Lecture 6

Digital Signal Processing (ESS040)

Chapter 4

Fourier transform of analog and digital signals

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Filtering

Input-output relations:

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f)H(f)$$

We can use the convolution operation to determine the output signal.

$$y(n) = h(n) * x(n) = \sum_k x(k)h(n-k) \tag{1}$$

If the Fourier transforms of both the input signal and the output signal exist, then we can also do

$$Y(\omega) = H(\omega)X(\omega) \tag{2}$$

and then calculate the inverse Fourier transform.

We often classify filters according to the characteristic of $H(\omega)$.

Filter type	Description
Low pass filter	Passes low frequencies and stops high frequencies.
High pass filter	Passes high frequencies and stops low frequencies.
Band pass filter	Passes a limited frequency band.
Band stop filter	Stops a limited frequency band.

Relation to the z-transform

We assume a causal impulse response $h(n)$. Causal means that $h(n) = 0$ for $n < 0$. The Fourier transform, DTFT, of the impulse response is then according to the definition:

$$H(\omega) = \sum_{n=0}^{\infty} h(n)\mathrm{e}^{-\mathrm{j}\omega n} \tag{3}$$

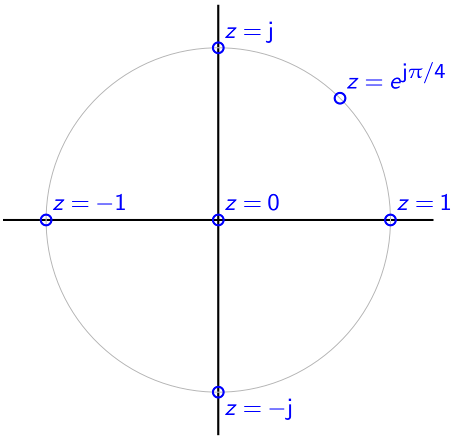
We defined the z-transform of the impulse response as

$$H(z) = \sum_{n=0}^{\infty} h(n)z^{-n} \tag{4}$$

where $z = \mathrm{e}^{\mathrm{j}\omega}$. The Fourier transform is the z-transform evaluated on the unit circle if $h(n)$ is causal and stable.

$$\boxed{H(\omega) = H(z \mid z = \mathrm{e}^{\mathrm{j}\omega})}$$

The unit circle is the frequency axis in the discrete domain.



Point in the z-plane	Frequency	
$z = 1$	$f = 0$	$\omega = 0$
$z = \mathrm{j}$	$f = 0.25$	$\omega = \pi/2$
$z = -1$	$f = \pm 0.5$	$\omega = \pm\pi$
$z = -\mathrm{j}$	$f = -0.25$	$\omega = -\pi/2$
$z = \mathrm{e}^{\mathrm{j}\pi/4}$	$f = 0.125$	$\omega = \pi/4$

Expanding the Fourier transform

Cosine (unstable signal):

$$x(n) = \cos(\omega_0 n) = \frac{1}{2} \cdot \left(\mathrm{e}^{\mathrm{j}\omega_0 n} + \mathrm{e}^{-\mathrm{j}\omega_0 n}\right) \tag{5}$$

$$X(\omega) = \frac{1}{2} \cdot [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \tag{6}$$

Sine (unstable signal):

$$x(n) = \sin(\omega_0 n) = \frac{1}{2\mathrm{j}} \cdot \left(\mathrm{e}^{\mathrm{j}\omega_0 n} - \mathrm{e}^{-\mathrm{j}\omega_0 n}\right) \tag{7}$$

$$X(\omega) = \frac{1}{2\mathrm{j}} \cdot [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] \tag{8}$$

Step:

$$x(n) = u(n) \tag{9}$$

$$X(\omega) = \frac{1}{1 - \mathrm{e}^{-\mathrm{j}\omega}} + \frac{1}{2} \cdot \delta(\omega) \tag{10}$$

The expressions of $X(\omega)$ are valid only for $-\pi < \omega < \pi$ and have to be made periodic for other ω.

Filtering with ideal low pass filter

We want to construct a filter that removes the high frequencies and keeps only the low frequencies.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f)H(f)$$

We have

$$y(n) = h(n) * x(n) = \sum_k x(k)h(n-k) \tag{11}$$

and

$$Y(\omega) = H(\omega)X(\omega) \tag{12}$$

An ideal low pass filter (non-causal) is defined as

$$H_{\text{ideal}}(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Frequencies less than ω_c (the cut-off frequency) are passed through unchanged while frequencies greater than ω_c are blocked entirely. What does the impulse response look like?

