Scalar quantization with memory

Speech and images have redundancy or memory that scalar quantization can not exploit.

Scalar quantization for source with memory provides $R(D)$ which is rather far from $H(D)$.

VQ could attain better $R(D)$ but usually at the cost of significant increasing of computational complexity.

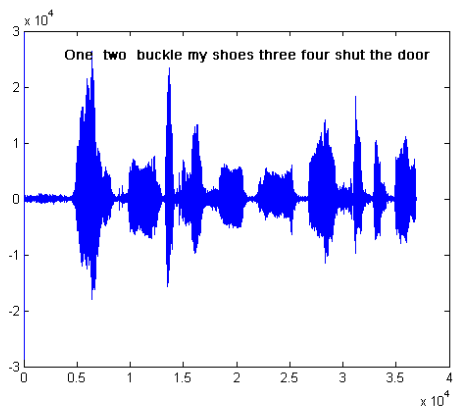
Another approach leading to better $R(D)$ and saving rather low computational complexity combines linear processing with scalar quantization.

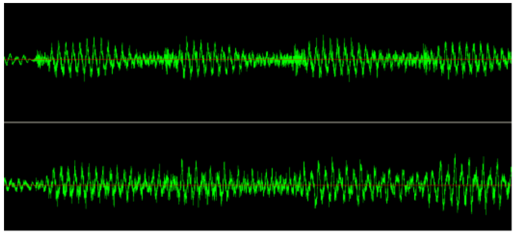
Scalar quantization with memory

Input signal → Preprocessing block → Scalar quantizer → Quantized data

Scalar quantization with memory

- Predictive coding
  - Speech coding
- Transform coding
  - Audio, images, video coding





Discrete-time filters

Analog or continuous system is something that transforms an input signal $x(t)$ into an output signal $y(t)$, where both signals are continuous functions of time.

A discrete-time system transforms an input sequence $x(nT_s)$ into output sequence $y(nT_s)$.

Any discrete-time system can be considered as a discrete filter.

Thus predictive coding and transform coding are kinds of discrete filtering.

A discrete-time system (filter) L is said to be linear if the following condition holds

$$L(\alpha x_1(nT_s) + \beta x_2(nT_s)) = \alpha L(x_1(nT_s)) + \beta L(x_2(nT_s)),$$

where α and β are arbitrary real or complex constants.

Discrete-time filters

Description by means of recurrent equations

A discrete filter of N th order is described by the following linear recurrent equation

$$y(nT_s) = \sum_{i=0}^{M} a_i x(nT_s - iT_s) + \sum_{j=1}^{N} b_j y(nT_s - jT_s), \qquad (4.1)$$

where $x(nT_s)$, $y(nT_s)$ are samples of input and output signals, a_i, b_j are constants which do not depend on $x(nT_s)$.

The discrete-time counterpart to Dirac's delta function is called Kronecker's delta function $\delta(nT_s)$

$$\delta(nT_s) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} .$$

If $x(nT_s) = \delta(nT_s)$ then $y(nT_s) = h(nT_s)$, where $h(nT_s)$

is the discrete-time pulse response.

Discrete-time filters

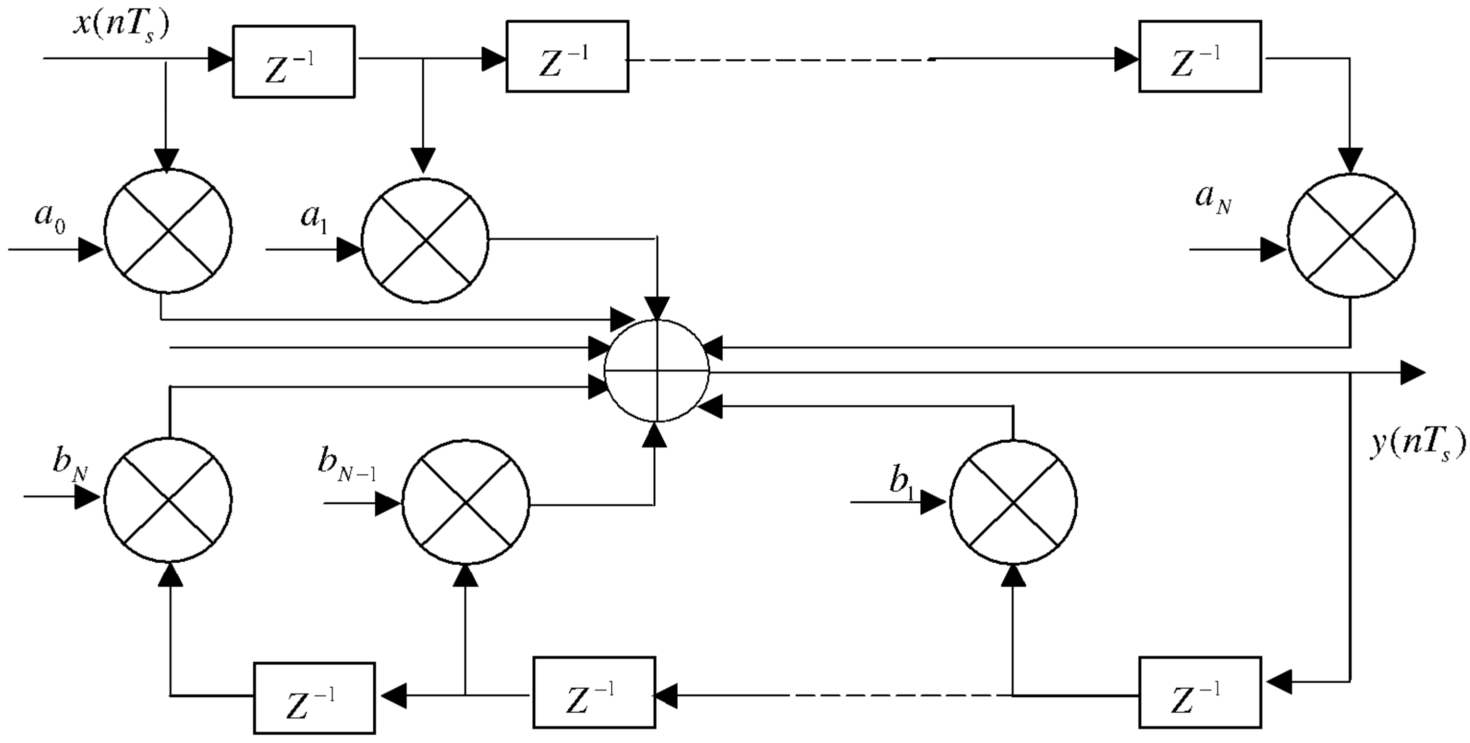


Fig.4.1 An Nth order time-invariant discrete filter

Discrete-time filters

The output $y(nT_s)$ can be expressed via the input $x(nT_s)$ and $h(nT_s)$ as a discrete-time convolution

$$y(nT_s) = \sum_{m=0}^{n} x(mT_s)h(nT_s - mT_s) = \sum_{m=0}^{n} h(mT_s)x(nT_s - mT_s). \quad (4.2)$$

If $h(nT_s)$ has nonzero samples only for $n = 0,\ldots,M$ we call the filter a finite response (FIR) filter.

If in (4.1) we set $b_j = 0,\ j = 1,\ldots,N$ then we obtain a FIR filter of M th order.

Since FIR filters do not contain feedback paths they are also called nonrecursive filters.

Discrete-time filters

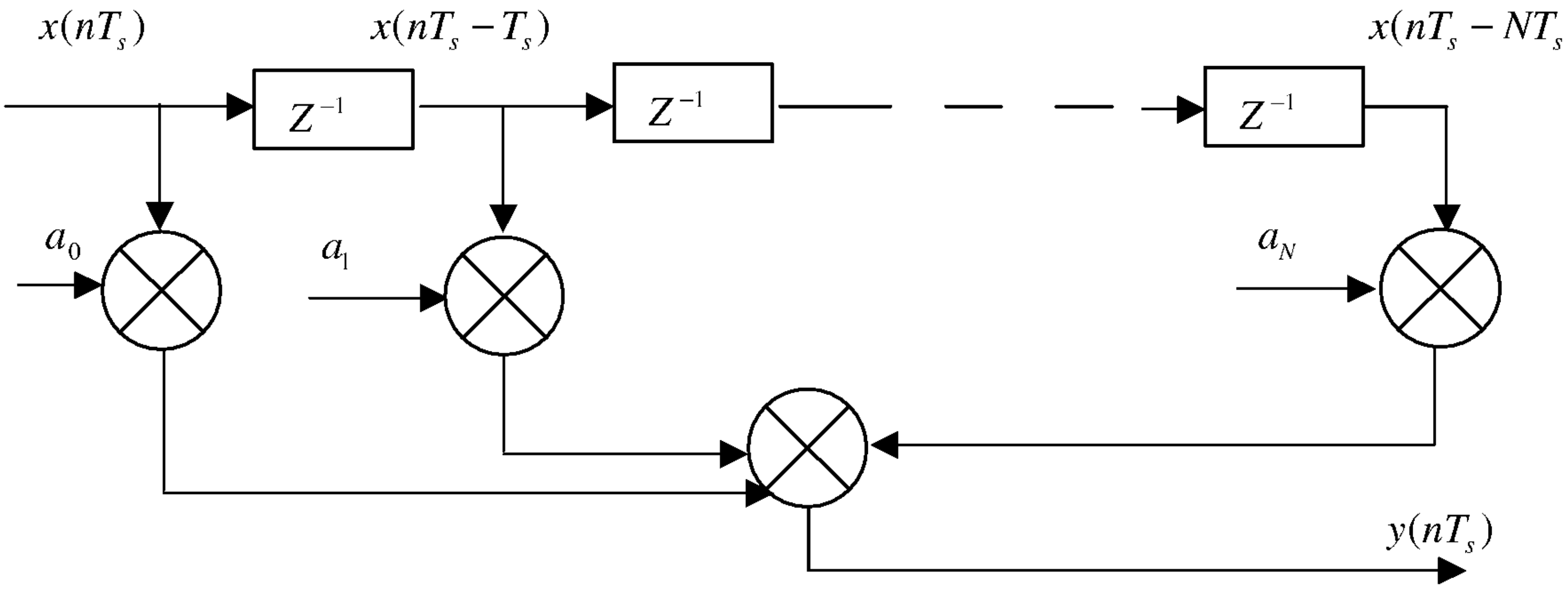


Fig.4.2 An Nth order FIR filter

Discrete-time filters

It follows from (4.1) that the output of the FIR filter is

$$y(nT_s) = a_0 x(nT_s) + a_1 x(nT_s - T_s) + \ldots + a_M x(nT_s - MT_s). \qquad (4.3)$$

Formula (4.2) in the case of FIR filter has the form

$$y(nT_s) = \sum_{m=0}^{M} x(mT_s) h(nT_s - mT_s) = \sum_{m=0}^{M} h(mT_s) x(nT_s - mT_s). \qquad (4.4)$$

Comparing (4.3) and (4.4) we obtain that $h(iT_s) = a_i, i = 0,1,\ldots M.$

A filter with an infinite (im)pulse response is called IIR filter. In general case the recursive filter described by (4.1) represents IIR filter.

Discrete-time filters

In order to compute the pulse response of the discrete IIR filter it is necessary to solve the corresponding recurrent equation (4.1).

Example. The first order IIR filter is described by the equation: $y(nT_s) = ax(nT_s) + by(nT_s - T_s).$

It is evident that $y(0) = ax(0) + by(-T_s),$

$$y(T_s) = b^2 y(-T_s) + abx(0) + ax(T_s).$$

Continuing in such a manner we obtain

$$y(nT_s) = b^{n+1} y(-T_s) + \sum_{i=0}^{n} b^i ax(nT_s - iT_s).$$

Setting $y(-T_s) = 0$ and $x(nT_s) = \delta(nT_s)$ we get $h(nT_s) = ab^n.$

Discrete-time filters

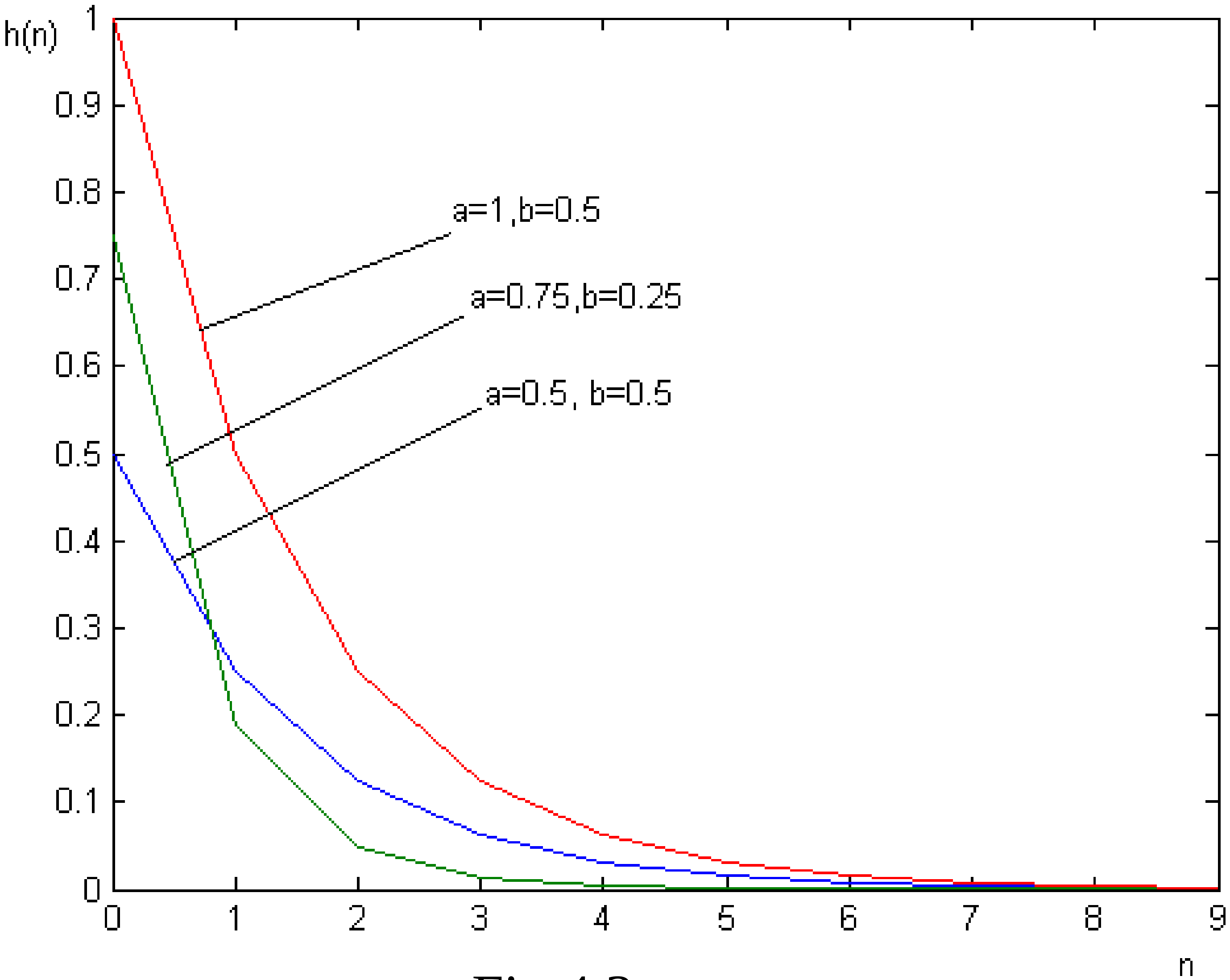


Fig.4.3

Discrete-time filters

Description by means of *z*-transform

The *z*-transform of a given discrete-time signal $x(nT_s)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT_s) z^{-n}, \; z \in E_x,$$

where z is a complex variable, $X(z)$ denotes a function of z, E_x is called the existence region.

If $x(nT_s)$ is causal then one-sided *z*-transform is defined as

$$X(z) = \sum_{n=0}^{\infty} x(nT_s) z^{-n}, \; z \in E_x. \qquad (4.5)$$

z-transform allows to replace recurrent equations over sequences by algebraic equations over z-transforms like the Laplace transform allows to replace differential equations by algebraic equations.

Discrete-time filters

Properties of *z*-transform

Linearity: If $X(z)$ is the *z*-transform of $x(nT_s)$ and $Y(z)$ is *z*-transform of $y(nT_s)$ then $ax(nT_s) + by(nT_s)$ has *z*-transform $aX(z) + bY(z)$.

Discrete convolution: If $X(z)$ is *z*-transform of $x(nT_s)$ and $Y(z)$ is *z*-transform of $y(nT_s)$ and $R(z) = X(z)Y(z)$ then $R(z)$ is *z*-transform of $r(nT_s)$

$$r(nT_s) = \sum_{m=0}^{n} x(mT_s)y(nT_s - mT_s) = \sum_{m=0}^{n} x(nT_s - mT_s)y(mT_s)$$

Delay: If $X(z)$ is *z*-transform of $x(nT_s)$ then *z*-transform of $y(nT_s) = x(nT_s - mT_s)$ is $Y(z) = X(z)z^{-m}$.

Name of the sequence	Sequence	z-transform
Unit impulse	$x(nT_s) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$	$X(z) = 1$
Unit step	$x(nT_s) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$	$X(z) = \dfrac{1}{1 - z^{-1}}$
Exponent	$x(nT_s) = Ae^{-anT_s}$	$X(z) = \dfrac{A}{1 - e^{-aT_s} z^{-1}}$
Damped sine	$x(nT_s) = e^{-anT_s} \sin(\omega nT_s)$	$X(z) = \dfrac{z^{-1} e^{-aT_s} \sin \omega T_s}{1 - z^{-1} 2e^{-aT_s} \cos \omega T_s + e^{-2aT_s} z^{-2}}$
Damped cosine	$x(nT_s) = e^{-anT_s} \cos(\omega nT_s)$	$X(z) = \dfrac{1 - e^{-aT_s} z^{-1} \cos \omega T_s}{1 - z^{-1} 2e^{-aT_s} \cos \omega T_s + e^{-2aT_s} z^{-2}}$
Response of the 2nd order filter	$x(nT_s) = (\gamma_1^{n+1} - \gamma_2^{n+1}),$ $\gamma_{1,2} = a/2 \pm \sqrt{\dfrac{a^2}{4} + b}$	$X(z) = \dfrac{1}{1 - z^{-1} a - z^{-2} b} = \dfrac{1}{(z - \gamma_1)(z - \gamma_2)}$
Complex exponent	$x(nT_s) = \begin{cases} e^{j\omega nT_s}, & n \geq 0 \\ 0, & n < 0 \end{cases}$	$X(z) = \dfrac{1}{1 - e^{j\omega T_s} z^{-1}}$

Example

Let $X(z) = \dfrac{1}{\frac{1}{2} z^{-2} - \frac{3}{2} z^{-1} + 1}.$

Decompose $X(z)$ into sum of simple fractions. We obtain

$$X(z) = \frac{1}{(1 - z^{-1})\left(1 - \frac{1}{2} z^{-1}\right)} = \frac{A}{1 - z^{-1}} + \frac{B}{1 - \frac{1}{2} z^{-1}} \qquad (4.6)$$

Multiplying both parts of (4.6) by $(1 - z^{-1})\left(1 - \frac{1}{2} z^{-1}\right)$ we get

$$1 = A\left(1 - \frac{1}{2} z^{-1}\right) + B(1 - z^{-1}).$$

Equating coefficients of like powers of z we obtain

$$\begin{cases} A + B = 1 \\ -\dfrac{A}{2} - B = 0 \end{cases}$$

and

$$X(z) = \frac{1}{(1 - z^{-1})\left(1 - \frac{1}{2} z^{-1}\right)} = \frac{2}{1 - z^{-1}} - \frac{1}{1 - \frac{1}{2} z^{-1}}.$$

Example

Comparing the obtained results with examples in the table we obtain that the first summand corresponds to

$$x(nT_s) = \begin{cases} 2, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

.

and the second summand corresponds to $x(nT_s) = -2^{-n}.$

Thus we obtain

$$x(nT_s) = \begin{cases} 2 - 2^{-n}, & n \geq 0 \\ 0, & n < 0 \end{cases} .$$

Inverse *z*-transform

There is one-to-one correspondence between a discrete sequence and its *z*-transform.

According to definition $x(nT_s)$ represents the inverse *z*-transform of $X(z)$.

It can be found from (4.5). First multiply both parts by z^{k-1} and then take the contour integral of both parts of equation.

The contour integral is taken along a counterclockwise arbitrary closed path that encloses all the finite poles of $X(z)z^{k-1}$ and lies entirely in E_x.

$$\oint X(z)z^{k-1}dz = \sum_{n=0}^{\infty} x(nT_s)\oint z^{k-n-1}dz \qquad (4.7)$$

Inverse z-transform

It follows from the Cauchy theorem that if the path of integration encloses the origin of coordinates then

$$\oint z^{k-n-1} dz = 0$$

for all k except $k = n$. For $k = n$ it is equal to

$$\oint z^{-1} dz = 2\pi j.$$

Thus (4.7) reduces to

$$x(kT_s) = \frac{1}{2\pi j} \oint X(z) z^{k-1} dz \tag{4.8}$$

which describes the inverse z-transform.

Inverse z-transform

The contour integral (4.8) can be evaluated via the theorem of residues.

Theorem. Let function $f(z)$ be unique and analytic (differentiable) everywhere along the contour L and inside it except the so-called singular points z_k, $(k=1,\ldots,N)$, lying inside L. Then

$$\oint_{(L)} f(z)dz = 2\pi j \sum_{k=1}^{N} \mathrm{Res}\left[f(z), z_k\right],$$

that is, the contour integral is equal to the sum of residues of integrand function taken in the singular points lying inside the path of integration, multiplied by $2\pi j$.

Inverse z-transform

A residue in the m th order pole z_0 can be calculated as

$$\operatorname{Res}\left[f(z), z_0\right] = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right]$$

If function $f(z)$ represents a ratio of two finite functions $f(z) = g(z)/h(z)$ and it being known that $h(z)$ has zero of the 1st order in $z = a$ and that $g(a) \neq 0$ then the residue in $z = a$ can be computed as

$$\operatorname{Res}\left[f(z), a\right] = \frac{g(a)}{h'(a)} \qquad (4.9)$$

If $X(z)z^{n-1}$ has K finite poles at $z = a_i$, $i = 1, 2, \ldots K$ then

$$x(nT_s) = \sum_{i=1}^{K} \operatorname{Res}\left[X(z) z^{n-1}, z = a_i \right]$$

Example

Continuing our example we obtain that

$$x(nT_s) = \frac{1}{2\pi j}\oint \frac{2z^{n-1}dz}{1-z^{-1}} - \frac{1}{2\pi j}\oint \frac{z^{n-1}dz}{1-\frac{1}{2}z^{-1}}.$$

Using (4.9) we compute

$$\mathrm{Res}\left[\frac{2z^n}{z-1},1\right] = \frac{2}{1} \qquad \mathrm{Res}\left[\frac{2z^n}{2z-1},1/2\right] = \frac{2(1/2)^n}{2} = 2^{-n}$$

Thus $x(nT_s) = 2 - 2^{-nT_s}.$

Example

Let us solve the equation

$$y(nT_s) = ax(nT_s) + by(nT_s - T_s).$$

using z-transform. Using properties (linearity and delay) we obtain

$$Y(z) = bY(z)z^{-1} + aX(z),$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a}{1 - bz^{-1}},$$

where $H(z)$ is the transfer function of the filter, $X(z)$ and $Y(z)$ are the z-transforms of the input and output. $H(z)$ is the z-transform of $h(nT_s)$ $H(z) = \sum_{n=0}^{\infty} h(nT_s) z^{-n}.$

Since $Y(z) = X(z)H(z)$ then if $x(nT_s) = \delta(nT_s)$, $X(z) = 1.$

This yields $Y(z) = H(z) = \dfrac{a}{1 - bz^{-1}} = \dfrac{az}{z - b}.$

Example

By applying the inverse z-transform to $H(z)$ if the integration path encloses the pole $z = b$ we obtain

$$y(nT_s) = ab^n.$$

Frequency function of the discrete filter

The complex frequency function of the discrete-time linear filter can be obtained by inserting $z = e^{j\omega T_s}$ into $H(z)$.

Let $x(nT_s) = e^{j\omega nT_s}$ then the corresponding output of the filter with pulse response $h(nT_s)$ is

$$y(nT_s) = \sum_{m=0}^{n} h(mT_s)x(nT_s - mT_s) = \sum_{m=0}^{n} h(mT_s)e^{j\omega(n-m)T_s} =$$

$$= e^{j\omega nT_s}\sum_{m=0}^{n} h(mT_s)e^{-j\omega mT_s}$$

When $n \to \infty$ $y(nT_s) = e^{j\omega nT_s}\sum_{m=0}^{\infty} h(mT_s)e^{-j\omega mT_s} = e^{j\omega nT_s}H(e^{j\omega T_s})$,

where $H(e^{j\omega T_s}) = \sum_{m=0}^{\infty} h(mT_s)e^{-j\omega mT_s}$ is the complex frequency function of the discrete-time linear filter.

Frequency function of the discrete filter

The complex frequency function can be represented in the form

$$H(e^{j\omega T_s}) = A(\omega)e^{j\varphi(\omega)} = \mathrm{Re}(H(e^{j\omega T_s})) + j\,\mathrm{Im}(H(e^{j\omega T_s})),$$

where $A(\omega) = \sqrt{\left\{\mathrm{Re}(H(e^{j\omega T_s}))\right\}^2 + \left\{\mathrm{Im}(H(e^{j\omega T_s}))\right\}^2}$

is the amplitude function and

$$\varphi(\omega) = \arctan\frac{\mathrm{Im}(H(e^{j\omega T_s}))}{\mathrm{Re}(H(e^{j\omega T_s}))}$$

is the phase function.

If $x(nT_s) = \sin(n\omega T_s)$ then $y(nT_s) = A(\omega)\sin(n\omega T_s + \varphi(\omega))$.

The amplitude and the phase functions described the change in amplitude and phase of the discrete sinusoid introduced by the discrete-time linear filter.

Frequency function of the discrete filter

Properties of the frequency function:

•The frequency function is a continuous function of frequency

•The frequency function is a periodic function of frequency with period equal to the sampling frequency.

•For discrete-time linear filters with coefficients a_i, b_i which are real numbers the amplitude function is an even function of frequency and the phase function is an odd function of frequency.

•The frequency function can be represented via the pulse response as

$$H(e^{j\omega T_s}) = \sum_{n=0}^{\infty} h(nT_s)e^{-j\omega n T_s} = \sum_{n=0}^{\infty} h(nT_s)e^{-j2\pi n f T_s} = \sum_{n=0}^{\infty} h(nT_s)\left(\cos(2\pi n f T_s) - j\sin(2\pi n f T_s)\right).$$

Frequency function of discrete filter

For convenience of comparison of different frequency functions instead of ω we consider the normalized frequency $\alpha = \omega / \omega_s$, where $\omega_s = 2\pi / T_s$ is the sampling frequency in radians. Then we obtain

$$H(e^{j\omega T_s}) = \sum_{n=0}^{\infty} h(nT_s) e^{-j2\pi n\alpha} = \sum_{n=0}^{\infty} h(nT_s)\left(\cos(2\pi n\alpha) - j\sin(2\pi n\alpha)\right).$$

Example

The transfer function of the discrete-time linear filter of the 1st order has the form

$$H(z) = \frac{a}{1 - bz^{-1}}.$$

By inserting $z = e^{j\omega T_s}$ we obtain the frequency function

$$H(e^{j\omega T_s}) = \frac{a}{1 - be^{-j\omega T_s}} = \frac{a}{1 - b\cos(\omega T_s) + jb\sin(\omega T_s)}.$$

The amplitude function is

$$A(\omega) = \frac{a}{\sqrt{(1 - b\cos(\omega T_s))^2 + b^2 \sin^2(\omega T_s)}} = \frac{a}{\sqrt{1 - 2b\cos(\omega T_s) + b^2}}.$$

The phase function

$$\varphi(\omega) = -\arctan\frac{b\sin(\omega T_s)}{1 - b\cos(\omega T_s)}.$$

Example

By normalizing frequency we obtain

$$A(\alpha) = \frac{a}{\sqrt{1 - 2b\cos(2\pi\alpha) + b^2}},$$

$$\varphi(\alpha) = -\arctan\frac{b\sin(2\pi\alpha)}{1 - b\cos(2\pi\alpha)}.$$

The amplitude function $A(\alpha)$ for $a = 1,\ b = 0.5$

is shown in Fig.4.4.

Example

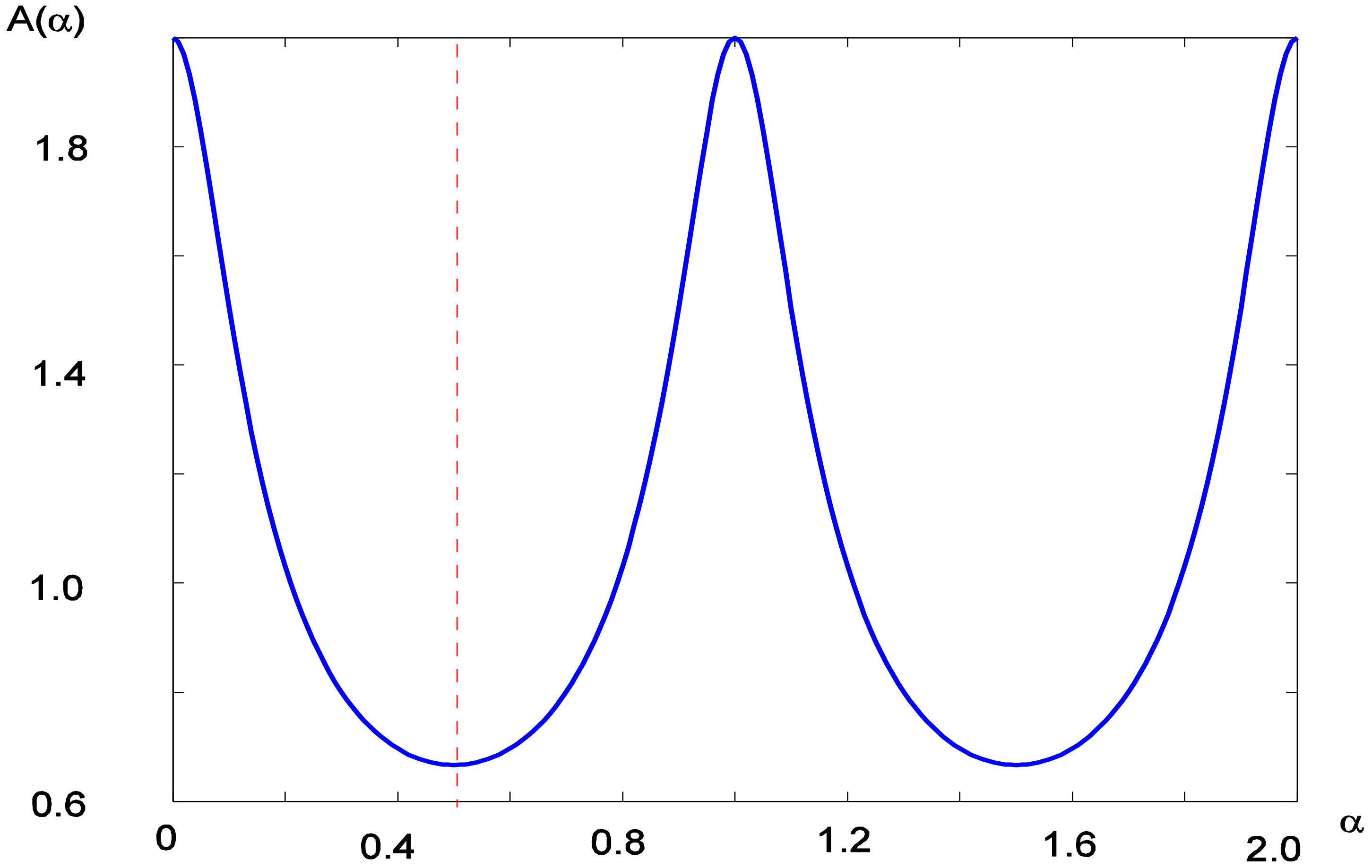


Fig.4.4