Lecture 5

Digital Signal Processing

Chapter 4

Fourier transforms of analog and digital signals

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rev. 2016

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Transforms in Digital Signal Processing

Analog signals

- Fourier transform of analog signals, FT (Laplace transform)

Discrete signals (sampled, digital signals)

- Fourier transform of digital signals, DTFT, chapter 4
- Discrete Fourier transform (DFT, FFT), chapter 7
- z-transform of digital signals, chapter 3.

Fourier series expansion

- Fourier series expansion of periodic signals

Fourier transform (page 236-238)

Fourier transform of analog signals

$$X(F) = \int_{-\infty}^{\infty} x(t) \mathrm{e}^{-\mathrm{j}2\pi F t} \,\mathrm{d}t \tag{1}$$

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) \mathrm{e}^{-\mathrm{j}\Omega t} \,\mathrm{d}t \qquad [\text{where } \Omega = 2\pi F] \tag{2}$$

$$x(t) = \int_{-\infty}^{\infty} X(F) \mathrm{e}^{\mathrm{j}2\pi F t} \,\mathrm{d}F = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} X(\Omega) \mathrm{e}^{\mathrm{j}\Omega t} \,\mathrm{d}\Omega \tag{3}$$

Convergence if $x(t)$ is stable:

$$\int |x(t)| \,\mathrm{d}t < \infty \tag{4}$$

Weaker convergence if

$$\int |x(t)| \,\mathrm{d}t \to \infty \tag{5}$$

when then signal has a limited energy:

$$\int |x(t)|^2 \,\mathrm{d}t < \infty \tag{6}$$

Fourier transform of discrete signals, DTFT (page 251)

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) \mathrm{e}^{-\mathrm{j}2\pi f n} \tag{7}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \mathrm{e}^{-\mathrm{j}\omega n} \qquad [\text{where } \omega = 2\pi f] \tag{8}$$

$$x(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f) \mathrm{e}^{\mathrm{j}2\pi f n} \, \mathrm{d}f = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} X(\omega) \mathrm{e}^{\mathrm{j}\omega n} \, \mathrm{d}\omega \tag{9}$$

Convergence if $x(n)$ is stable:

$$\sum_n |x(n)| < \infty \tag{10}$$

Weaker convergence if

$$\sum_n |x(n)| \rightarrow \infty \tag{11}$$

when the signal has a limited energy:

$$\sum_n |x(n)|^2 < \infty \tag{12}$$

Compare the z-transform and the Fourier transform.

$$X(z) = \sum_n x(n) z^{-n} \tag{13}$$

$$X(\omega) = \sum_n x(n) \mathrm{e}^{-\mathrm{j}\omega n} \tag{14}$$

Important:

$$\boxed{X(\omega) = X(z \mid z = \mathrm{e}^{\mathrm{j}\omega})}$$

The Fourier transform is the z-transform evaluated on the unit circle.

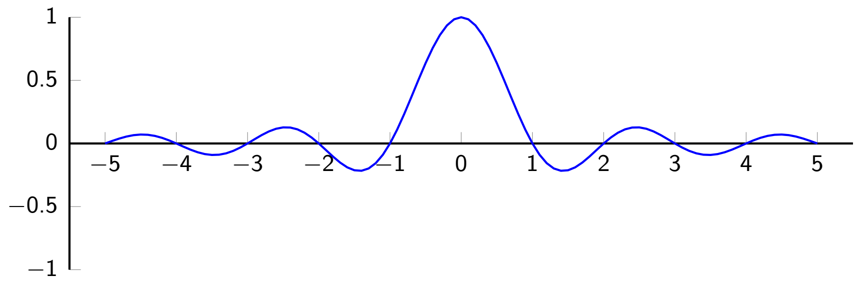
sinc function

A commonly occurring function is signal processing and other subjects is the sinc function.

