Lecture 7

Digital Signal Processing

Chapter 5

LTI system
Signals in linear systems

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Linear time invariant systems

Difference equations:

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \tag{1}$$

The z-transform:

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \tag{2}$$

Convolution:

$$y(n) = h(n) * x(n) \tag{3}$$

$$= \sum_{k} h(k) x(n-k) \tag{4}$$

The transform of the output signal $Y(z)$ is the product of the transforms of the input signal $X(z)$ and the filter $H(z)$.

$$Y(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \cdot X(z) = H(z)X(z) \tag{5}$$

We have two kinds of difference equations.

- An FIR system has $a_k = 0$ for all $k \neq 0$. An FIR system therefore has no feedback. The impulse response is $h(n) = \left\{ \begin{array}{cccc} b_0 & b_1 & \cdots & b_M \end{array} \right\}$ which is the same as the coefficients of the difference equation.
- An IIR system has $a_k \neq 0$ for some $k \neq 0$. An IIR system therefore has some feedback.

We often describe the system equation $H(z)$ with poled and zeros and draw them in a pole-zero diagram.

Fourier transform

If $h(n)$ is causal and stable we have the identity

$$H(\omega) = H(z) \qquad \text{where } z = \mathrm{e}^{\mathrm{j}\omega} \tag{6}$$

and therefore

$$Y(\omega) = \frac{b_0 + b_1 \mathrm{e}^{-\mathrm{j}\omega} + \cdots + b_M \mathrm{e}^{-\mathrm{j}\omega M}}{1 + a_1 \mathrm{e}^{-\mathrm{j}\omega} + \cdots + a_N \mathrm{e}^{-\mathrm{j}\omega N}} \cdot X(\omega) = H(\omega)X(\omega) \tag{7}$$

The output signal $Y(\omega)$ is the product of the input signal $X(\omega)$ and the filter $H(\omega)$. The filter $H(\omega)$ is called the frequency response. We often write $H(\omega)$ in polar coordinates and plot the amplitude and the phase of the frequency response.

Sinusoidal signals and LTI systems

What happens if we filter a sinusoidal signal? We know from experience that if we filter a sinusoid of a given frequency, then we get a sinusoid with the same frequency but with a different amplitude and phase. We will examine two cases:

- We start the signal at $n = 0$. We solve this using the z-transform and partial fraction expansion.
- We start the signal at $n = -\infty$ so that any initial conditions have dissipated. We solve this using convolution because the input signal is not causal so we cannot determine its z-transform.

Numerical solution in Matlab

First determine a numerical solution in Matlab.

Given: The input signal

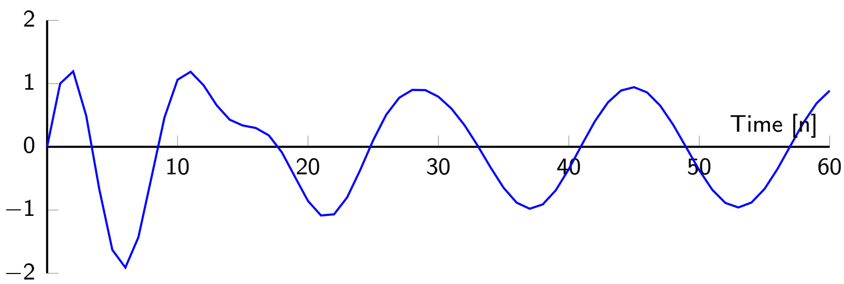
$$x(n) = \cos\left(2\pi \cdot \frac{1}{16} \cdot n\right) \cdot u(n) \tag{8}$$

and the system

$$H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}} \tag{9}$$

Find: Determine numerically the output signal $y(n) = x(n) * h(n)$.

```
>> n = 0:60;
>> b = [0, 1, -1];
>> a = [1, -1.27, 0.81];
>> x = cos(2*pi*n/16);
>> y = filter(b, a, x);
>> plot(n, y);
```



We get $y(n) =$ transient solution + stationary solution.

Solution using the z-transform

We start the signal at $n = 0$:

$$x(n) = \cos(\omega_0 n) \cdot u(n) \qquad \text{where } \omega_0 = 2\pi \cdot \frac{1}{16} \tag{10}$$

This signal is causal and we can determine its z-transform. The transforms of $x(n)$ and $h(n)$ are

$$X(z) = \frac{1 - \cos(\omega_0) z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}} \tag{11}$$

and

$$H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}} = \frac{N(z)}{D(z)} \tag{12}$$

We can now determine the output signal using the z-transform:

$$\begin{aligned} Y(z) &= H(z)X(z) && (13)\\ &= \frac{N(z)}{D(z)} \cdot \frac{1 - \cos(\omega_0) z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}} && (14)\\ &= \frac{N_1(z)}{D(z)} + \frac{C_0 + C_1 z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}} && (15) \end{aligned}$$

where the two terms are the transient solution and the stationary solution.