Inverse Fourier transform of a rectangular pulse

The ideal low pass filter is defined by a rectangular pulse. The impulse response is

$$\begin{aligned} h(n) &= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} H(\omega) \mathrm{e}^{\mathrm{j}\omega n} \, \mathrm{d}\omega & (14) \\ &= \frac{1}{2\pi} \cdot \int_{-\omega_c}^{\omega_c} H(\omega) \mathrm{e}^{\mathrm{j}\omega n} \, \mathrm{d}\omega & (15) \\ &= \frac{1}{2\pi} \cdot \frac{\mathrm{e}^{\mathrm{j}\omega_c n} - \mathrm{e}^{-\mathrm{j}\omega_c n}}{\mathrm{j}n} & (16) \\ &= \frac{\omega_c}{\pi} \cdot \frac{\sin(\omega_c n)}{\omega_c n} & (17) \\ &= \frac{\omega_c}{\pi} \cdot \operatorname{sinc}\left(\frac{\omega_c}{\pi} n\right) & (18) \end{aligned}$$

which is both non-causal and of infinite duration. We have to truncate it to make it causal.

Truncation of an ideal low pass filter

A causal low pass FIR filter can be obtained by selecting N values around the origin and then delay the impulse response by $(N-1)/2$ samples (choose N odd).

$$h(n) = \frac{\omega_c}{\pi} \cdot \frac{\sin\left(\omega_c\left(n - \frac{N-1}{2}\right)\right)}{\omega_c\left(n - \frac{N-1}{2}\right)} \qquad \text{for } 0 \le n < N \tag{19}$$

This is a very common approach to constructing FIR filters. But, how good are the results and can we improve it?

See the graphs below!

Why do we get problems when truncating?

The impulse response for an ideal low pass filter is of the form $\sin(x)/x$ that we have to truncate in order to obtain a causal filter. The truncation is the cause of the ringing in the amplitude response of the truncated filter, also called Gibb's phenomenon.

The condition for convergence of the Fourier transform was:

$$\sum_n |x(n)| < \infty \tag{20}$$

or

$$\sum_n |x(n)|^2 < \infty \quad \text{if} \quad \sum_n |x(n)| \to \infty \tag{21}$$

For $\operatorname{sinc}(x)$ we have

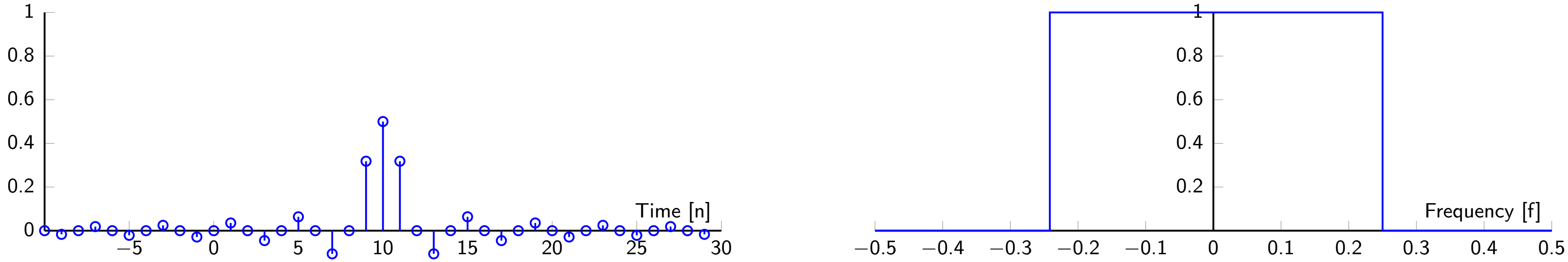
$$\sum_n |\operatorname{sinc}(n)| = \infty \tag{22}$$

