The discrete cosine transform

The DCT is a basis of all modern standards of image and video compression. The DCT was chosen as the standard solution for video compression problem because of the following reasons:

•For discrete-time signal $x(n)$ with covariance matrix in the form

$$\Lambda = \sigma^2 \begin{pmatrix} 1 & \rho & \rho^2 & \dots & \rho^{N-1} \\ \rho & 1 & \rho & \dots & \rho^{N-2} \\ \rho^2 & \rho & 1 & \dots & \rho^{N-3} \\ \dots & \dots & \dots & \dots & .. \\ \rho^{N-1} & \rho^{N-2} & \rho^{N-3} & \dots & 1 \end{pmatrix},$$

where $\rho = E\{(x(i) - E\{x(i)\})(x(i+1) - E\{x(i+1)\})\} / \sigma^2$ is

correlation coefficient, if $\rho \to 1$ eigenvectors are sampled sine waves.

The discrete cosine transform

As the result for images with highly-correlated samples $(\rho > 0.7)$ the efficiency of DCT in terms of localization signal energy is close to the efficiency of the KL transform.

- DCT represents the orthonormal separable transform which does not depend on the transformed image and thus its computational complexity is rather low.

- Transform coefficients are real numbers.

The discrete cosine transform

The DCT decompose the signal using a set of N different cosine waveforms sampled at N points.

There are two commonly used types of DCT: DCT-II and DCT-IV.

The DCT-II is used in JPEG and MPEG standards. It is defined as

$$X(k) = \frac{c(k)\sqrt{2}}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) \cos\left(\frac{(n+1/2)k\pi}{N} \right), \qquad (7.1)$$

where $c(k) = \begin{cases} 1/\sqrt{2}, & k = 0 \\ 1, & k \neq 0 \end{cases}$

The inverse transform is

$$x(n) = \sum_{k=0}^{N-1} X(k) \frac{c(k)\sqrt{2}}{\sqrt{N}} \cos\left(\frac{(n+1/2)k\pi}{N} \right) \qquad (7.2)$$

DCT-II

In matrix form (7.1) and (7.2) can be written as

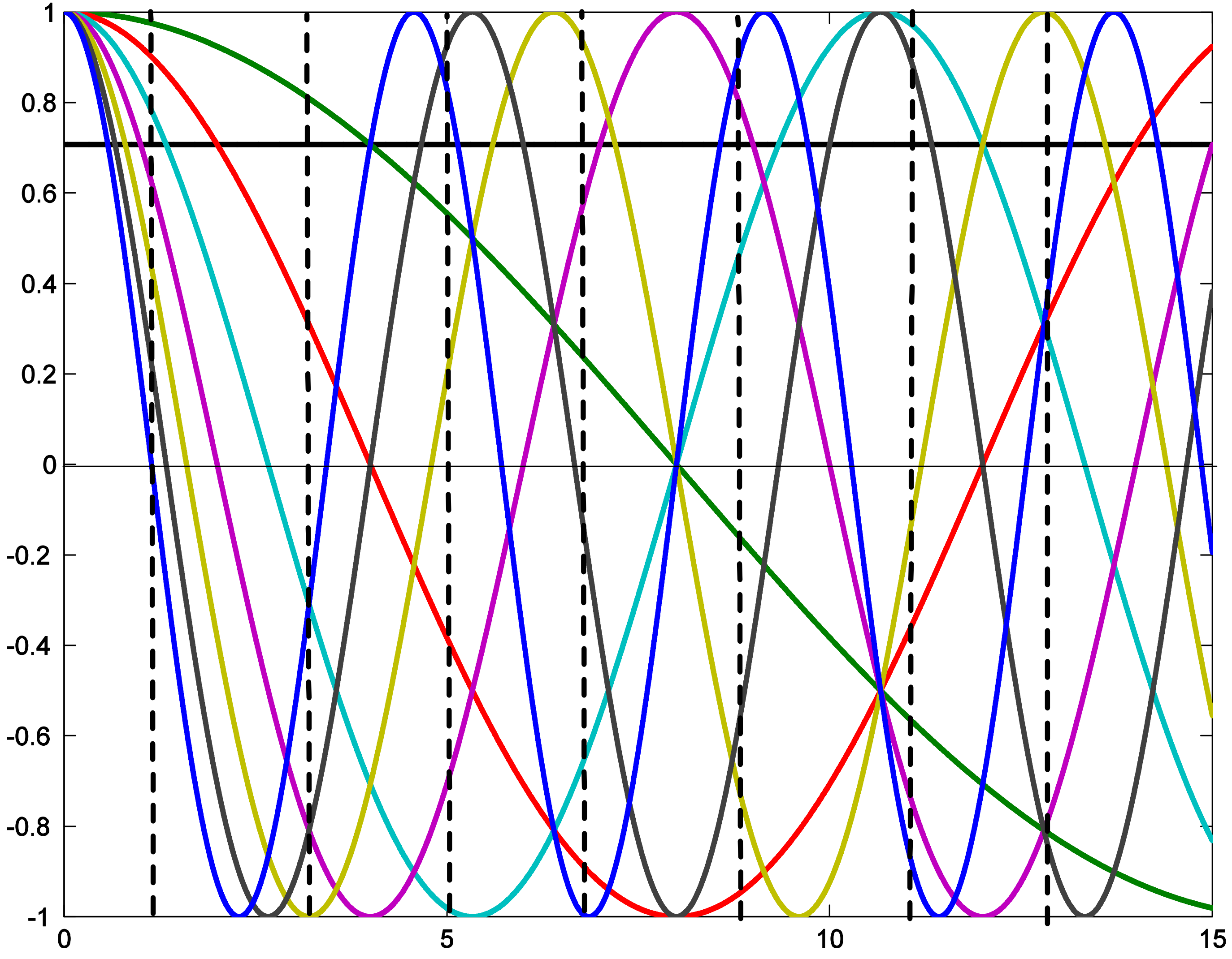
$$\mathbf{X} = T_D \mathbf{x}, \quad \mathbf{x} = T_D^{-1} \mathbf{X} = T_D^T \mathbf{X}, \qquad (7.3)$$

where $\mathbf{X} = (X(0), X(1), ..., X(N-1))^T$, $\mathbf{x} = (x(0), x(1), ..., x(N-1))^T$

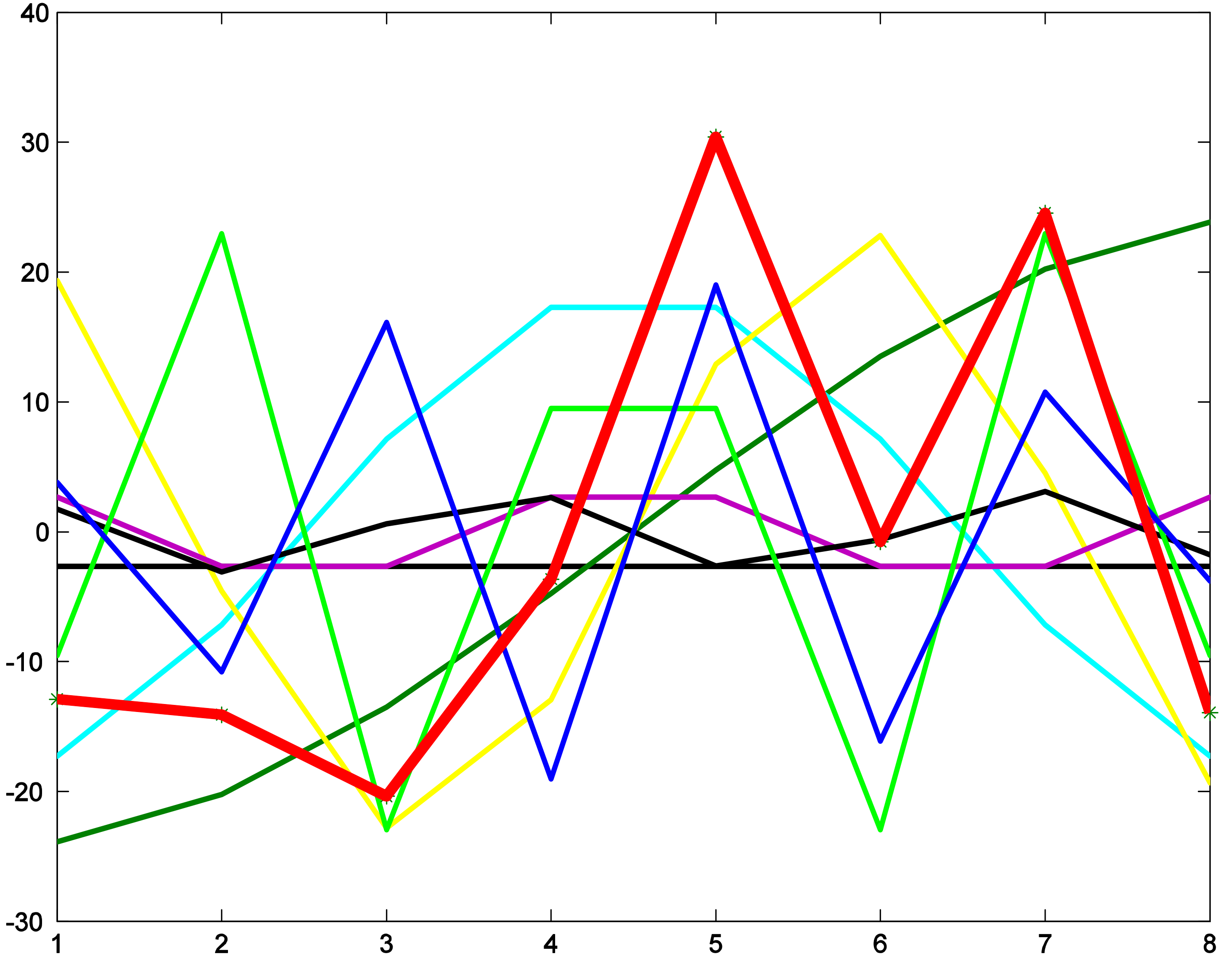
$$T_D = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & ... & 1/\sqrt{2} & 1/\sqrt{2} \\ \cos\left(\frac{\pi}{2N}\right) & \cos\left(\frac{3\pi}{2N}\right) & ... & \cos\left(\frac{(2N-3)\pi}{2N}\right) & \cos\left(\frac{(2N-1)\pi}{2N}\right) \\ ... & ... & ... & ... & ... \\ \cos\left(\frac{\pi(N-2)}{2N}\right) & \cos\left(\frac{3(N-2)\pi}{2N}\right) & ... & \cos\left(\frac{(2N-3)(N-2)\pi}{2N}\right) & \cos\left(\frac{(2N-1)(N-2)\pi}{2N}\right) \\ \cos\left(\frac{\pi(N-1)}{2N}\right) & \cos\left(\frac{3(N-1)\pi}{2N}\right) & ... & \cos\left(\frac{(2N-3)(N-1)\pi}{2N}\right) & \cos\left(\frac{(2N-1)(N-1)\pi}{2N}\right) \end{pmatrix}$$

DCT-II

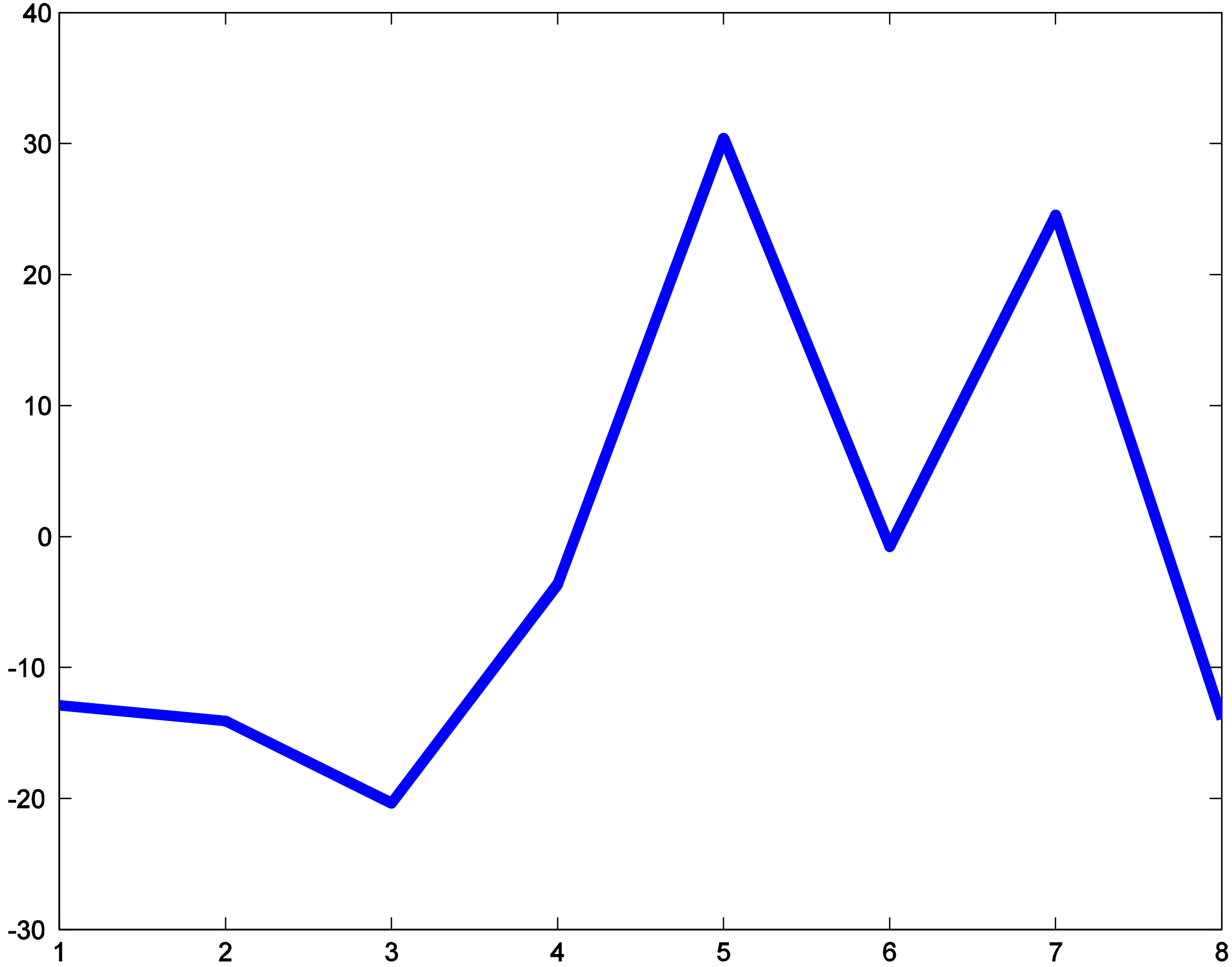
It follows from (7.3) that the DCT-II is the orthonormal transform. The cosine basis functions are orthogonal. For N=8:



DCT-II



DCT-II



DCT-II

The 2-dimensional DCT is determined as follows

$$\mathbf{X}(k,l)=\frac{2c(k)}{\sqrt{N}}\frac{c(l)}{\sqrt{M}}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\mathbf{x}(n,m)\cos\left(\frac{(n+1/2)k\pi}{N}\right)\cos\left(\frac{(m+1/2)l\pi}{M}\right). \tag{7.4}$$

The inverse 2-dimensional DCT can be written as

$$\mathbf{x}(n,m)=\sum_{k=0}^{N-1}\sum_{l=0}^{M-1}\mathbf{X}(k,l)\frac{2c(k)}{\sqrt{N}}\frac{c(l)}{\sqrt{M}}\cos\left(\frac{(n+1/2)k\pi}{N}\right)\cos\left(\frac{(m+1/2)l\pi}{M}\right),$$

where $\mathbf{x}(n,m)$ is the element of the input matrix $\mathbf{x}$ and $\mathbf{X}(k,l)$ is the element of the matrix of transform coefficients.

DCT-II

It is easy to see that (7.4) can be represented in the form

$$\mathbf{X}(k,l) = \frac{\sqrt{2}c(k)}{\sqrt{N}} \sum_{n=0}^{N-1} \mathbf{z}(n,l) \cos\left(\frac{(n+1/2)k\pi}{N}\right),$$

where $\mathbf{Z}$ is the output of 1-dimensional DCT performed over rows of $\mathbf{x}$, that is DCT is a separable transform.

In the matrix form

$\mathbf{Z} = \mathbf{x}T_D^T,$ $\mathbf{X} = T_D\mathbf{Z},$ $\mathbf{X} = T_D\mathbf{x}T_D^T.$

DCT-II

Let us connect DCT-II and DFT. Reorder $x(n)$:

$$y(n) = x(2n), \quad y(N-n-1) = x(2n+1), \quad n = 0,1,\ldots,\frac{N}{2}-1$$

Now take the DFT of $y(n)$. The DCT-II coefficients are

$$X(k) = \cos\left(\frac{\pi k}{2N}\right)\operatorname{Re}\{Y(k)\} + \sin\left(\frac{\pi k}{2N}\right)\operatorname{Im}\{Y(k)\} =$$

$$= \sum_{n=0}^{N-1} y(n)\left(\cos\left(\frac{2\pi nk}{N}\right)\cos\left(\frac{\pi k}{2N}\right) - \sin\left(\frac{2\pi nk}{N}\right)\sin\left(\frac{\pi}{2N}\right)\right) = \sum_{n=0}^{N-1} y(n)\cos\left(\frac{(4n+1)\pi k}{2N}\right)$$

where $Y(k),\ k = 0,1,\ldots,N-1$ are the DFT coefficients of $y(n)$, $n = 0,1,\ldots,N-1.$

DCT-II

It is evident that even samples $x(2n), \; n = 0,1,\ldots,\frac{N}{2}-1$

we multiply by $\cos\left(\frac{(4n+1)\pi k}{2N}\right)$ that coincides with the

required coefficient $\cos\left(\frac{(2n+1/2)\pi k}{N}\right)$.

Odd samples we multiply by

$$\cos\left(\frac{(4(N-n-1)+1)\pi k}{2N}\right) = \cos\left(\frac{(4N-(4n+3))\pi k}{2N}\right) = \cos\left(\frac{(2n+1+1/2)\pi k}{N}\right).$$

DCT-IV

The DCT-IV is used in the standard MPEG-audio as the basis of the so-called modified DCT (a kind of overlapped transform). It is defined as

$$X(k) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} x(n) \cos\left(\frac{(n+1/2)(k+1/2)\pi}{N} \right)$$

The inverse DCT-IV is

$$x(n) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} X(k) \cos\left(\frac{(n+1/2)(k+1/2)\pi}{N} \right).$$

This transform is also orthonormal and separable. Its basis functions (for N=8) are shown in Fig. 7.2

DCT-IV

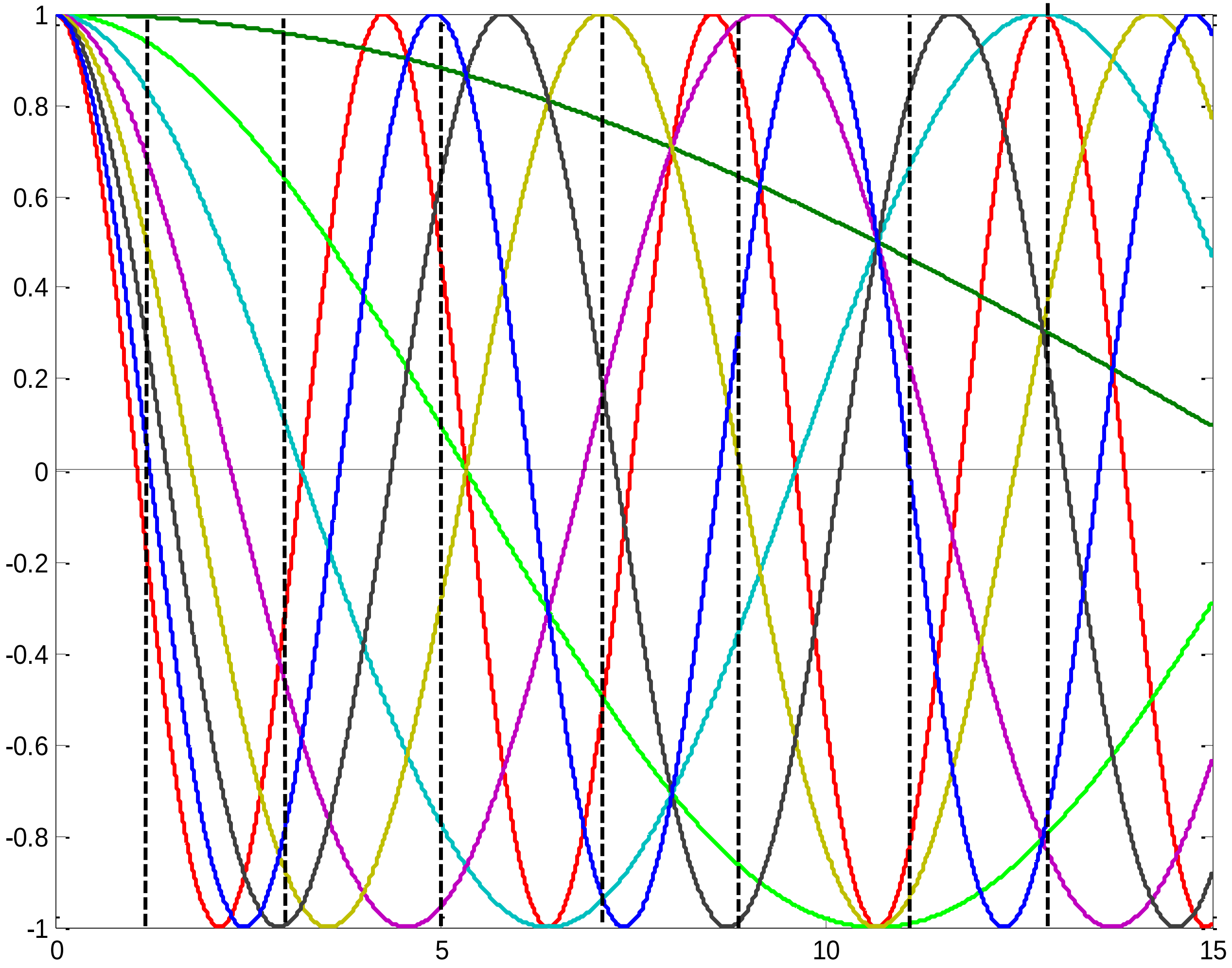


Fig.7.2

DCT-IV

There is no constant among its basis functions. The transform matrix is symmetric.

The DCT-IV also can be implemented via DFT of length $N/2$.

Reordering the input sequence $x(n)$ we create an auxiliary sequence of $N/2$ complex numbers

$$v(n) = (x(2n) + jx(N-1-2n))\exp\left(-j\frac{\pi}{N}\left(n+\frac{1}{4}\right)\right).$$

For the obtained sequence we compute DFT of length $N/2$.

$$V(k) = \sum_{n=0}^{N/2-1} v(n) e^{-j\frac{4nk\pi}{N}}$$ and introduce $$C(k) = \sqrt{\frac{2}{N}}V(k).$$

Coefficients of DCT-IV can be obtained by formulas

$$X(2k) = \operatorname{Re}\left\{C(k)\exp\left(-jk\frac{\pi}{N}\right)\right\},$$

$$X(N-1-2k) = -\operatorname{Im}\left\{C(k)\exp\left(-jk\frac{\pi}{N}\right)\right\},$$

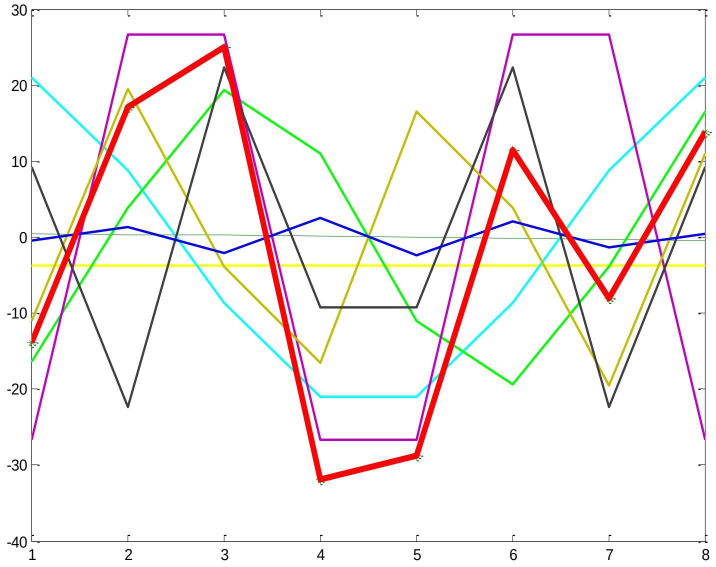
Filter banks

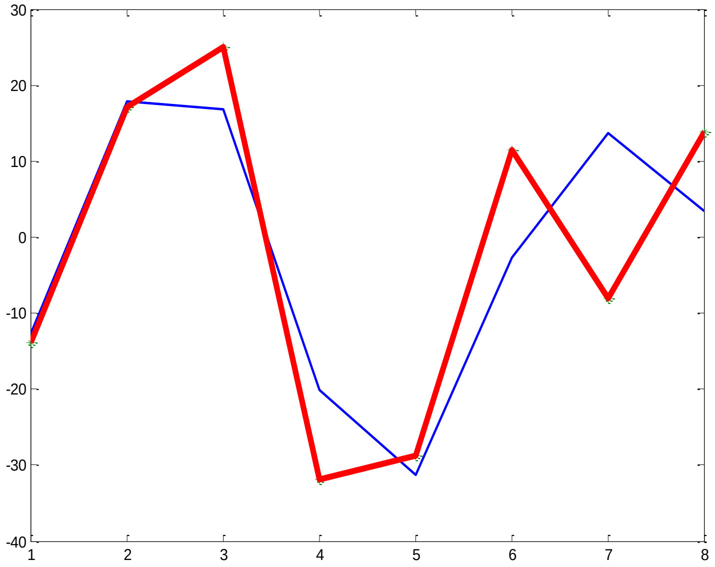
DFT and DCT are two linear transforms which are based on decomposition of the input signal over a system of orthogonal harmonic functions.

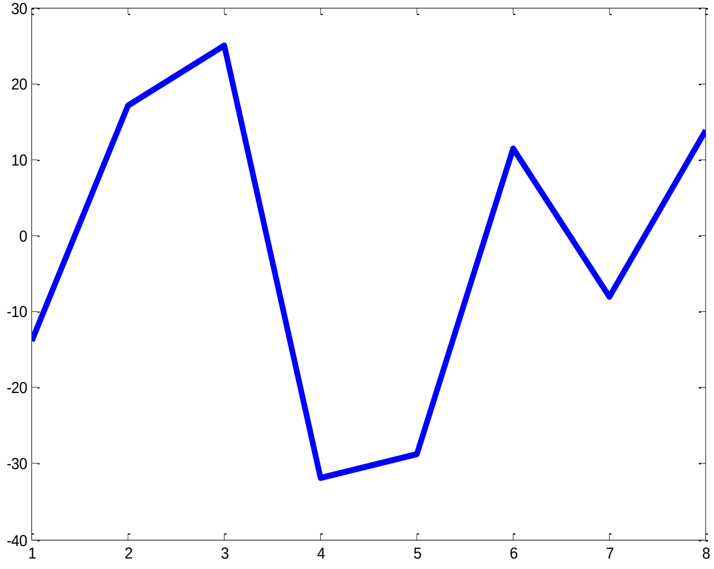
The main shortcoming of these transforms is that basis functions are uniformly distributed over frequency axis.

All frequencies of the input signal are considered as equally important in the sense of recovering original signal from the transform coefficients. On the other hand it is clear that low-frequency components of the signal are more important than high-frequency components, that is, resolution of system of basis functions should be non-uniform over frequency axis.

Filter banks







Filter banks

The problem of constructing a transform with basis functions which are non-uniformly distributed over frequency axis is solved by using filter banks.

One of the most efficient transforms of this type is based on wavelet filter banks and called wavelet filtering.

The output of discrete-time filter with the pulse response $h(n)$ is

$$y(n) = \sum_{k=0}^{n} h(k)x(n-k) = \sum_{k=0}^{n} h(n-k)x(k)$$

or in matrix form

$$\mathbf{y} = T\mathbf{x},$$

$$T = \begin{pmatrix} h(0) & 0 & 0 & 0 & \ldots \\ h(1) & h(0) & 0 & 0 & \ldots \\ h(2) & h(1) & h(0) & 0 & \ldots \\ h(3) & h(2) & h(1) & h(0) & \ldots \\ \ldots & h(3) & h(2) & h(1) & \ldots \end{pmatrix}.$$

Filter banks

It is said that filtering is equivalent to the linear transform described by constant-diagonal matrix T.

Consider the simplest low-pass filter. Its output at time $t = n$ is the average of the input $x(n)$ at that time and the input $x(n-1)$ at previous time:

$$y(n) = \frac{1}{2}x(n) + \frac{1}{2}x(n-1).$$

It is moving average, because the output averages the current component $x(n)$ with the previous one. Its coefficients are $h(0) = 1/2,\ h(1) = 1/2$. This filter is FIR filter.

Filter banks

The filter matrix has the form

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Filter banks

Let us find the frequency response. The frequency response function $H(e^{j\omega T_s})$ can be expressed via pulse response $h(n)$ as follows

$$H(e^{j\omega T_s}) = \sum_n h(n) e^{-j\omega n T_s} = \frac{1}{2} + \frac{1}{2} e^{-j\omega T_s}$$

We factor out $e^{-j\omega T_s/2}$ and obtain $\left(e^{-j\omega T_s/2} + e^{j\omega T_s/2}\right)/2.$

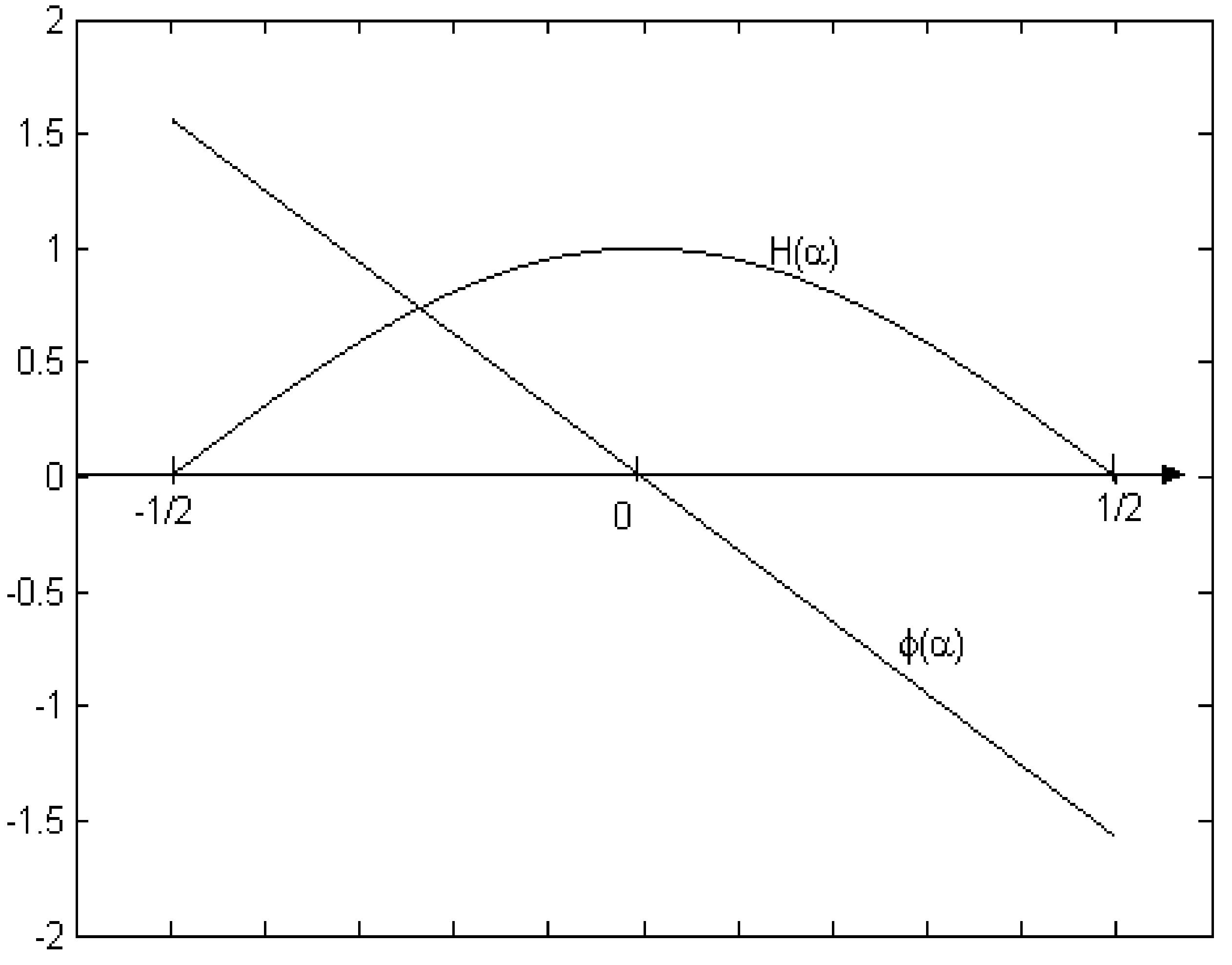
This quantity is a perfect cosine, that is, we get

$$H(e^{j\omega T_s}) = \left(\cos\left(\frac{\omega T_s}{2}\right)\right) e^{-j\omega T_s/2}.$$

The amplitude and the phase are determined as

$$H(\omega) = \cos\left(\frac{\omega T_s}{2}\right), \; H(\alpha) = \cos(\pi\alpha), \; \varphi(\omega) = -\frac{\omega T_s}{2}, \; \varphi(\alpha) = -\pi\alpha.$$

Filter banks



Lowpass filter

The lowest frequency $\alpha = 0$ which is the DC term is exactly preserved because cos0=1.

The filter smoothes out the bumps in the signal. A bump is a high-frequency component, which the lowpass filter reduces or removes.

The frequency response is small or zero near the highest discrete-time frequency $\alpha = 1/2$.

This is an example of linear phase filter. This property means that the filter does not distort the phase of the output signal $y(n)$ and just delays it with respect to the input signal $x(n)$.

Highpass filter

A highpass filter takes "differences". It picks out the bumps in the signal. The smooth parts are low-frequency components, which the highpass filter reduces or removes.

The simplest highpass filter computes moving differences:

$$y(n) = \frac{1}{2}x(n) - \frac{1}{2}x(n-1)$$

The filter coefficients are $h(0) = \frac{1}{2}$ and $h(1) = -\frac{1}{2}$.

This filter is also FIR filter. Its matrix has the form

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \dots & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \dots \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Highpass filter

The frequency response of this highpass filter is

$$H(e^{j\omega T_s}) = \sum_n h(n) e^{-j\omega n T_s} = \frac{1}{2} - \frac{1}{2} e^{-j\omega T_s}.$$

We take out a factor $e^{-j\omega T_s/2}$ and obtain the frequency response in the form

$$H(e^{j\omega T_s}) = \sin\left(\frac{\omega T_s}{2}\right) j e^{-j\omega T_s/2}.$$

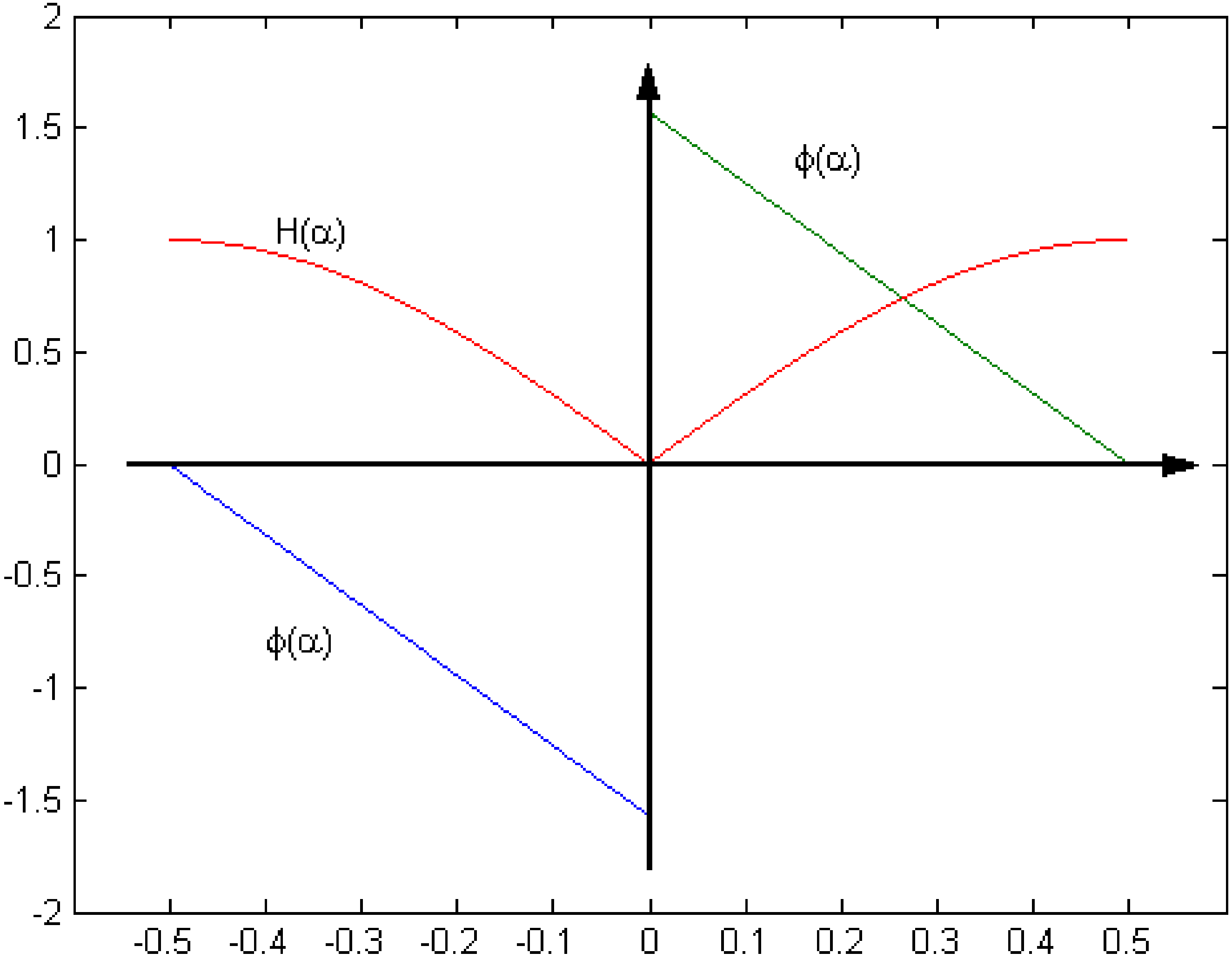
Thus the amplitude is

$$H(\omega) = \left|\sin\left(\frac{\omega T_s}{2}\right)\right|, \; H(\alpha) = \left|\sin(\pi\alpha)\right|.$$

The phase is

$$\varphi(\omega) = \begin{cases} \dfrac{\pi}{2} - \dfrac{\omega T_s}{2} & \text{for } 0 < \omega < \pi \\ -\dfrac{\pi}{2} - \dfrac{\omega T_s}{2} & \text{for } -\pi < \omega < 0 \end{cases}, \; \varphi(\alpha) = \begin{cases} \dfrac{\pi}{2} - \pi\alpha & \text{for } 0 < \alpha < \dfrac{1}{2} \\ -\dfrac{\pi}{2} - \pi\alpha & \text{for } -\dfrac{1}{2} < \alpha < 0 \end{cases}.$$

Highpass filter



Highpass filter

1. The filter has zero response to the sequence (…,1,1,1,1,…).
2. It has unit response to the sequence (…,1,-1,1,-1,1,…).
3. The phase of the filter has a discontinuity. It jumps from $-\pi/2$ to $\pi/2$ at $\alpha = 0$. At other points the graph is linear and we do not pay attention to this discontinuity and say that the filter is still linear phase.

Denote as H_0 and H_1 the frequency response of the lowpass filter and highpass filter, respectively. Let T_0 and T_1 be the corresponding transform matrices.

Filter banks

The transform matrices look like

$$T_0 = \begin{pmatrix} 1/2 & 0 & 0 & 0 & \ldots \\ 1/2 & 1/2 & 0 & 0 & \ldots \\ 0 & 1/2 & 1/2 & 0 & \ldots \\ 0 & 0 & 1/2 & 1/2 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

and

$$T_1 = \begin{pmatrix} 1/2 & 0 & 0 & 0 & \ldots \\ -1/2 & 1/2 & 0 & 0 & \ldots \\ 0 & -1/2 & 1/2 & 0 & \ldots \\ 0 & 0 & -1/2 & 1/2 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

Filter banks

The averaging and differencing filters are not invertible.

The lowpass filter wipes out the alternating signal

$$x(n) = \ldots, 1, -1, 1, -1, \ldots$$

The highpass filter wipes out the constant signal

$$x(n) = \ldots, 1, 1, 1, 1, \ldots$$

A linear operator can recover only zero input from zero output.

A frequency response of an invertible filter must have

$H(\alpha) \neq 0$ at all frequencies.

Our filters are not invertible because $H_0(1/2) = 0$ and $H_1(0) = 0.$

Filter banks

When the inverse filter for the filter described by $H(\omega)$ (matrix T) exists, it has frequency response $1/H(\omega)$ (matrix T^{-1}). Formally we can write the frequency response for the inversion of the moving average

$$\frac{1}{\frac{1}{2}+\frac{1}{2}e^{-j2\pi\alpha}} = 2\left(1-e^{-j2\pi\alpha}+e^{-j4\pi\alpha}-e^{-j6\pi\alpha}+\ldots\right).$$

The inverse matrix is

$$T_0^{-1}T_0 = \begin{pmatrix} 2 & 0 & 0 & \ldots & \ldots \\ -2 & 2 & 0 & 0 & \ldots \\ 2 & -2 & 2 & 0 & 0 \\ -2 & 2 & -2 & 2 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}\begin{pmatrix} 1/2 & 0 & \ldots & \ldots & \ldots \\ 1/2 & 1/2 & 0 & \ldots & \ldots \\ 0 & 1/2 & 1/2 & 0 & \ldots \\ 0 & 0 & 1/2 & 1/2 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} = I$$

Filter banks

We seem to have an inverse but it is not a legal filter. The filter is not stable. The bounded input $y(n) = (\ldots, -1, 1, -1, 1, \ldots)$

would produce unbounded output

$$T_0^{-1}\left(y(0), y(1), \ldots\right)^T = (-2, 4, -6, 8, -12, 16, \ldots)$$

The series expansion breaks down completely at $\alpha = 1/2$

$$\frac{1}{0} = \frac{1}{\frac{1}{2} + \frac{1}{2}e^{-j\pi}} = 2(1+1+1+1+\ldots).$$

Inverse filter is very rarely FIR, because $1/H(\omega)$ is not a polynomial.