Lecture 4

Digital Signal Processing

Chapter 3

z-transforms

Mikael Swartling Nedelko Grbic Bengt Mandersson

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Department of Electrical and Information Technology
Lund University

[1]Update of pole-zero table below eq. (17) on 2021-03-30 - Ove Edfors

z-transform

We defined the z-transform of the impulse response $h(n)$ as

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} h(n) z^{-n} \tag{1}$$

where $h(n) = 0$ for $n < 0$ and where $z = r\mathrm{e}^{\mathrm{j}\omega}$.

We used the z-transform on the second order difference equation

$$y(n) - 1.27y(n-1) + 0.81y(n-2) = x(n-1) - x(n-2) \tag{2}$$

with the z-transform

$$Y(z) - 1.27z^{-1}Y(z) + 0.81z^{-2}Y(z) = z^{-1}X(z) - z^{-2}X(z) \tag{3}$$

This gave us

$$Y(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}} \cdot X(z) = H(z)X(z) \tag{4}$$

We will now continue with this example.

Poles and zeros

From the difference equation we can always get $H(z)$ which is a ratio of two polynomials in z^{-1}. We want to analyze this ratio by factorizing $H(z)$.

$$H(z) = \frac{B(z)}{A(z)} \tag{5}$$

$$= \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}} \tag{6}$$

$$= \frac{z - 1}{z^2 - 1.27z + 0.81} \tag{7}$$

Finding the roots to the denominator polynomial

$$z^2 - 1.27z + 0.81 = 0 \tag{8}$$

gives us roots in

$$z = \frac{1.27}{2} \pm \sqrt{\left(\frac{1.27}{2}\right)^2 - 0.81} \tag{9}$$

$$= 0.64 \pm \mathrm{j}0.64 \tag{10}$$

$$= 0.9\mathrm{e}^{\pm \mathrm{j}\frac{\pi}{4}} \tag{11}$$

Finding the roots to the numerator polynomial

$$z - 1 = 0 \qquad \rightarrow \qquad z = 1 \tag{12}$$

The factoring of $H(z)$ can now be written as

$$H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}} \tag{13}$$

$$= \frac{z-1}{z^2 - 1.27z + 0.81} \tag{14}$$

$$= \frac{z-1}{\left(z - 0.9\mathrm{e}^{\mathrm{j}\frac{\pi}{4}}\right)\left(z - 0.9\mathrm{e}^{-\mathrm{j}\frac{\pi}{4}}\right)} \tag{15}$$

The roots of the numerator polynomial are called *zeros* and roots to the denominator polynomial are called *poles*. We draw the poles and the zeros in a complex number plane called a *pole-zero diagram* where poles are marked by $\times$ and zeros are marked by $\circ$.

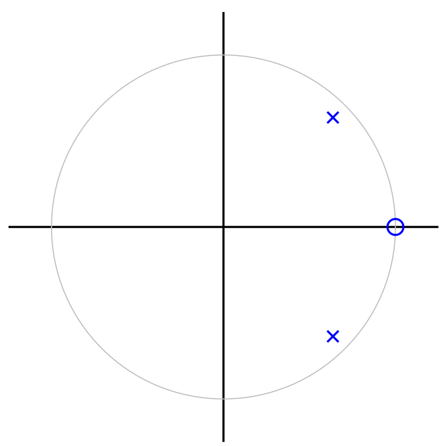
Poles:

$$p_{1,2} = 0.9\mathrm{e}^{\pm\mathrm{j}\frac{\pi}{4}} \tag{16}$$

Zeros:

$$z_1 = 1 \tag{17}$$

Pole-zero diagram:



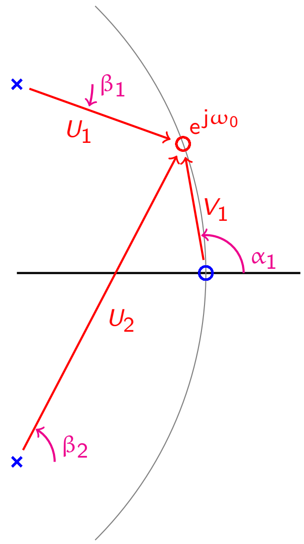
Filter response	When...
$\lvert H(z)\rvert = 0$	z is a zero.
$\lvert H(z)\rvert = \infty$	z is a pole.
$\lvert H(z)\rvert \approx 0$	z is close to a zero.
$\lvert H(z)\rvert \gg 1$	z is close to a pole.