but

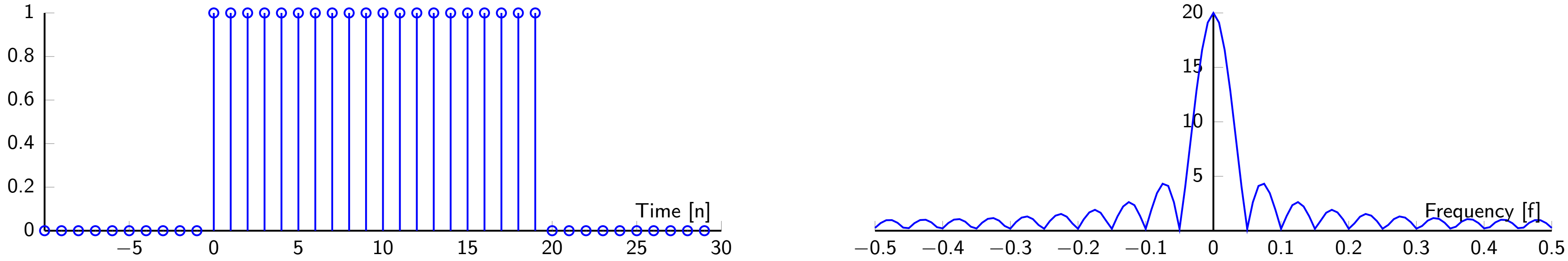
$$\sum_n |\operatorname{sinc}(n)|^2 < \infty \tag{23}$$

We therefore have the weaker form of convergence. The effect of the weaker convergence is shown when we truncate the filter.

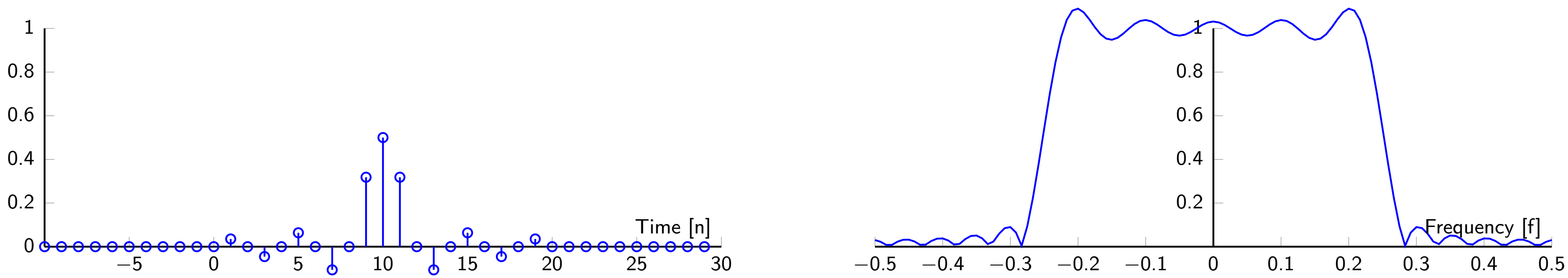
Ideal low pass filter.



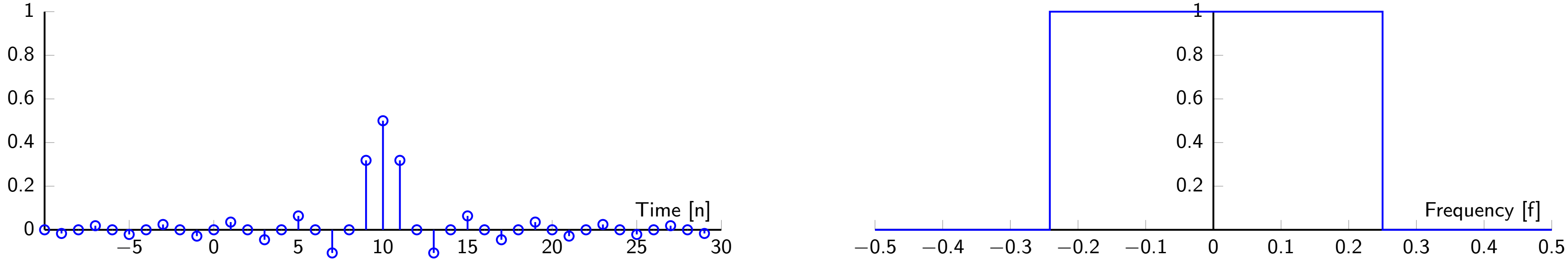
Rectangular window.