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \tag{15}$$

where

$$\lim_{x \to 0} \frac{\sin(\pi x)}{\pi x} = 1 \tag{16}$$



Fourier transform of rectangular pulse (page 257-258)

Analog rectangular pulse (rectangular window)

$$x(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} \, dt \tag{18}$$

$$= \int_{0}^{T} 1 \cdot e^{-j2\pi F t} \, dt \tag{19}$$

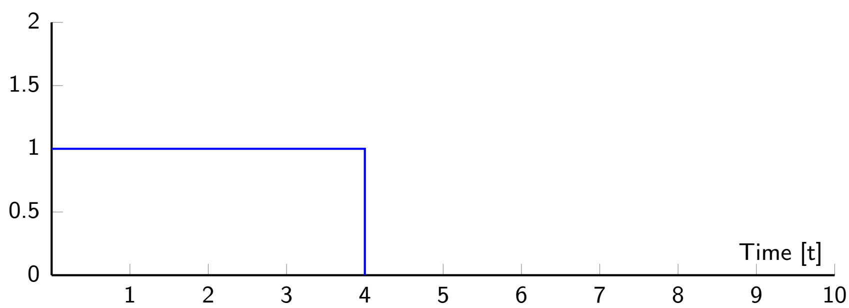
$$= \frac{e^{-j2\pi F T} - 1}{-j2\pi F} \tag{20}$$

$$= \frac{e^{-j2\pi \cdot \frac{T}{2} \cdot F} \left(e^{j2\pi \cdot \frac{T}{2} \cdot F} - e^{-j2\pi \cdot \frac{T}{2} \cdot F} \right)}{j2\pi F} \tag{21}$$

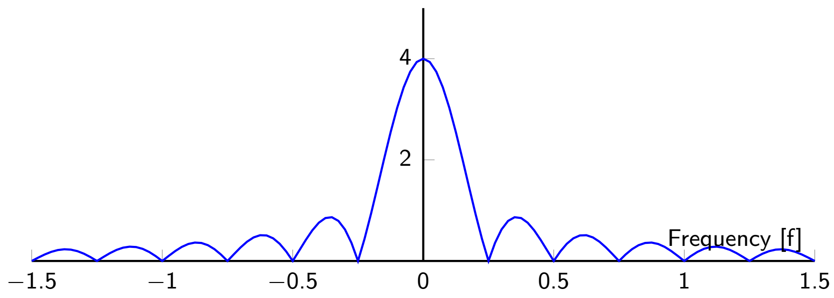
$$= T \cdot \frac{\sin\left(2\pi \cdot \frac{T}{2} \cdot F\right)}{2\pi \cdot \frac{T}{2} \cdot F} \cdot e^{-j2\pi \cdot \frac{T}{2} \cdot F} \tag{22}$$

$$= T \cdot \operatorname{sinc}\left(2 \cdot \frac{T}{2} \cdot F\right) \cdot e^{-j2\pi \cdot \frac{T}{2} \cdot F} \tag{23}$$

The signal $x(t)$ when $T = 4$.



The amplitude function $|X(F)|$ of the signal $x(t)$.



Discrete rectangular pulse (rectangular window)

$$x(n) = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\mathrm{e}^{-\mathrm{j}\omega n} \tag{25}$$

$$= \sum_{n=0}^{N-1} 1 \cdot \mathrm{e}^{-\mathrm{j}\omega n} \tag{26}$$

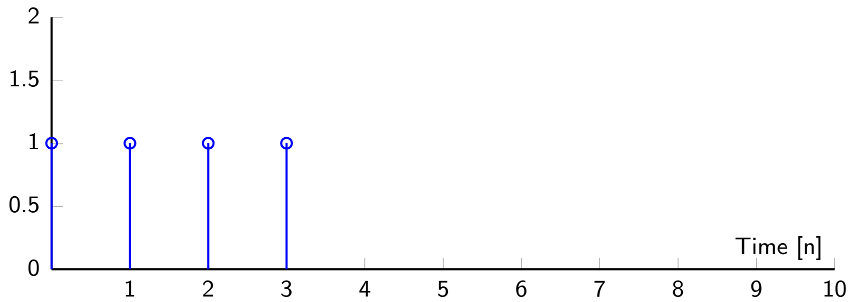
$$= \frac{1 - \mathrm{e}^{-\mathrm{j}\omega N}}{1 - \mathrm{e}^{-\mathrm{j}\omega}} \tag{27}$$

$$= \frac{\mathrm{e}^{\mathrm{j}\omega N/2}\left(\mathrm{e}^{\mathrm{j}\omega\cdot\frac{N}{2}} - \mathrm{e}^{-\mathrm{j}\omega\cdot\frac{N}{2}}\right)}{\mathrm{e}^{\mathrm{j}\omega/2}\left(\mathrm{e}^{\mathrm{j}\omega\cdot\frac{1}{2}} - \mathrm{e}^{-\mathrm{j}\omega\cdot\frac{1}{2}}\right)} \tag{28}$$

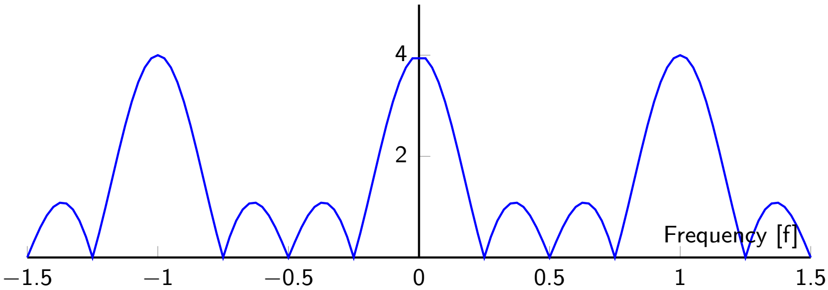
$$= N \cdot \frac{\sin\left(\omega \cdot \frac{N}{2}\right)}{N \sin\left(\omega \cdot \frac{1}{2}\right)} \cdot \mathrm{e}^{-\mathrm{j}\omega(N-1)/2} \tag{29}$$

Observe that $X(\omega)$ is periodic with the period $f = 1$ or $\omega = 2\pi$.

The signal $x(n)$ for $N = 4$.



The amplitude function $|X(f)|$ of the signal $x(n)$.



Example of transforms (DTFT)

Discrete signal:

$$x(n) = \left\{ \begin{array}{ccc} 3 & 2 & 1 \end{array} \right\} \tag{30}$$

$$X(\omega) = 3 + 2\mathrm{e}^{-\mathrm{j}\omega} + \mathrm{e}^{-\mathrm{j}2\omega} \tag{31}$$

Delta function:

$$x(n) = \delta(n) = \left\{ \begin{array}{c} 1 \end{array} \right\} \tag{32}$$

$$X(\omega) = 1 \tag{33}$$

Time shift:

$$y(n) = x(n - n_0) \tag{34}$$

$$Y(\omega) = \mathrm{e}^{-\mathrm{j}\omega n_0} \cdot X(\omega) \tag{35}$$

Convolution becomes multiplication:

$$y(n) = h(n) * x(n) = \sum_k x(k)h(n-k) \tag{36}$$

$$\begin{aligned} Y(\omega) &= \sum_n y(n)\mathrm{e}^{-\mathrm{j}\omega n} && (37) \\ &= \sum_n \sum_k x(k)h(n-k)\mathrm{e}^{-\mathrm{j}\omega n} && (38) \\ &= \sum_n \sum_k x(k)h(n-k)\mathrm{e}^{-\mathrm{j}\omega(n-k)}\mathrm{e}^{-\mathrm{j}\omega k} && (39) \\ &= \sum_k x(k)\mathrm{e}^{-\mathrm{j}\omega k} \sum_n h(n-k)\mathrm{e}^{-\mathrm{j}\omega(n-k)} && (40) \\ &= H(\omega)X(\omega) && (41) \end{aligned}$$

Appendix

Show the Fourier transform

$$x_{\text{inverse}}(n) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} X(\omega) \mathrm{e}^{\mathrm{j}\omega n} \,\mathrm{d}\omega \tag{42}$$

$$= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} x(k) \mathrm{e}^{-\mathrm{j}\omega k} \cdot \mathrm{e}^{\mathrm{j}\omega n} \,\mathrm{d}\omega \tag{43}$$

$$= \frac{1}{2\pi} \cdot \sum_{k=-\infty}^{\infty} x(k) \cdot \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{j}\omega k} \cdot \mathrm{e}^{\mathrm{j}\omega n} \,\mathrm{d}\omega \tag{44}$$

$$= \frac{1}{2\pi} \cdot \sum_{k=-\infty}^{\infty} x(k) \cdot \underbrace{\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{j}\omega (k-n)} \,\mathrm{d}\omega}_{2\pi \text{ if } k=n, \text{ 0 otherwise}} = x(n) \tag{45}$$

Show the Fourier transform numerically

Given:

$$H(\omega) = \frac{2\mathrm{e}^{\mathrm{j}\omega}}{1 - 0.5\mathrm{e}^{-\mathrm{j}\omega}} \tag{46}$$

Show:

$$\frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} H(\omega) \,\mathrm{d}\omega = 1 \tag{47}$$

```
>> N = 1001;
>> w = linspace(-pi, pi, N);
>> dw = 2*pi/(N-1);
>> H = 2*exp(1j*w) ./ (1-0.5*exp(-1j*w));
>> sum(trapz(H)*dw/2/pi)
ans =
   1.0000 - 0.0000i
```