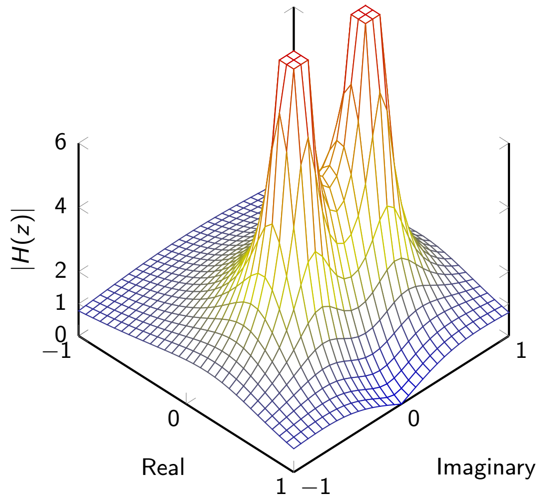
Describing a system by poles and zeros is very useful.

Determine $H(\omega)$ from the pole-zero plot (page 315–320)

The frequency response $H(\omega)$ is determined by the location of the poles and zeros relative to the unit circle.

Define the units $U_n = \mathrm{e}^{\mathrm{j}\omega_0} - p_n$ and $V_n = \mathrm{e}^{\mathrm{j}\omega_0} - z_n$ for some frequency ω_0. The units corresponds to the vectors from the poles or the zeros, respectively, to the point on the unit circle corresponding to the frequency ω_0.





The amplitude response is:

$$|H(\omega_0)| = \frac{\prod |V_n|}{\prod |U_n|} = \frac{|V_1|}{|U_1| \cdot |U_2|} \tag{18}$$

The phase response is:

$$\angle H(\omega_0) = \sum \angle V_n - \sum \angle U_n = \angle V_1 - \angle U_1 - \angle U_2 = \alpha_1 - \beta_1 - \beta_2 \tag{19}$$

z-transform of second order system

Second order system with complex roots..

Sine

$$h(n) = r^n \cdot \sin(\omega n) u(n) \tag{20}$$

$$= r^n \cdot \frac{1}{2\mathrm{j}} \cdot \left(\mathrm{e}^{\mathrm{j}\omega n} - \mathrm{e}^{-\mathrm{j}\omega n}\right) u(n) \tag{21}$$

$$H(z) = \frac{1}{2\mathrm{j}} \cdot \left(\frac{1}{1 - r\mathrm{e}^{\mathrm{j}\omega} z^{-1}} - \frac{1}{1 - r\mathrm{e}^{-\mathrm{j}\omega} z^{-1}}\right) \tag{22}$$

$$= \frac{r \sin(\omega) z^{-1}}{1 - 2r\cos(\omega) z^{-1} + r^2 z^{-2}} \tag{23}$$

Cosine

$$h(n) = r^n \cdot \cos(\omega n) u(n) \tag{24}$$

$$= r^n \cdot \frac{1}{2} \cdot \left(\mathrm{e}^{\mathrm{j}\omega n} + \mathrm{e}^{-\mathrm{j}\omega n}\right) u(n) \tag{25}$$

$$H(z) = \frac{1}{2} \cdot \left(\frac{1}{1 - r\mathrm{e}^{\mathrm{j}\omega} z^{-1}} + \frac{1}{1 - r\mathrm{e}^{-\mathrm{j}\omega} z^{-1}}\right) \tag{26}$$

$$= \frac{1 - r\cos(\omega) z^{-1}}{1 - 2r\cos(\omega) z^{-1} + r^2 z^{-2}} \tag{27}$$