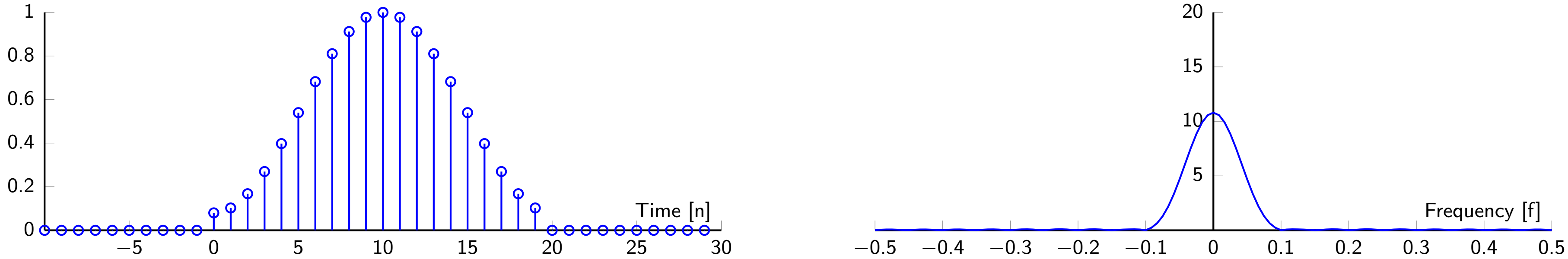
Truncated filter.



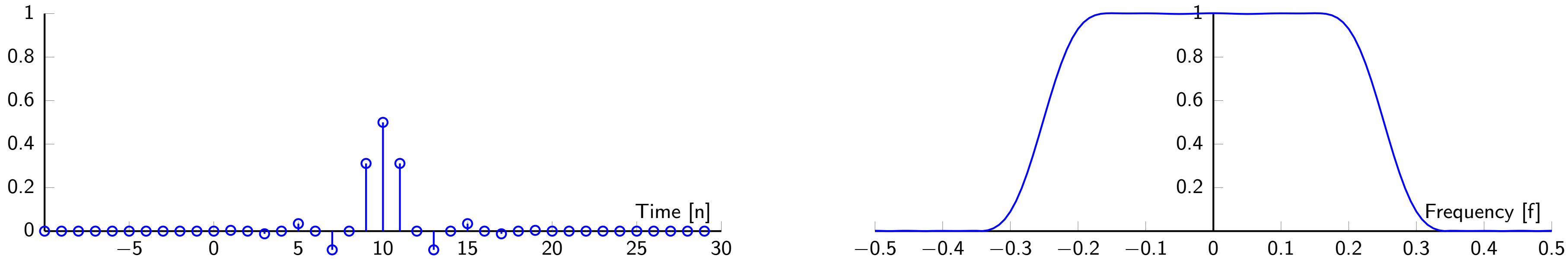
Ideal low pass filter.



Hamming window.



Truncated filter.



Fourier series expansion of periodic signals

Harmonic composition

We know that any periodic signal can be written as a sum of harmonic sinusoids.

$$x(n) = A_0 + A_1 \cos(\omega_0 n + \Theta_1) + A_2 \cos(2\omega_0 n + \Theta_2) + \cdots \tag{24}$$

We already know the Fourier transform of a rectangle pulse. See the figures for the time plot and the spectrum of a pulse with pulse duration 4 and a pulse period of 16.

Make a harmonic signal by repeat the pulse 4 and 16 times. See the figures for the repeated signals and their spectra and compare with the overlayed spectrum of the fundamental rectangle pulse.

As the duration of the repeated signal increases (the number of repetitions of the fundamental rectangle pulse increases) the energy of the harmonic signal approaches infinity. When the signal is infinitely long we can no longer calculate its Fourier transform.

Instead we can describe the signal as a sum of the harmonic components A_n and Φ_n directly related to the Fourier transform of the fundamental pulse.

