Lecture 9

Digital Signal Processing

Chapter 6

Sampling

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Sampling

We will not look closer at sampling, especially the connection between the spectrum before and after sampling.

$$x(t) \longrightarrow \underbrace{\boxed{\text{LP}} \longrightarrow \boxed{\text{A/D}}}_{\text{Sampling}} \longrightarrow \underbrace{\boxed{\text{Proc.}}}_{\text{Processing}} \longrightarrow \underbrace{\boxed{\text{D/A}} \longrightarrow \boxed{\text{LP}}}_{\text{Reconstruction}} \longrightarrow y(t)$$

We will show the sampling theorem and the importance of the low pass filters during sampling and reconstruction.

A signal is read (audio signal is sampled, or pictures in a movie) at a regular interval

$$t = n \cdot T_s = n \cdot \frac{1}{F_s} \tag{1}$$

where T_s is the time period between each sample and F_s is the sampling rate.

TV frame rate	$F_s = 50\,\text{frames/s}$	
CD audio	$F_s = 44\,100\,\text{Hz}$	Studio audio up to 48 kHz.
Telephony	$F_s = 8000\,\text{Hz}$	GSM and analog systems.

Rotating spokes on a wheel in a movie sometimes look like they are standing still or rotating in reverse; why is that?

Sampling

Read a continuous signal F_s times per second:

$$x(n) = x(t \mid t = n \cdot T_s = n/F_s) \tag{2}$$

Read at least two times per period of the signal. If the highest frequency in $x(t)$ is $F_{\max}$ then choose the sampling frequency as $F_s > 2 \cdot F_{\max}$ as we can reconstruct the analog signal $x(t)$ exactly.

Example

Given:

$$x(t) = \cos(2\pi 400 t) \tag{3}$$

where $F_0 = 400$ and $F_s = 1000$ gives

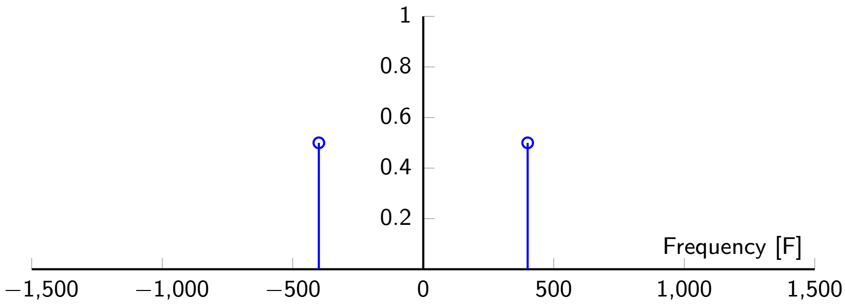
$$x(n) = \cos\left(2\pi \cdot \frac{400}{1000} \cdot n\right) = \cos(2\pi 0.4 n) \tag{4}$$

But, we also get

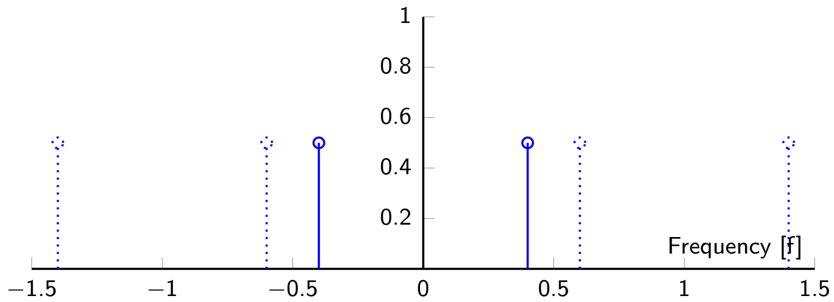
$$x(n) = \cos(2\pi 0.4 n) = \cos(2\pi(0.4 + k)n) \qquad k \text{ integer} \tag{5}$$

because n is an integer. We therefore also get the new frequencies $f_0 = \pm 0.4 \pm k$ where k is an integer.

The spectrum $|X(F)|$ before sampling:



The spectrum $|X(f)|$ after sampling:



Important:

The spectrum of a digital signal is periodic with the period $f = 1$ or $\omega = 2\pi$.

Example of folding distortion

Consider a signal with two cosine terms.

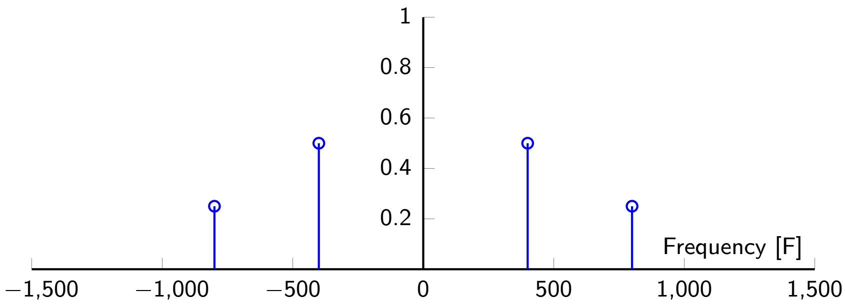
$$x(t) = \cos(2\pi 400 t) + 0.5\cos(2\pi 800 t) \tag{6}$$

where $F_s = 1000$ such that

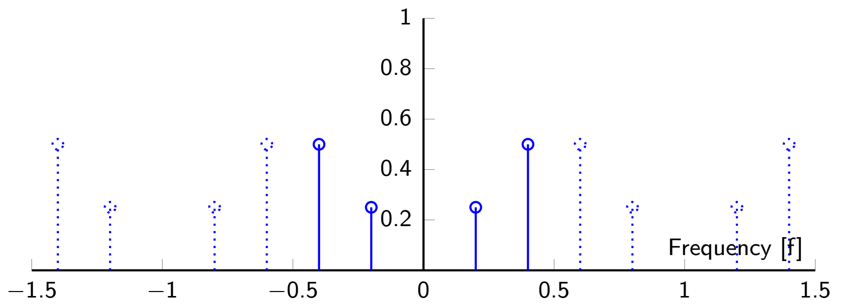
$$\begin{aligned} x(n) &= \cos(2\pi 0.4 n) + 0.5\cos(2\pi 0.8 n) && (7) \\ &= \cos(2\pi 0.4 n) + 0.5\cos(2\pi(-0.2)n) \qquad \text{[because of periodicity]} && (8) \end{aligned}$$

The sampled signal contains the frequencies $f_0 = \pm 0.4 \pm k$ and $f_1 = \pm 0.2 \pm k$ where k is an integer.

The spectrum before sampling



The spectrum after sampling:



Folding distortion yields a “false” frequency at $f = \pm 0.2$ corresponding to $F = \pm 200\,\text{Hz}$. For discrete signals in general, a frequency at f_i also appears at $f_i \pm 1$, $f_i \pm 2$, and so on.

Spectrum for a sampled signal

To express the spectrum of a sampled signal with a formula, we look at the Fourier transform. If the Fourier transform exists, then we can derive a simple formula for the spectrum after sampling. The Fourier transform is defined as

$$X(F) = \int_{-\infty}^{\infty} x(t) \mathrm{e}^{-\mathrm{j}2\pi F t}\, \mathrm{d}t \tag{9}$$

Let $x(n) = x(t)$ where $t = n/F_s$ and change the integral for a sum

$$X(F) \approx \sum_{n=-\infty}^{\infty} x(n) \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{F}{F_s} \cdot n} \cdot \Delta t = X(f) \cdot \frac{1}{F_s} \tag{10}$$

meaning that

$$X(f) = X(F) \cdot F_s \tag{11}$$

This applies as long as there is no folding distortion.

If we consider that every frequency component in the analog signal appears periodically in the sampled signal, we can assume that the final formula for the spectrum of a signal after sampling is

$$X(f) = F_s \cdot [\ldots + X(F - 2F_s) + X(F - F_s) + X(F) + X(F + F_s) + X(F + 2F_s) + \ldots] \tag{12}$$

$$= F_s \cdot \sum_{k=-\infty}^{\infty} X(F - kF_s) \tag{13}$$

This applies if the Fourier transform exists.

Proof

Compare the spectrum of the analog signal $x_a(t)$ and the discrete signal $x(n)$.

$$X_a(F) = \int_{-\infty}^{\infty} x_a(t)\mathrm{e}^{-\mathrm{j}2\pi F t}\,\mathrm{d}t \qquad \Leftrightarrow \qquad x_a(t) = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{\mathrm{j}2\pi F t}\,\mathrm{d}F \tag{14}$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)\mathrm{e}^{-\mathrm{j}2\pi f n} \qquad \Leftrightarrow \qquad x(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f)\mathrm{e}^{\mathrm{j}2\pi f n}\,\mathrm{d}f \tag{15}$$

Periodic sampling imposes a relation between t and n of

$$t = n \cdot T_s = \frac{n}{F_s} \tag{16}$$

If $x(n) = x_a(n \cdot T_s) = x_a(n/F_s)$ then

$$x(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f)\mathrm{e}^{\mathrm{j}2\pi f n}\,\mathrm{d}f \qquad [= x_a(n/F_s)] \tag{17}$$

$$= \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{\mathrm{j}2\pi\cdot\frac{F}{F_s}\cdot n}\,\mathrm{d}F \tag{18}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[F_s \cdot \sum_{k=-\infty}^{\infty} X_a((f-k)\cdot F_s)\right] \cdot \mathrm{e}^{\mathrm{j}2\pi f n}\,\mathrm{d}f \tag{19}$$

and

$$X(f) = F_s \cdot \sum_{k=-\infty}^{\infty} X_a((f-k)\cdot F_s) \tag{20}$$

which implies that $X(f)$ is a periodic repetition of $X_a(F)$.

Observer that the spectrum is often a complex function and that the addition is complex addition.

Example with folding distortion and phase addition

When folding occurs, we get a sum of different parts of the analog spectrum. We have to take the phase of the signals into consideration during this summation.

Assume a signal with a cosine and a sine term.

$$x(t) = \cos(2\pi 400 t) + \sin(2\pi 600 t) \tag{21}$$

$$= \frac{1}{2}\cdot \mathrm{e}^{\mathrm{j}2\pi 400 t} + \frac{1}{2}\cdot \mathrm{e}^{-\mathrm{j}2\pi 400 t} + \frac{1}{\mathrm{j}2}\cdot \mathrm{e}^{\mathrm{j}2\pi 600 t} - \frac{1}{\mathrm{j}2}\cdot \mathrm{e}^{-\mathrm{j}2\pi 600 t} \tag{22}$$

Sampling with $F_s = 1000$ yields

$$x(n) = \cos(2\pi 0.4n) + \sin(2\pi 0.6n) \tag{23}$$

$$= \frac{1}{2}\cdot e^{j2\pi 0.4n} + \frac{1}{2}\cdot e^{-j2\pi 0.4n} + \frac{1}{j2}\cdot e^{j2\pi 0.6n} - \frac{1}{j2}\cdot e^{-j2\pi 0.6n} \tag{24}$$

$$\left[e^{j2\pi 0.6n} = e^{-j2\pi 0.4n} \quad \text{and} \quad \frac{1}{j2} = \frac{1}{2}\cdot e^{-j\pi/2} \right] \tag{25}$$

$$= \frac{1}{2}\cdot e^{j2\pi 0.4n} + \frac{1}{2}\cdot e^{-j2\pi 0.4n} + \frac{1}{2}\cdot e^{-j\pi/2} e^{-j2\pi 0.4n} - \frac{1}{2}\cdot e^{-j\pi/2} e^{j2\pi 0.4n} \tag{26}$$

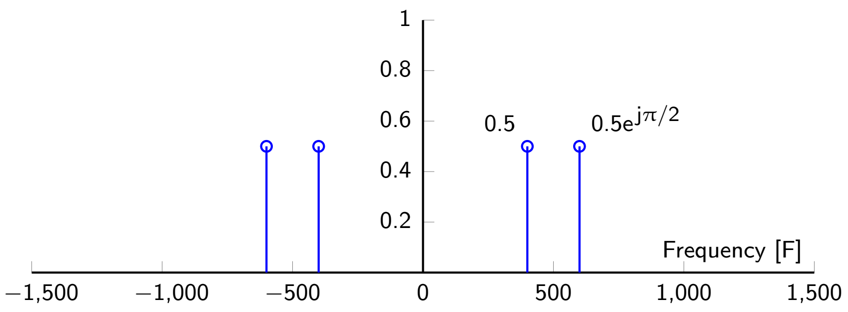
$$= \frac{1}{2}\cdot\left(1 - e^{-j\cdot\pi/2}\right)\cdot e^{j2\pi 0.4n} + \frac{1}{2}\cdot\left(1 + e^{-j\cdot\pi/2}\right)\cdot e^{-j2\pi 0.4n} \tag{27}$$

$$= \frac{\sqrt{2}}{2}\cdot e^{j\left(2\pi 0.4n + \frac{\pi}{4}\right)} + \frac{\sqrt{2}}{2}\cdot e^{-j\left(2\pi 0.4n + \frac{\pi}{4}\right)} \tag{28}$$

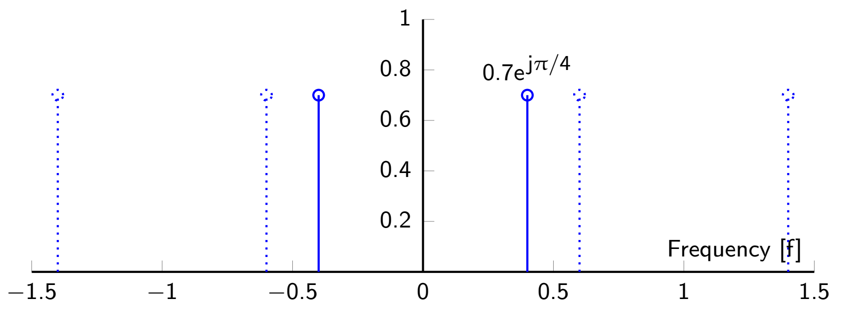
$$= \sqrt{2}\cdot\cos\left(2\pi 0.4n + \frac{\pi}{4}\right) \tag{29}$$

We draw the magnitude spectrum as usual but also note the phases in the figure.

Spectrum before sampling (the analog signal):



Spectrum after sampling (the discrete signal):



How do we do reconstruction and D/A-conversion? (page 387–388, 395–397)

We select the part of the spectrum in the interval

$$-0.5 < f < 0.5 \tag{30}$$

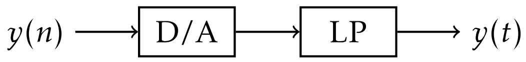
or

$$-\pi < \omega < \pi \tag{31}$$

equivalent to

$$-\frac{F_s}{2} < F < \frac{F_s}{2} \qquad \text{in Hz} \tag{32}$$

using a low pass filter.



This is a convolution and in the time domain it becomes (see the appendix and the end)

$$y(t) = \sum_{n=-\infty}^{\infty} x(n) \cdot \frac{\sin\left(\frac{\pi}{T} \cdot (t - nT)\right)}{\frac{\pi}{T} \cdot (t - nT)} \qquad \text{where } n = F_s \cdot t \tag{33}$$

This is a convolution with the filter $h_{\mathrm{LP}}(t)$ where

$$h_{\mathrm{LP}}(t) = \frac{\sin\left(\frac{\pi}{T} \cdot t\right)}{\frac{\pi}{T} \cdot t} = T \cdot \frac{\sin\left(\frac{\pi}{T} \cdot t\right)}{\pi t} \tag{34}$$

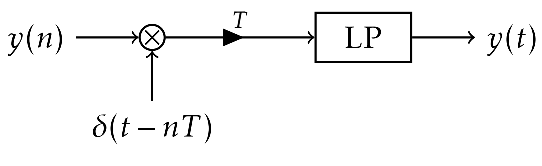
is a low pass filter with the gain T and cut-off frequency $F_s/2$. The spectrum of the output signal is given by (convolution becomes multiplication)

$$Y(F) = \frac{1}{F_s} \cdot Y(\omega) \cdot H_{\mathrm{LP}}(\omega) \tag{35}$$

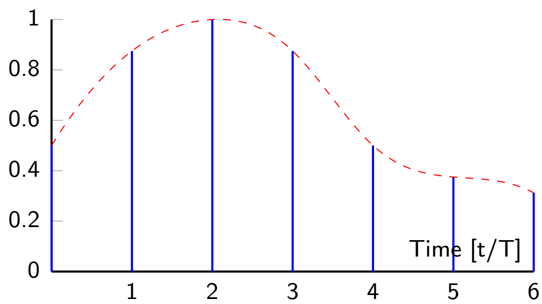
The sampling theorem states that a signal $x_a(t)$ can be sampled and then reconstructed exactly of the sampling frequency is chosen to be at least twice the highest frequency component of the signal.

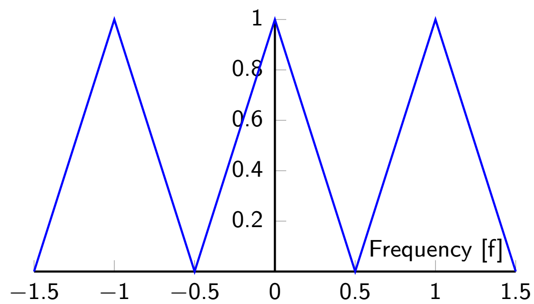
Ideal reconstruction

Select a period of $X(\omega)$ using a low pass filter.

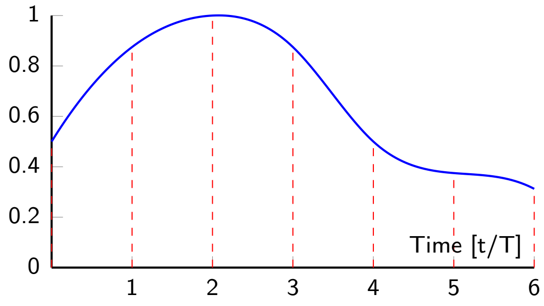


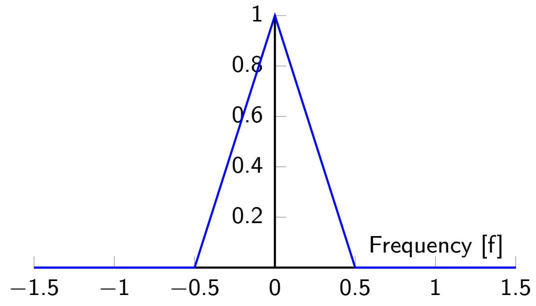
Analog impulse signal of the discrete signal:





Low pass filtered signal:





Reconstruction with sample-and-hold

During ideal reconstruction, the output signal energy is low because the impulses hold very little energy. We can increase the energy by widening the pulses. If the pulses are widened to match the sampling period, we get a *sample-and-hold* or *zero-order-hold* system.

We can interpret the sample-and-hold unit as a filter with a rectangular impulse response. The result is that the spectrum of the output signal is also multiplied by the frequency response of this filter.

$$h_{\mathrm{SH}}(t) = \begin{cases} 1 & 0 \leq t < T \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

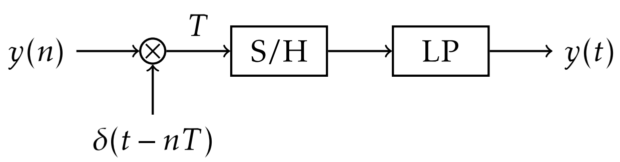
The Fourier transform is

$$H_{\mathrm{SH}}(F) = T \cdot \frac{\sin(\pi F T)}{\pi F T} \cdot \mathrm{e}^{-\mathrm{j}2\pi F T/2} \tag{37}$$

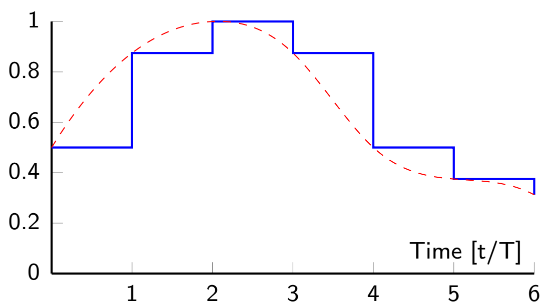
and the spectrum of the output signal becomes

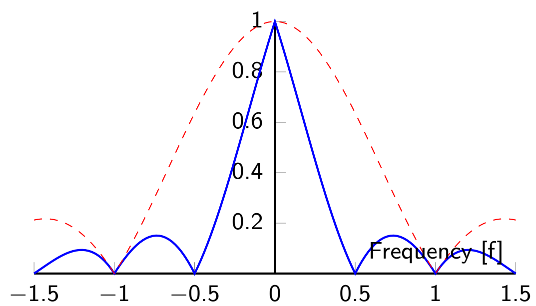
$$Y(F) = Y(\omega) \cdot H_{\mathrm{SH}}(F) \cdot H_{\mathrm{LP}}(F) \tag{38}$$

We therefore introduce a scaling distortion. The error is zero for $F = 0$ and $\sin(\pi/2)/\pi/2 \approx 0.64$ for $F = F_s/2$.

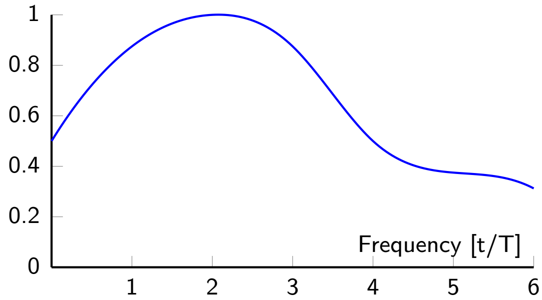


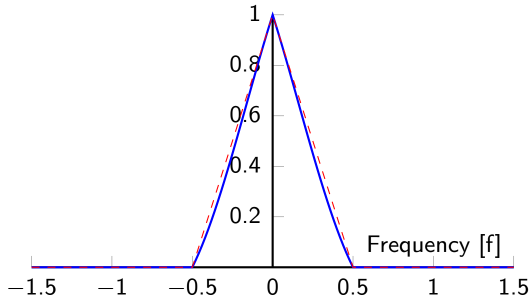
Analog sample-and-hold signal of the discrete signal:





Low pass filtered signal:





Interpolation and decimation

Interpolation increases the sample rate of a signal.

$$x(n) \longrightarrow \boxed{\uparrow I} \longrightarrow y(n) \qquad F_s' = F_s \cdot I$$

Decimation reduces the sample rate of a signal.

$$x(n) \longrightarrow \boxed{\downarrow D} \longrightarrow y(n) \qquad F_s' = F_s / D$$

Interpolation

Given:

$$x(n) = \sin(2\pi f_0 n) = \left\{ \begin{array}{cccccc} \dots & x(-1) & x(0) & x(1) & x(2) & \dots \end{array} \right\} \quad \text{where } f_0 = 0.4 \tag{39}$$

Find: $Y(f)$ of the sequence

$$y(n) = \left\{ \begin{array}{cccccccccc} \dots & x(-1) & 0 & x(0) & 0 & x(1) & 0 & x(2) & \dots \end{array} \right\} \tag{40}$$

Solution:

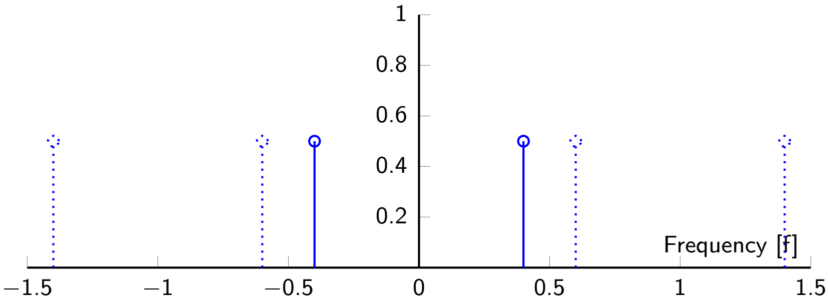
$$y(n) = \begin{cases} x(n/2) & \text{for } n \text{ even} \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

$$Y(\omega) = \sum_n y(n) \mathrm{e}^{-\mathrm{j}\omega n} \qquad [\text{let } n' = n/2 \text{ and } n = 2n'] \tag{42}$$

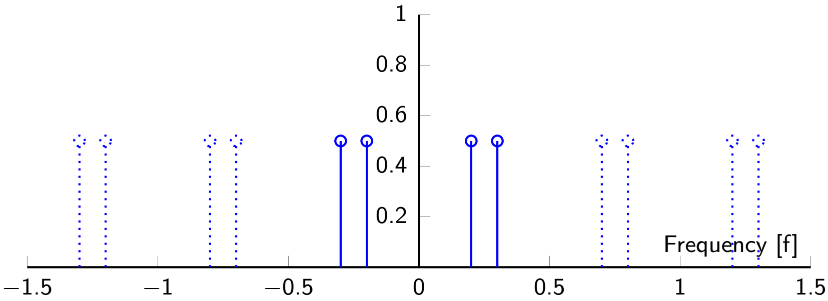
$$= \sum_{n'} x(n') \mathrm{e}^{-\mathrm{j}\omega 2n'} = X(2\omega) \tag{43}$$

We get a scaling of the frequency axis. One period of $Y(\omega)$ corresponds to 2 periods of $X(\omega)$.

Before interpolation:



After interpolation:



Therefore, we get

$$y(n) = \sin(2\pi 0.2n) + \sin(2\pi 0.3n) \tag{44}$$

Frequencies after interpolation are

$$f' = \frac{\pm\frac{F}{F_s} \pm k}{I} \qquad -0.5 < f' < 0.5 \tag{45}$$

Decimation

Given:

$$x(n) = \sin(2\pi f_0 n) \tag{46}$$

Find: $Y(\omega)$ of the sequence

$$y(n) = x(n \cdot D) \qquad \text{for } D = 4 \tag{47}$$

Solution:

$$Y(\omega) = \sum_{n'} y(n') \mathrm{e}^{-\mathrm{j}\omega \cdot \frac{n'}{D}} = X(\omega/D) \qquad \text{where } n' = n \cdot D \tag{48}$$

Assume a sinusoid of 100 Hz is sampled with a frequency of 8000 Hz.

$$x(n) = \sin\left(2\pi \cdot \frac{1}{80} \cdot n\right) \tag{49}$$

The signal after decimation by $D = 4$ is

$$y(n) = x(Dn) = \sin\left(2\pi 4 \cdot \frac{1}{80} \cdot n\right) = \sin\left(2\pi \cdot \frac{1}{20} \cdot n\right) \tag{50}$$

The frequency of the sampled signals are $f_{x(n)} = \pm 0.0125 \pm k$ and $f_{y(n)} = \pm 0.05 \pm k$. This corresponds to 100 Hz i both the sample rate of the input signal and the output signal.

Frequencies after decimation are

$$f' = \pm \frac{F}{F_s} \cdot D \pm k \tag{51}$$

Decimation with folding

Given: Previous example with $f_0 = 3200\,\text{Hz}$ and $D = 3$.

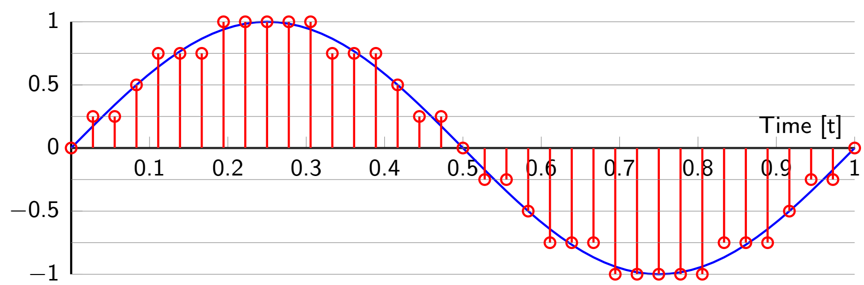
$$y(n) = x(n \cdot D) = \sin\left(2\pi 3 \cdot \frac{2}{5} \cdot n\right) = \sin\left(2\pi \cdot \frac{6}{5} \cdot n\right) = \sin\left(2\pi \cdot \frac{1}{5} \cdot n\right) \tag{52}$$

The frequencies of the sampled signals are $f_{x(n)} = \pm 0.0125 \pm k$ and $f_{y(n)} = \pm 0.05 \pm k$. This corresponds to 3200 Hz for the input signal but 533 Hz for the output signal due to folding.

Quantization during A/D-conversion (page 403–408)

The quantization effect:

$$x(t) = A \cdot \sin(\omega_0 t) \tag{53}$$



A resolution of b binary bits yields a total of 2^b signal levels. If the maximum amplitude is A then the dynamic range becomes $2A$.

Quantization resolution:

$$\Delta = \frac{2A}{2^b} \tag{54}$$

Quantization error power: The variance of a random variable with a rectangular probability density function.

$$P_q = \frac{\Delta^2}{12} \tag{55}$$

Signal power: Sinusoid with amplitude A.

$$P_s = \frac{A^2}{2} \tag{56}$$

Signal to Quantization noise ratio:

$$\mathrm{SQNR} = 10 \cdot \log_{10} \frac{P_s}{P_q} = 1.76 + 2b \approx 6 \times \text{number of bits} \qquad \text{[in dB]} \tag{57}$$

Decibel

Decibel [dB] is a logarithmic measure of a power-ratio between a value and a reference. It can be seen as the gain or attenuation of a system.

$$\text{gain in dB} = 10 \cdot \log_{10} \left(\frac{\text{output power}}{\text{input power}} \right) \tag{58}$$

The ratio is either:

- A relative power ratio of two signal. For example the power of an input signal and an output signal.
- An absolute signal power with an implied standardised reference and unit. For example dBV has a reference of 1 V and dB SPL has a reference of 20 µPa sound pressure level.

The decibel is always unitless and a relative measure.

Example with a gain

Given: A system with a gain of 2.

$$x(n) \xrightarrow{2} y(n)$$

Find: The system gain in dB.

Solution: Calculate input and output powers.

$$P_{\text{in}} = \mathrm{E}\left[x(n) \cdot x^*(n)\right] = \sigma_x^2 \tag{59}$$

$$P_{\text{out}} = \mathrm{E}\left[y(n) \cdot y^*(n)\right] = \mathrm{E}\left[2 \cdot x(n) \cdot 2 \cdot x^*(n)\right] = 4 \cdot \sigma_x^2 \tag{60}$$

$$\text{gain in dB} = 10 \cdot \log_{10} \left(\frac{4 \cdot \sigma_x^2}{\sigma_x^2} \right) = 10 \cdot \log_{10} \left(\frac{4}{1} \right) \approx 6.02\,\mathrm{dB} \tag{61}$$

A linear gain of 2 corresponds to a gain of approximately 6 dB.

Example with an attenuation

Given: A system with a gain of 1/4.

$$x(n) \xrightarrow{1/4} y(n)$$

Find: The system gain in dB.

Solution:

$$\text{gain in dB} = 10 \cdot \log_{10}\left(\frac{\sigma_x^2}{16 \cdot \sigma_x^2}\right) = 10 \cdot \log_{10}\left(\frac{1}{16}\right) \approx -12.04\,\text{dB} \tag{62}$$

A linear gain of 0.25 corresponds to a gain of approximately $-12\,\text{dB}$, or equivalently an attenuation of $12\,\text{dB}$.

Example with calculating linear gain from decibel

Given: A system with a gain of $3\,\text{dB}$.

$$x(n) \xrightarrow{3\,\text{dB} = a} y(n)$$

Find: The linear gain of the system.

Solution: Solve for the desired linear gain a in

$$3 = 10 \cdot \log_{10}\left(a^2\right) \tag{63}$$

$$\frac{3}{10} = \log_{10}\left(a^2\right) \tag{64}$$

$$10^{\frac{3}{10}} = a^2 \qquad \Rightarrow \qquad a \approx \sqrt{2} \tag{65}$$

Examples of decibel values

	Linear	Decibel	
	1000	60 dB	
	100	40 dB	
	10	20 dB	
	4	12 dB	(approximate)
	2	6 dB	(approximate)
$\sqrt{2}$	1.414	3 dB	(approximate)
	1	0 dB	
$\sqrt{0.5}$	0.707	−3 dB	(approximate)
	0.5	−6 dB	(approximate)
	0.25	−12 dB	(approximate)
	0.1	−20 dB	
	0.01	−40 dB	
	0.001	−60 dB	