$$y(n) = \text{transient solution} + A\cos(\omega_0 n) + B\sin(\omega_0 n) \tag{16}$$

If we want the whole solution we have to determine the partial fraction expansions $N_1(z)$ and $N_2(z) = C_0 + C_1 z^{-1}$ and do the inverse z-transforms.

$$\begin{aligned} Y(z) &= \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}} \cdot \frac{1 - \cos(\omega_0) z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}} && (17)\\ &= -0.35 \cdot \frac{1 - 4.177z^{-1}}{1 - 1.27z^{-1} + 0.81z^{-2}} + 0.35 \cdot \frac{1 - 1.896z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}} && (18)\\ &= -0.35 \cdot \frac{1 - 0.9\cos\left(\frac{\pi}{4}\right) z^{-1}}{1 - 1.27z^{-1} + 0.81z^{-2}} && (19)\\ &\quad + 0.35 \cdot \frac{5.5629 \cdot \sin\left(\frac{\pi}{4}\right) z^{-1}}{1 - 1.27z^{-1} + 0.81z^{-2}} && (20)\\ &\quad + 0.35 \cdot \frac{1 - \cos(\omega_0) z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}} && (21)\\ &\quad + 0.35 \cdot \frac{(\cos(\omega_0) - 1.896)\, z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}} && (22) \end{aligned}$$

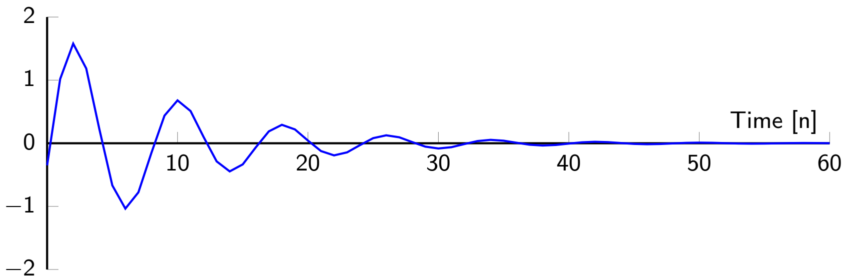
$$y(n) = -0.35 \cdot 0.9^n \cdot \cos\left(2\pi \cdot \frac{1}{8} \cdot n\right) + 0.35 \cdot 5.562 \cdot 0.9^n \sin\left(2\pi \cdot \frac{1}{8} \cdot n\right) \tag{23}$$

$$+ 0.35 \cdot \cos(\omega_0 \cdot n) - 0.35 \cdot 2.5392 \cdot \sin(\omega_0 \cdot n) \tag{24}$$

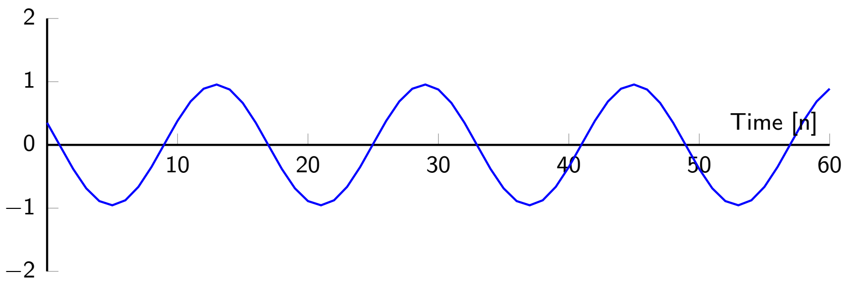
Plot the solution in Matlab.

```
>> n = 0:80;
>> yt = -0.35*0.9.^n.*cos(2*pi*n/8) + 0.35*5.562*0.9.^n.*sin(2*pi*n/8);
>> ys = 0.35*cos(2*pi*n/16) - 0.35*2.5392*sin(2*pi*n/16);
>> subplot(3, 1, 1); plot(n, yt);
>> subplot(3, 1, 2); plot(n, ys);
>> subplot(3, 1, 3); plot(n, yt+ys);
```

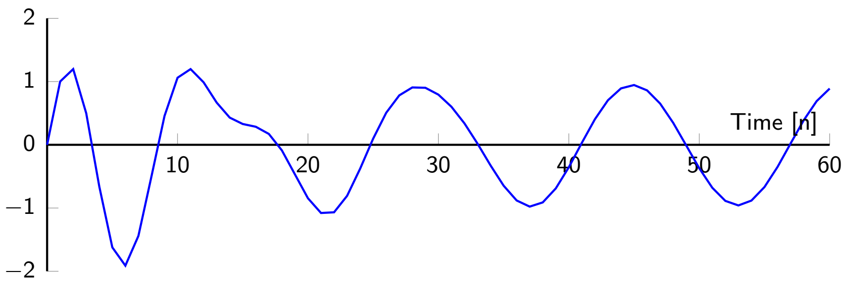
The transient solution:



The stationary solution:



The output signal as the sum of the stationary and the transient solutions.



We can see that the input signal $x(n)$ yields the stationary solution

$$y_{st}(n) = 0.95 \cdot \cos(\omega_0 n - 1.19) \tag{25}$$

We will show that the stationary solution is given by

$$y_{st}(n) = |H(z_0)| \cdot \cos(\omega_0 n + \angle H(z_0)) \tag{26}$$

where $z_0 = e^{j\omega_0}$.

A sinusoidal input signal with the frequency ω_0 yields a sinusoidal output signal with the same frequency, but with the amplitude changed by the magnitude of $H(z)$ and the phase changed by the argument of $H(z)$ for $z = e^{j\omega_0}$.

Solution without the transient state

The sinusoid is started at $n = -\infty$ and the transient part of the solution has now dissipated.

We start with a complex sinusoidal signal, see page 301-306.

$$x_0(n) = e^{j\omega_0 n} \tag{27}$$

The input signal is not causal so we use convolution.

$$y_0(n) = x_0(n) * h(n) \tag{28}$$

$$= \sum_{k=-\infty}^{\infty} h(k)x_0(n-k) \tag{29}$$

$$= \sum_{k=-\infty}^{\infty} h(k)e^{j\omega_0(n-k)} \tag{30}$$

$$= \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega_0 k}e^{j\omega_0 n} \tag{31}$$

$$= H(\omega_0) \cdot e^{j\omega_0 n} \tag{32}$$

The filter $h(n)$ has to be stable.

For the whole sinusoidal signal, using both terms of Euler's formula, we get

$$x(n) = \cos(\omega_0 n) = \frac{1}{2} \cdot \left[e^{j\omega_0 n} + \cdot e^{-j\omega_0 n}\right] = \frac{1}{2} \cdot \left[x_0(n) + x_0^*(n)\right] \tag{33}$$

which gives us the output signal

$$y(n) = \frac{1}{2} \cdot \left[H(\omega_0) \cdot e^{j\omega_0 n} + H^*(\omega_0) \cdot e^{-j\omega_0 n}\right] \tag{34}$$

$$= |H(\omega_0)| \cdot \cos(\omega_0 n + \angle H(\omega_0)) \tag{35}$$

We can determine and plot the amplitude and the phase for $H(\omega)$ using Matlab and just evaluate the frequency response at $\omega = \omega_0$.

```
>> w0 = 2*pi/16;
>> num = exp(-i*w0) - exp(-i*2*w0);
>> den = 1-1.27*exp(-i*w0)+0.81*exp(-i*2*w0);
>> H0 = num/den;
>> abs(H0), angle(H0)
```

```
ans =
    0.9546
ans =
    1.1956
```

Compare the magnitude and the phase with the stationary solution from before.

$$y_{st}(n) = 0.95 \cdot \cos(\omega_0 n - 1.19) \tag{36}$$

NOTE: This only applies after any initial conditions have dissipated from the system. For an FIR filter of length L, this is after $L-1$ samples. This is called the stationary solution, or the *steady state* solution.

NOTE: This only applies for sinusoidal signals, or for a composite signal (the sum of two or more sinusoidal signals) by computing the response for each component individually.

Linear phase

We often want a filter with linear phase.

$$x(n) \longrightarrow \boxed{H(\omega) = A(\omega)\mathrm{e}^{\mathrm{j}\Phi(\omega)}} \longrightarrow y(n)$$

$$x(n) = \sin(\omega_0 n) \tag{37}$$

$$y(n) = A(\omega_0)\sin(\omega_0 n + \Phi(\omega_0)) \tag{38}$$

$$= A(\omega_0)\sin\left(\omega_0\left(n + \frac{\Phi(\omega_0)}{\omega_0}\right)\right) \tag{39}$$

If $\Phi(\omega_0)/\omega_0$ is constant for all ω_0, then $\Phi(\omega)$ is a straight line in ω. In other words, the filter has linear phase. A filter with linear phase delays all frequencies by the same amount. The time

$$\tau_g = -\frac{\mathrm{d}\Phi(\omega)}{\mathrm{d}\omega} \tag{40}$$

is called the *group delay*.

Example of a filter with linear phase

Given: The impulse response $h(n) = \left\{ \begin{matrix} 1 & 2 & 1 \end{matrix} \right\}$.

Find: The phase response of $H(\omega)$.

Solution:

$$H(\omega) = 1 + 2\mathrm{e}^{-\mathrm{j}\omega} + \mathrm{e}^{-\mathrm{j}2\omega} \tag{41}$$

$$= \mathrm{e}^{-\mathrm{j}\omega} \cdot \left(\mathrm{e}^{\mathrm{j}\omega} + 2 + \mathrm{e}^{-\mathrm{j}\omega}\right) \tag{42}$$

$$= \mathrm{e}^{-\mathrm{j}\omega} \cdot (2 + 2\cos(\omega)) \tag{43}$$

$$= A(\omega) \cdot \mathrm{e}^{\mathrm{j}\Phi(\omega)} \tag{44}$$

$$\Phi(\omega) = -\omega \tag{45}$$