Analog harmonic signals

An analog signal is periodic if

$$x(t) = x(t + T_p) \tag{25}$$

where T_p is the fundamental period and $\Omega_0 = 2\pi/T_p$ is the fundamental frequency of the signal. The period T_p is chosen as the smallest possible period. The periodic signal can be rewritten as a sum of sinusoids as

$$x(t) = \frac{1}{T_p} \cdot \sum_{k=-\infty}^{\infty} X(k \cdot \Omega_0) \mathrm{e}^{\mathrm{j}\Omega_0 \cdot kt} \tag{26}$$

$$= \frac{1}{T_p} \cdot X(0) + \frac{2}{T_p} \cdot \sum_{k=1}^{\infty} |X(k \cdot \Omega_0)| \cdot \cos(\Omega \cdot kt + \angle X(k \cdot \Omega_0)) \tag{27}$$

where $X(\Omega)$ is the Fourier transform of the fundamental period of the signal defined earlier as

$$X(\Omega) = \int_{\frac{-T_p}{2}}^{\frac{T_p}{2}} x(t) \mathrm{e}^{-\mathrm{j}\Omega t} \, \mathrm{d}t \tag{28}$$

Discrete harmonic signals

A discrete signal can also be composed from harmonic components. Periodicity is defined as

$$x(n) = x(n + N) \tag{29}$$

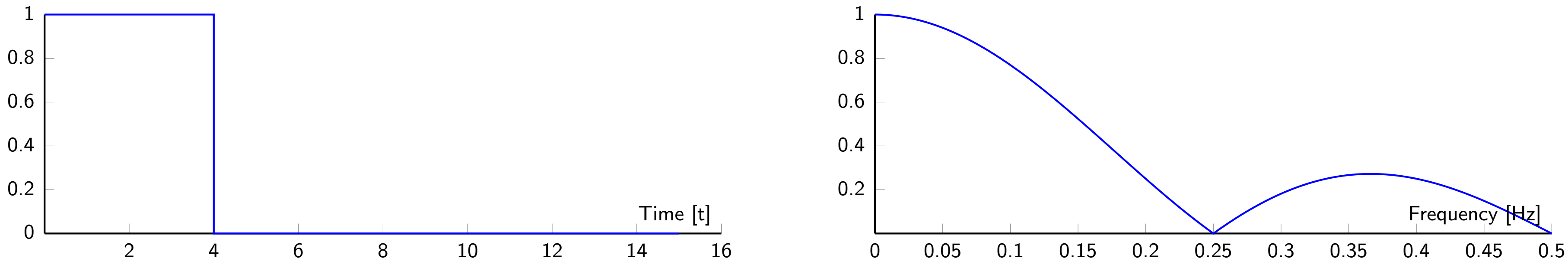
where N is the fundamental period and $\omega_0 = 2\pi/N$ is the fundamental frequency of the signal. A periodic signal can be composed by sinusoids as

$$x(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k \cdot \omega_0) \mathrm{e}^{\mathrm{j}\omega_0 \cdot kn} \tag{30}$$

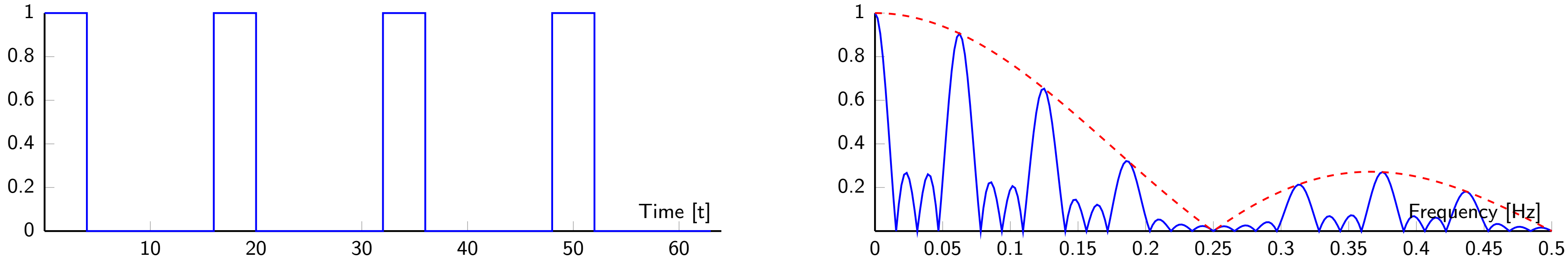
where

$$X(\omega) = \sum_{n=0}^{N-1} x(n) \mathrm{e}^{-\mathrm{j}\omega n} \tag{31}$$

Rectangle pulse:



Rectangle pulse repeated 4 times:



Rectangle pulse repeated 16 times: