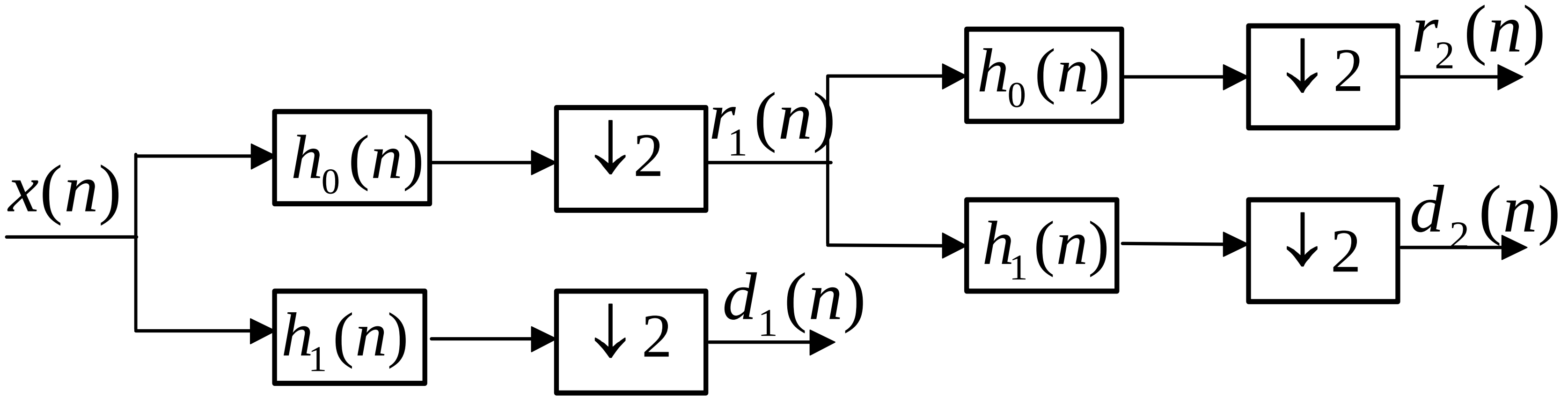


Fig.8.2 Multiresolution wavelet decomposition

Wavelet filtering

In the theory of wavelet filter banks such pairs of filters $h_0(n)$ and $h_1(n)$ are found that there exist pairs of the inverse filters $g_0(n)$ and $g_1(n)$ providing the perfect reconstruction of the input signal.

The wavelet filtering provides perfect reconstruction of the input signal, that is, the output signal is determined as follows:

$$\boxed{y(n) = Ax(n-\alpha)} \;,$$

where A is the gain factor, and α is the delay.

At the $L-th$ level of decomposition we obtain $r_L(n)$ with resolution 2^L times scaled down compared to the resolution of the input signal and $d_L(n), d_{L-1}(n)$,…, $d_1(n)$ with resolution 2^j

$j = L, L-1,...,1$ times scaled down compared to the input signal.

Wavelet filtering

At the $L-th$ level of decomposition the total length of reference and detail subsequences is :

$$2^{-L}N+2^{-L}N+2^{-(L-1)}N+2^{-(L-2)}N+...+2^{-1}N=$$

$$=N\left(\sum_{i=1}^{L}2^{-i}+2^{-L}\right)=N\left(2^{-1}\frac{1-2^{-L}}{1-2^{-1}}+2^{-L}\right)=N$$

The synthesis wavelet filter banks always consist of FIR linear phase filters that is very convenient from the implementation point of view.

Wavelet and scaling function

The remarkable properties of wavelet filter banks follow from the dilation equation and the wavelet equation which connect pulse responses of wavelet filters and special functions called the wavelet and the scaling function:

$$\varphi_A(t) = \sqrt{2}\sum_n h_0(n)\varphi_A(2t-n), \qquad (8.1)$$

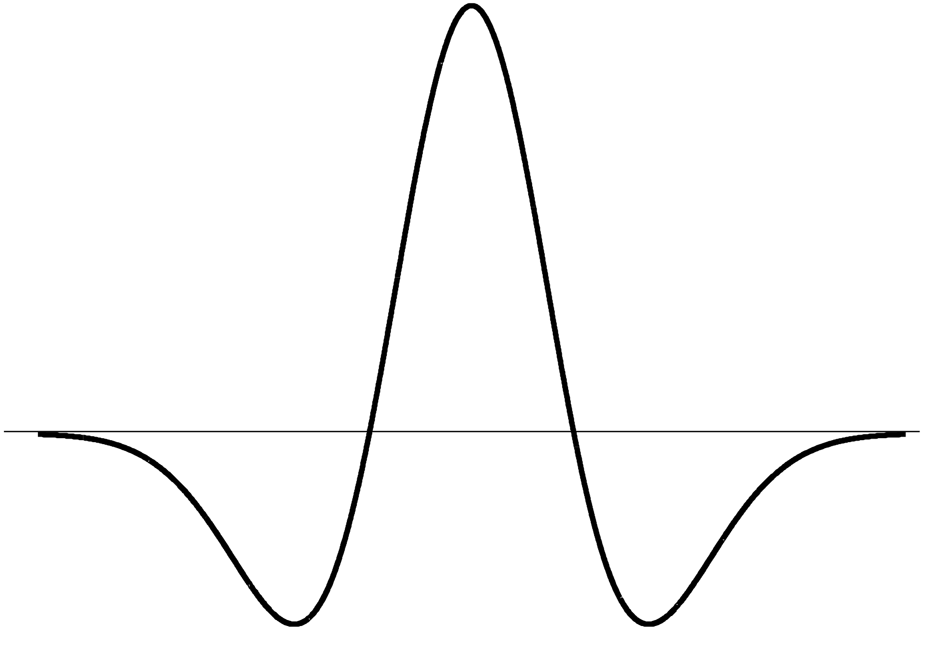
$$\varphi_S(t) = \sqrt{2}\sum_n g_0(n)\varphi_S(2t-n), \qquad (8.2)$$

$$\psi_A(t) = \sqrt{2}\sum_n h_1(n)\varphi_A(2t-n), \qquad (8.3)$$

$$\psi_S(t) = \sqrt{2}\sum_n g_1(n)\varphi_s(2t-n), \qquad (8.4)$$

where $\varphi_A(t)$, $\varphi_S(t)$ denote scaling functions and $\psi_A(t)$, $\psi_S(t)$ are the wavelets. (8.1),(8.2) are dilation equations and (8.3),(8.4) are wavelet equations.

Typical wavelet



Wavelets

Wavelets are basis functions $\psi_{jk}(t)$ in continuous time. A basis is a set of lineary independent functions that can be used to produce all admissible functions $f(t)$:

$$f(t) = \sum_{j,k} b_{jk} \psi_{jk}(t),$$

$$b_{JK} = \int_{-\infty}^{\infty} f(t) \psi_{JK}(t) dt.$$

The wavelet transform operates in continuous time (on functions) $f(t)$ and in discrete time (on vectors) $x(n)$. The output is the set of coefficients b_{jk}. For infinite signals the basis is necessarily infinite. For finite length vectors with N components there will be N basis vectors and N coefficients. The DWT is expressed by an $N \times N$ matrix.

Wavelets

The special feature of the wavelet basis is that all functions

$\psi_{jk}(t)$ are constructed from a single mother wavelet $\psi(t)$.

This wavelet is a small wave (a pulse). Normally it starts at time $t = 0$ and ends at time $t = N$.

The shifted wavelets ψ_{0k} start at time $t = k$ and end at time

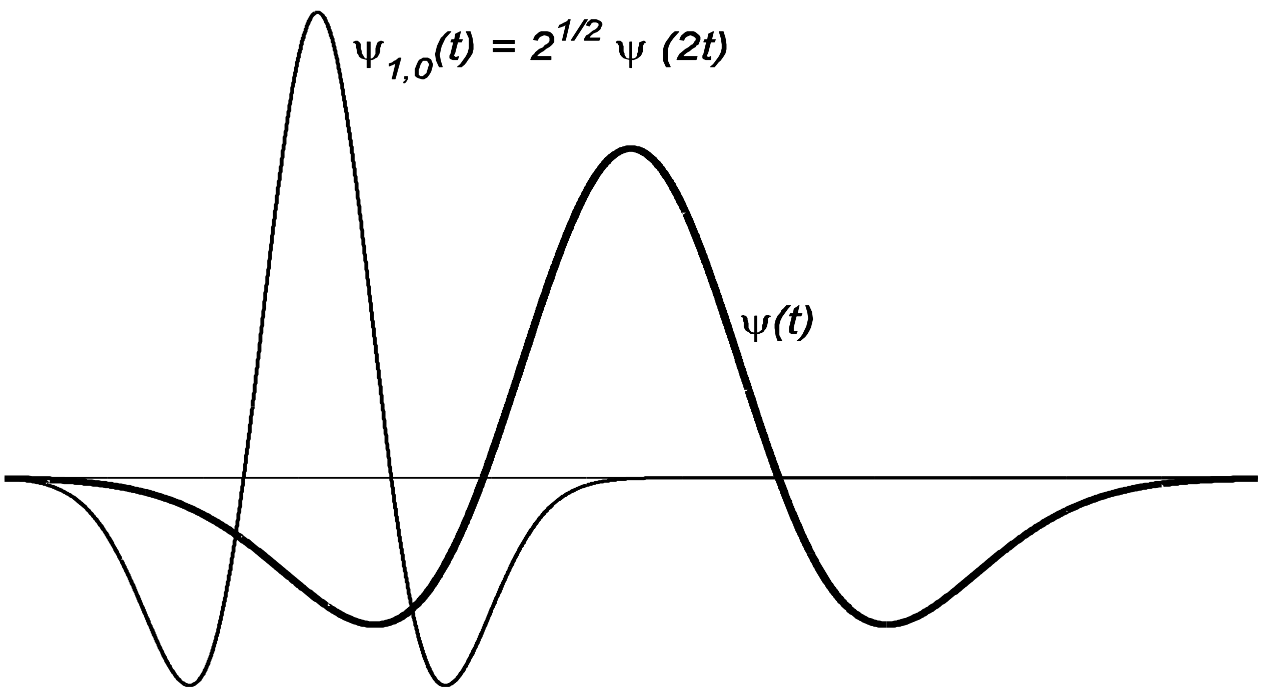
$t = k + N$.

The rescaled wavelets ψ_{j0} start at time $t = 0$ and end at time $t = N / 2^j$.

A typical wavelet ψ_{jk} is compressed 2^j times and shifted k times.

$$\psi_{jk}(t) = \psi(2^j t - k)$$

Scaled wavelet



Wavelet and scaling function

Let W_j be a subspace generated by wavelets ψ_{jk} for a given j then the following equality holds

$$L^2 = \dot{\sum_j} W_j ,$$

$$f(t) = \sum_j w_j, w_j = \sum_k b_{j,k} \psi_{jk}(t)$$

where L^2 is the space containing all functions for which $\left(\int_{-\infty}^{\infty} | f(t)|^2 dt \right)^{1/2}$ is finite and $\dot{\sum}$ is the direct sum of subspaces, that is, any function from L^2 can be represented as a sum of functions from W_j.

The scaling functions $\varphi(2^j t - k)$ with a given j are a basis for the set of signals V_j, $v_j = \sum_k a_{j,k} \varphi_{j,k}(t)$

The chain of subspaces V_j satisfies the conditions:

$$\ldots \subset V_j \subset V_{j+1} \ldots \subset L^2 ,$$

$$V_{j+1} = W_j \dot{+} V_j$$

$$f(t) = v_{J+1}(t) + \sum_{j=J+1} w_j(t)$$

Wavelet filtering

$$\varphi_{j,k}(t) = \sum_k h_0(k)\varphi_{j+1,k}(t) \qquad \psi_{j,k}(t) = \sum_k h_1(k)\varphi_{j+1,k}(t)$$

$$v_{j+1}(t) = \sum_k a_{j+1,k}\varphi_{j+1,k}(t) = \sum_k a_{j,k}\varphi_{j,k}(t) + \sum_k b_{j,k}\psi_{j,k}(t)$$

$$a_{j,k} = \int_{-\infty}^{\infty} v_{j+1}(t)\varphi_{j,k}(t)dt \qquad b_{j,k} = \int_{-\infty}^{\infty} v_{j+1}(t)\psi_{j,k}(t)dt$$

$$a_{j,k} = \sum_n h_0(n-2k)a_{j+1,k}$$

$$b_{j,k} = \sum_n h_1(n-2k)a_{j+1,k}$$

Wavelet and scaling function

When we perform the wavelet decomposition of the signal it is divided into different scales of resolution, rather than different frequencies. Multiresolution divides the frequencies into octave bands with bandwidth corresponding to the decomposition level, instead of uniform bands from ω to $\omega + \Delta\omega$

as the Fourier transform does.

At each level of the wavelet decomposition time steps are reduced with a factor of 2 and the frequency steps are increased twice or in other words $\Delta t = 2^{-j}$ and $\Delta\omega = 2^{j}$.

Haar example

For lowpass filter with $h_0(0) = \frac{1}{2}$ and $h_0(1) = \frac{1}{2}$ the dilation equation

$$\varphi(t) = \varphi(2t) + \varphi(2t-1).$$

It can be shown that the solution is the box function:

$$\varphi(t) = \begin{cases} 1, & \text{for } 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

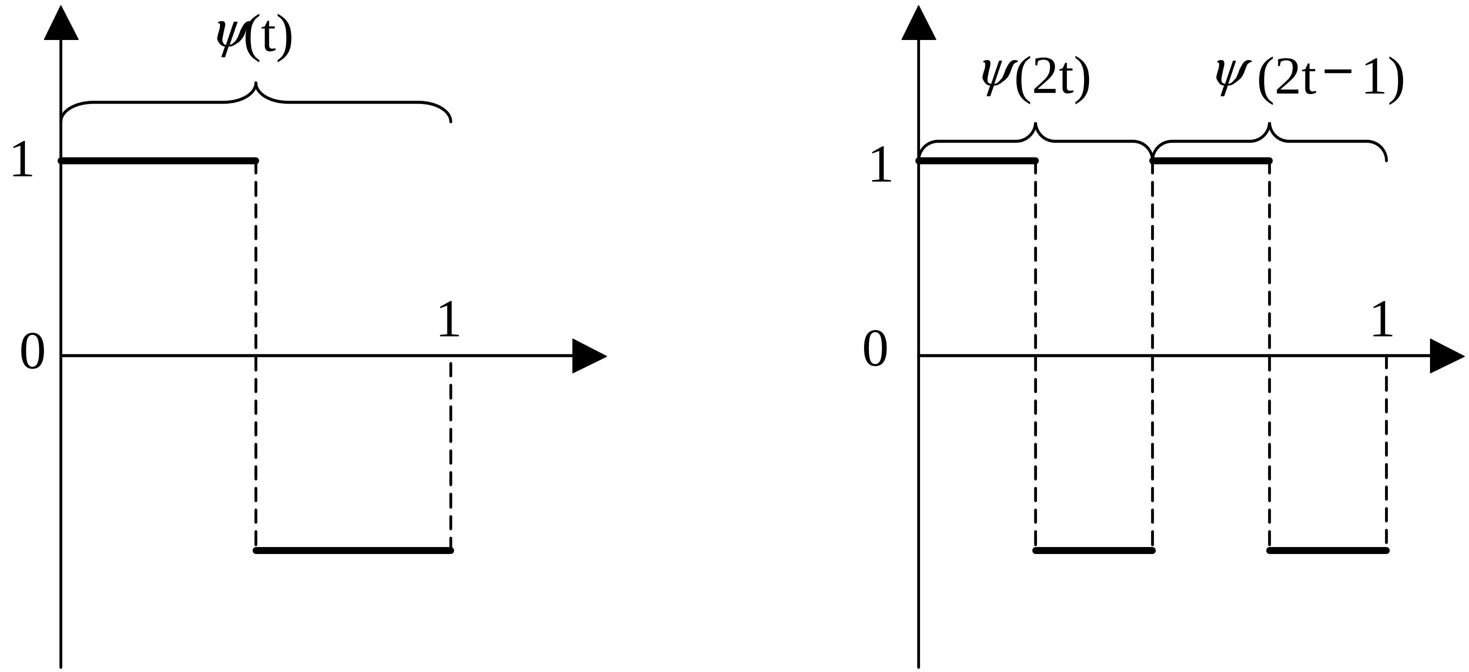
The coefficients of the wavelet equation are $h_1(0) = \frac{1}{2}$ and $h_1(1) = -\frac{1}{2}$. The wavelet is a difference of half-boxes:

$$\psi(t) = \varphi(2t) - \varphi(2t-1).$$

$$\psi(t) = \begin{cases} 1 & \text{for } 0 \le t \le 1/2 \\ -1 & \text{for } 1/2 \le t < 1 \end{cases}$$ is the Haar wavelet!

If wavelet is a small wave then the Haar wavelet is a square wave.

Haar example



Haar example

Let our input sequence $x(n)$ have length $N = 4$ then in terms of matrices the hierarchical filtering can be written as

$$\mathbf{y} = T\mathbf{x},$$

where

$$T = \begin{pmatrix} r^2 & r^2 & r^2 & r^2 \\ -r^2 & -r^2 & r^2 & r^2 \\ -r & r & 0 & 0 \\ 0 & 0 & -r & r \end{pmatrix},$$

is the matrix with rows equal to the Haar wavelets

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad \text{scaled by} \quad r = \frac{1}{\sqrt{2}}.$$

Haar example

The wavelet transform based on the Haar wavelets is orthonormal since the rescaled Haar wavelets $\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k)$ form an orthonormal basis and the inverse matrix used for synthesis represents the transposed matrix T.

We can write $V_{j+1} = V_j \oplus W_j$, where $\oplus$ denotes the orthogonal sum.

$V_j =$ all combinations $\sum_k a_{jk}\varphi_{jk}(t)$ of scaling functions at level j,

$W_j =$ all combinations $\sum_k b_{jk}\psi_{jk}(t)$ of wavelets at level j.

We have two orthogonal bases for V_{j+1}, either the φs at level $j+1$ or φ s and ψ s at level j.

Wavelet filtering

For decomposition with three levels we get

$$V_3 = V_2 \oplus W_2 = V_1 \oplus W_1 \oplus W_2 = V_0 \oplus W_0 \oplus W_1 \oplus W_2 .$$

The functions in V_0 are constant on (0,1]. The functions in

W_0, W_1, W_2 are combinations of wavelets.

The function $f(t)$ in V_3 has a piece $f_j(t)$ in each wavelet subspace W_j (plus V_0):

$$f(t) = \sum_k a_{0k}\varphi_{ok}(t) + \sum_k b_{0k}\psi_{0k}(t) + \sum_k b_{1k}\psi_{1k}(t) + \sum_k b_{2k}\psi_{2k}(t).$$