Lecture 8

Digital Signal Processing

Chapter 5

LTI systems

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Convolution and the z-transform

Example E3.3

Given: The system

$$y(n) - y(n-1) + \frac{3}{16} \cdot y(n-2) = x(n) \tag{1}$$

where

$$x(n) = \left(\frac{1}{2}\right)^n \cdot u(n) + \sin\left(2\pi \cdot \frac{1}{4} \cdot n\right) \tag{2}$$

Find: The output signal $y(n)$.

$$H(z) = \frac{1}{1 - z^{-1} + \frac{3}{16} \cdot z^{-2}} = \frac{z^2}{z^2 - z + \frac{3}{16}} \tag{3}$$

Solution: The poles of the filter is given by

$$z^2 - z + \frac{3}{16} = 0 \quad \rightarrow \quad p_1 = 0.25 \quad \text{and} \quad p_2 = 0.75 \tag{4}$$

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} = \frac{-0.5}{1 - p_1 z^{-1}} + \frac{1.5}{1 - p_2 z^{-1}} \tag{5}$$

$$h(n) = (-0.5 \cdot 0.25^n + 1.5 \cdot 0.75^n)\, u(n) \tag{6}$$

Split the input signal into the two terms and determine the output signal for each term individually.

First term

$$x_1(n) = 0.5^n \cdot u(n) \tag{7}$$

$$Y_1(z) = \frac{1}{(1 - 0.25z^{-1})(1 - 0.75z^{-1})} \cdot \frac{1}{1 - 0.5z^{-1}} \tag{8}$$

$$= \frac{\frac{1}{2}}{1 - 0.25z^{-1}} + \frac{\frac{9}{2}}{1 - 0.75z^{-1}} - \frac{4}{1 - 0.5z^{-1}} \tag{9}$$

The inverse transform gives

$$y_1(n) = 0.5 \cdot 0.25^n \cdot u(n) + 4.5 \cdot 0.75^n \cdot u(n) - 4 \cdot 0.5^n \cdot u(n) \tag{10}$$

Second term

$$x_2(n) = \sin(\omega_0 n) \qquad \text{where } \omega_0 = 2\pi \cdot \frac{1}{4} \tag{11}$$

$$y_2(n) = |H(\omega_0)| \cdot \sin(\omega_0 n + \angle H(\omega_0)) \tag{12}$$

Calculate

$$H(\omega_0) = H(z \mid z = \mathrm{e}^{\mathrm{j}\omega_0}) = \frac{1}{1 - \mathrm{e}^{-\mathrm{j}\omega_0} + \frac{3}{16} \cdot \mathrm{e}^{-\mathrm{j}\omega_0 \cdot 2}} = 0.77 \cdot \mathrm{e}^{-\mathrm{j}0.88} \tag{13}$$

which gives

$$y_2(n) = 0.77 \sin\left(2\pi \cdot \frac{1}{4} \cdot n - 0.88\right) \tag{14}$$

First and second part

The final solution is given by the sum of the two individual solutions.

$$y(n) = y_1(n) + y_2(n) \tag{15}$$

$$= 0.5 \cdot 0.25^n \cdot u(n) + 4.5 \cdot 0.75^n \cdot u(n) - 4 \cdot 0.5^n \cdot u(n) + + 0.77 \sin\left(2\pi \cdot \frac{1}{4} \cdot n - 0.88\right) \tag{16}$$

Filters

We will now look at how we can use time-discrete filters. We start by an example of a notch filter.

Suppose that we have a signal that is disturbed by a sinusoidal signal.

$$x(t) = s(t) + \sin(\Omega_0 t) \tag{17}$$

where $\Omega_0 = 2\pi \cdot 1250$ or $F_0 = 1250\,\mathrm{Hz}$. We will eliminate the sinusoidal disturbance with a time-discrete filter. Start by sampling the signal with the sampling frequency $F_s = 10\,000\,\mathrm{Hz}$. We get

$$x(n) = s(n) + \sin(\omega_0 n) \tag{18}$$

where

$$\omega_0 = 2\pi \cdot \frac{F_0}{F_s} = 2\pi \cdot 0.125 \tag{19}$$

We want to construct a filter that stops the frequency $\omega_0 = 2\pi \cdot 0.125$. The filter shall therefore have an amplification of zero at this particular frequency.

$$H(\omega_0) = 0 \tag{20}$$

In terms of poles and zeros, the filter must have zeros at

$$n_{1,2} = \mathrm{e}^{\pm\mathrm{j}\omega_0} = \mathrm{e}^{\pm\mathrm{j}2\pi \cdot 0.125} \tag{21}$$

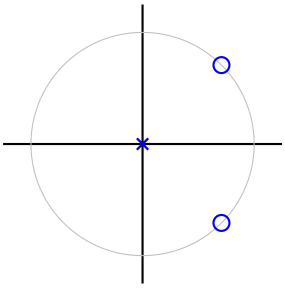
We can choose between an FIR filter or an IIR filter.

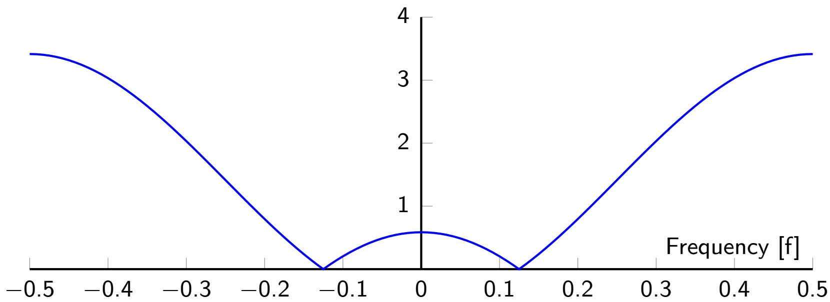
Notch FIR filter

An FIR filter has all its poles at the origin. We place two poles at the origin which yields the filter

$$H(z) = \frac{\left(z - e^{-j\omega_0}\right)\left(z - e^{j\omega_0}\right)}{z^2} = 1 - 2\cos(\omega_0)z^{-1} + z^{-2} \tag{22}$$

Pole-zero diagram and amplitude response:



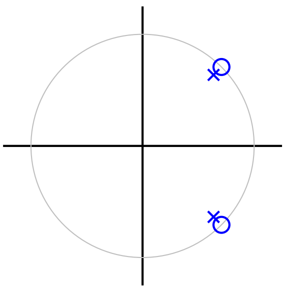


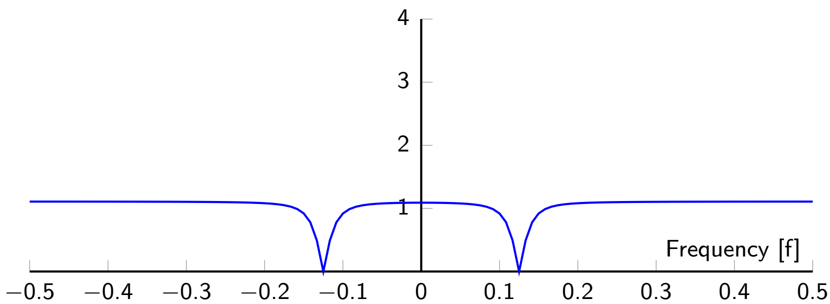
Notch IIR-filter

The poles of an IIR filter can be placed arbitrarily inside the unit circle. We place two poles just inside the unit circle near the zeros, yielding the following filter:

$$H(z) = \frac{\left(z - e^{-j\omega_0}\right)\left(z - e^{j\omega_0}\right)}{\left(z - \alpha e^{-j\omega_0}\right)\left(z - \alpha e^{j\omega_0}\right)} = \frac{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}{1 - 2\cdot 0.95\cdot\cos(\omega_0)z^{-1} + 0.95^2\cdot z^{-2}} \tag{23}$$

where $0 \leq \alpha < 1$ but is typically chosen near 1. Pole-zero diagram and amplitude response:





This will be explored further during the labs.

FIR filter with linear phase

An FIR filter with linear phase has a symmetric impulse response.

Symmetry	Description	Filter property
$h(n) = h(-n)$	Symmetry around 0.	$H(\omega)$ real.
$h(n) = h(N-n)$	Symmetry around $N/2$.	$H(\omega)$ linear phase.
$h(n) = -h(N-n)$	Anti-symmetry around $N/2$.	$H(\omega)$ linear phase.

Example:

- Symmetric impulse response around $n = 0$.

$$h(n) = \left\{ \begin{array}{ccccc} 1 & 2 & \underline{3} & 2 & 1 \end{array} \right\} \tag{24}$$

- Causal symmetric impulse response.

$$h(n) = \left\{ \begin{array}{ccccc} \underline{1} & 2 & 3 & 2 & 1 \end{array} \right\} \tag{25}$$

- Causal anti-symmetric impulse response.

$$h(n) = \left\{ \begin{array}{ccccc} \underline{1} & 2 & 0 & -2 & -1 \end{array} \right\} \tag{26}$$

Show that $H(\omega)$ has a linear phase. Assume a causal symmetric impulse response:

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4} \tag{27}$$

$$H(\omega) = 1 + 2\mathrm{e}^{-\mathrm{j}\omega} + 3\mathrm{e}^{-\mathrm{j}2\omega} + 2\mathrm{e}^{-\mathrm{j}3\omega} + \mathrm{e}^{-\mathrm{j}4\omega} \tag{28}$$

$$= \left(\mathrm{e}^{\mathrm{j}2\omega} + 2\mathrm{e}^{\mathrm{j}\omega} + 3 + 2\mathrm{e}^{-\mathrm{j}\omega} + \mathrm{e}^{-\mathrm{j}2\omega}\right) \cdot \mathrm{e}^{-\mathrm{j}2\omega} \tag{29}$$

$$= (3 + 4\cos\omega + 2\cos 2\omega) \cdot \mathrm{e}^{-\mathrm{j}2\omega} \tag{30}$$

$$= |(3 + 4\cos\omega + 2\cos 2\omega)| \cdot \mathrm{e}^{-\mathrm{j}2\omega + \mathrm{j}\pi\cdot k} \qquad \text{for } k \text{ integer} \tag{31}$$

where k are eventual phase jumps in π. The first factor is real values and the second factor is linear phase.

Pole-zero diagram

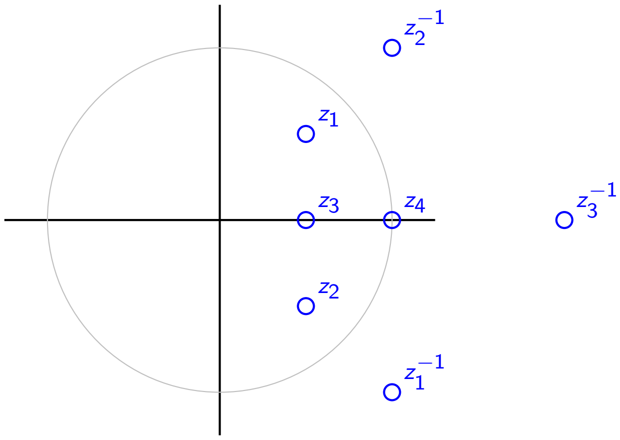
What does linear phase look like in a pole-zero diagram?

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4} \tag{32}$$

$$= z^{-4} \cdot \left(z^4 + 2z^3 + 3z^2 + 2z + 1\right) \tag{33}$$

$$= z^{-4} \cdot H(z^{-1}) \tag{34}$$

$H(z)$ and $H(z^{-1})$ must therefore be zero for the same values of z. If z is a zero then z^{-1} must also be a zero.



Filter types

Ideal low pass filter

A non-causal ideal low pass filter is defined as

$$H_{\text{ideal}}(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

The impulse response is

$$h(n) = \frac{\omega_c}{\pi} \cdot \frac{\sin \omega_c n}{\omega_c n} \tag{36}$$

A causal low pass FIR filter can be obtained by selecting N (choose N odd) values around the origin and then delay the impulse response by $(N-1)/2$.

$$h(n) = \frac{\omega_c}{\pi} \cdot \frac{\sin \omega_c \left(n - \frac{N-1}{2}\right)}{\omega_c \left(n - \frac{N-1}{2}\right)} \tag{37}$$

Ideal high pass filter

A non-causal ideal high pass filter is defined as

$$H_{\text{ideal}}(\omega) = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & \text{otherwise} \end{cases} \tag{38}$$

$$= 1 - \text{low pass filter} \tag{39}$$

The impulse response is

$$h(n) = \delta(n) - \frac{\omega_c}{\pi} \cdot \frac{\sin \omega_c n}{\omega_c n} \tag{40}$$

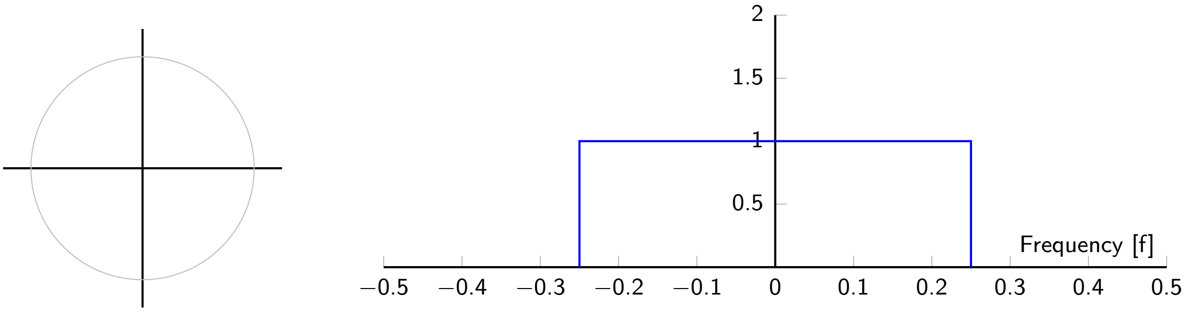
A causal high pass FIR filter can be obtained by selecting N (choose N odd) values around the origin and then delay the impulse response by $(N-1)/2$.

$$h(n) = \delta\left(n - \frac{N-1}{2}\right) - \frac{\omega_c}{\pi} \cdot \frac{\sin \omega_c \left(n - \frac{N-1}{2}\right)}{\omega_c \left(n - \frac{N-1}{2}\right)} \tag{41}$$

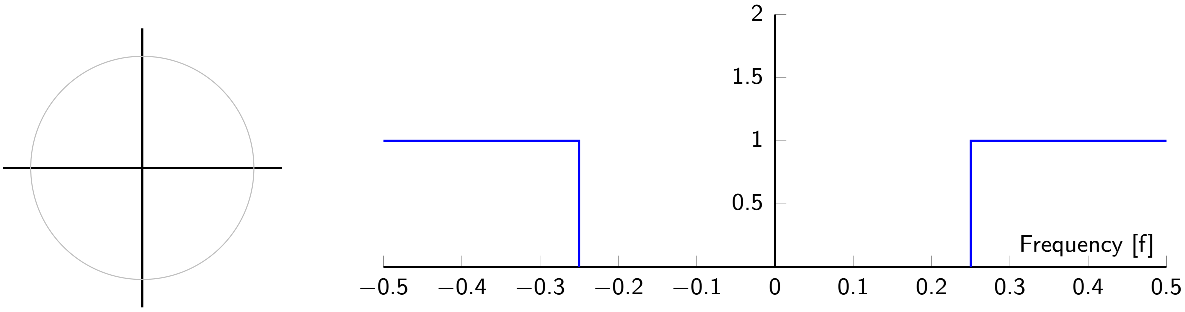
Classification of filters

Suggest a pole-zero placement for the filter types (page 330–346).

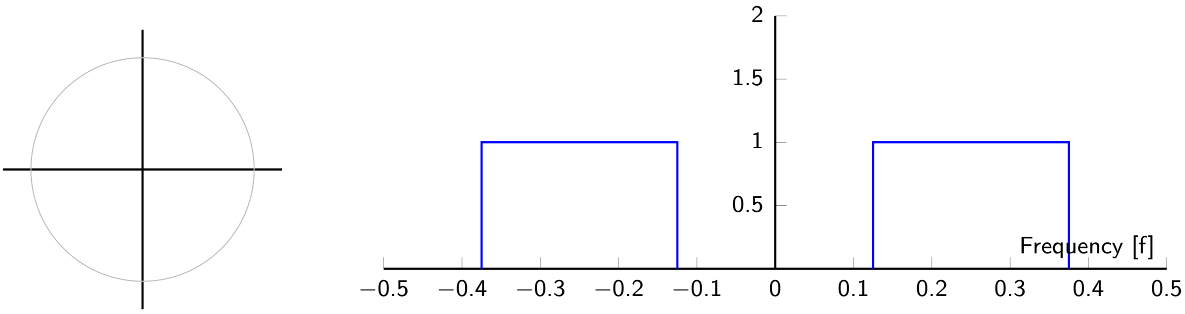
Low pass filter:



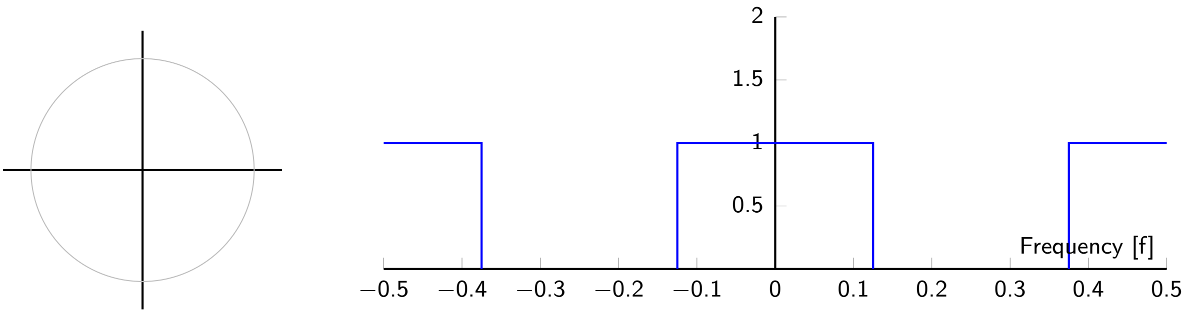
High pass filter:



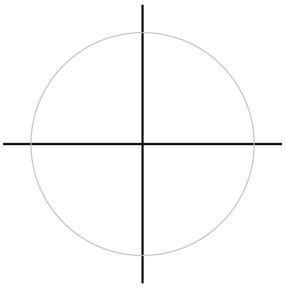
Band pass filter:

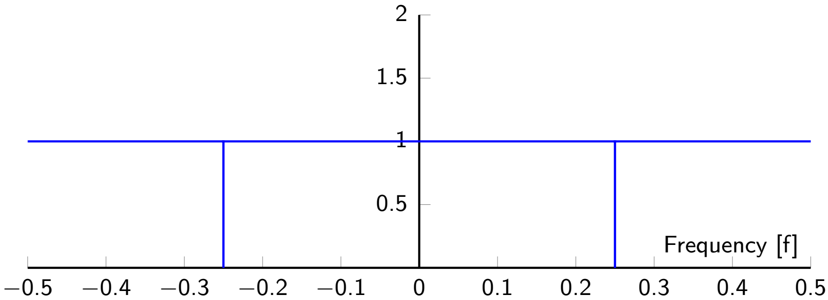


Band stop filter:

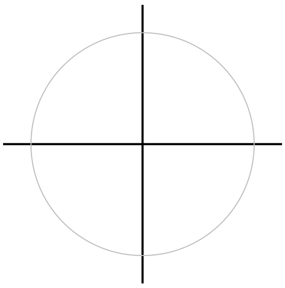


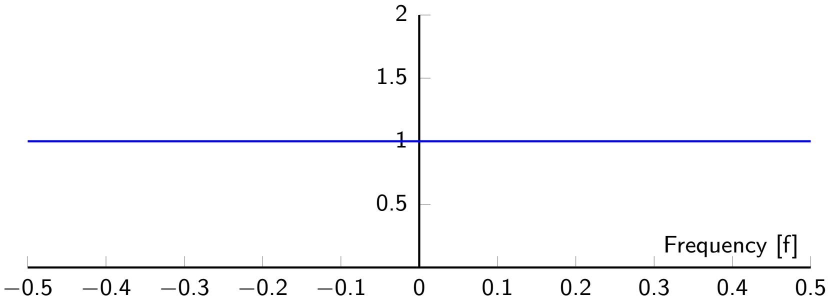
Notch filter:



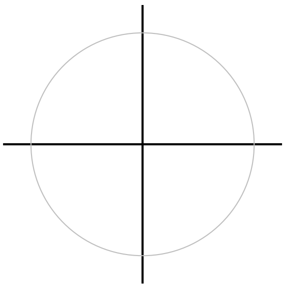


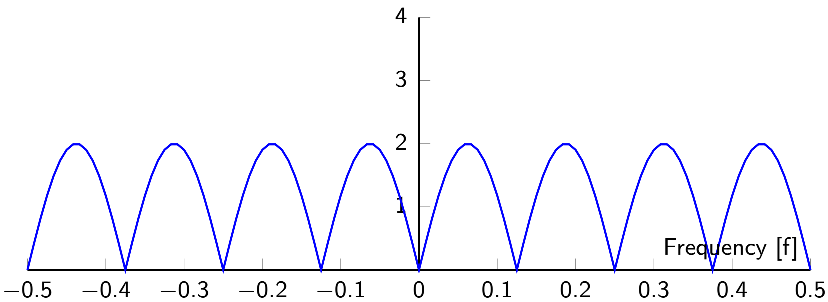
All pass filter:





Comb filter:





Minimum and maximum phase systems

A common classification of systems are *minimum* and *maximum* phase systems.

A system $H(z)$ with all the zeros inside the unit circle is called a minimum phase system. The name comes from the phase response which is as small as possible. This can be seen from a pole-zero diagram. When a zero is located inside the unit circle, the phase contribution to the frequency response for that zero is minimum. It can also be seen from the impulse response where most of the energy of the filter is located near $n = 0$.

Likewise, a system $H(z)$ with all the zeros outside the unit circle is called a maximum phase system. Zeros outside the unit circle contribute the most to the phase response of the filter. Most of the energy of a maximum phase filter is located near the end of the filter.

A system $H(z)$ with zeros both inside and outside the unit circle is called a *mixed phase* system.

We often want minimum or linear phase systems.

Causal FIR and IIR filters

FIR filters:

- The impulse response is of finite duration.
- The filter is always stable.
- All poles are at the origin.
- The filter may have linear phase.

IIR filters:

- The impulse response is of infinite duration.
- The filter is stable if and only if all poles lie inside the unit circle.
- The filter cannot have linear phase.

Connection between the number of poles N_p, the number of zeros N_z and the impulse response $h(n)$:

$N_p = N_z$ For example a zero and a pole, with the impulse response starting at $n = 0$:

$$h(n) = \left\{ \ \underline{1} \quad 1 \quad 0 \quad \ldots \ \right\} \quad \Rightarrow \quad H(z) = 1 + z^{-1} = \frac{z-1}{z} \tag{42}$$

$N_p = N_z + 1$ For example a zero and two poles, with the impulse response starting at $n = 1$:

$$h(n) = \left\{ \ \underline{0} \quad 1 \quad 1 \quad 0 \quad \ldots \ \right\} \quad \Rightarrow \quad H(z) = z^{-1} + z^{-2} = \frac{z-1}{z^2} \tag{43}$$

$N_p = N_z + 2$ For example a zero and three poles, with the impulse response starting at $n = 2$:

$$h(n) = \left\{ \begin{array}{cccccc} \underline{0} & 0 & 1 & 1 & 0 & \dots \end{array} \right\} \quad \Rightarrow \quad H(z) = z^{-2} + z^{-3} = \frac{z-1}{z^3} \tag{44}$$

If the number of poles is greater or equal to the number of zeros, the system is causal. This applies generally for both causal FIR filters and causal IIR filters.