Example

We had

$$H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}} \tag{28}$$

$$= z^{-1} \cdot \frac{1 - z^{-1}}{1 - 1.27z^{-1} + 0.81z^{-2}} \tag{29}$$

Now determine $h(n)$, the inverse z-transform of $H(z)$, with the help of the transforms for the sine and the cosine. Rewrite $H(z)$ so that we can identify the sine and cosine terms.

$$H(z) = z^{-1} \cdot \frac{1 - z^{-1}}{1 - 1.27z^{-1} + 0.81z^{-2}} \tag{30}$$

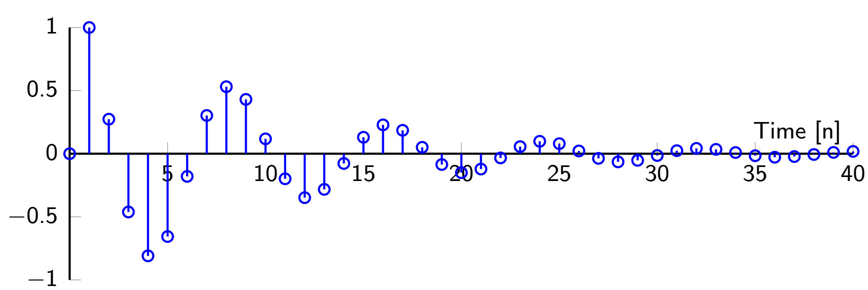
Identify $1.27 = 2r\cos(\omega_0)$ and $0.81 = r^2$. Therefore $\omega_0 = \pi/4$ and $r = 0.9$.

$$H(z) = z^{-1} \cdot \frac{1 - r\cos(\omega_0)z^{-1} + \left(r\cos(\omega_0)z^{-1} - z^{-1}\right) \cdot \frac{r\sin(\omega_0)}{r\sin(\omega_0)}}{1 - 1.27z^{-1} + 0.81z^{-2}} \tag{31}$$

The transforms for the sine and the cosine now gives

$$h(n) = r^{n-1} \cdot \left[\cos(\omega_0 \cdot (n-1)) + \frac{r\cos(\omega_0) - 1}{r\sin(\omega_0)} \cdot \sin(\omega_0 \cdot (n-1))\right] \cdot u(n-1) \tag{32}$$

$$= 0.9^{n-1} \cdot \left[\cos\left(\frac{\pi}{4} \cdot (n-1)\right) - 0.57\sin\left(\frac{\pi}{4} \cdot (n-1)\right)\right] \cdot u(n-1) \tag{33}$$



Stability

A system is stable of an input signal of limited amplitude yields an output signal that is also of limited amplitude. This is called BIBO-stability (bounded input, bounded output). A sufficient requirement is that

$$\sum_n |h(n)| < \infty \tag{34}$$

This is equivalent to all poles being inside the unit circle in the pole-zero diagram.

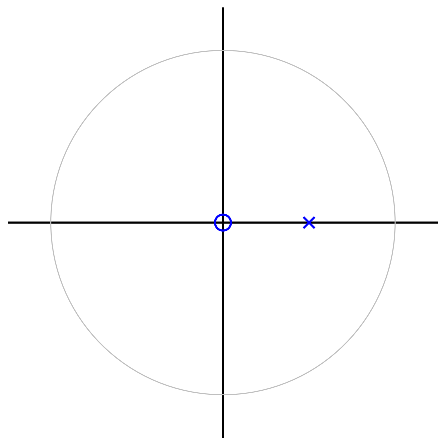
FIR-filter

FIR filters have all their poles at the origin and are always BIBO-stable: $h(n)$ has a finite duration and $\sum_n |h(n)|$ is always limited.

First order IIR-filter

$$H(z) = \frac{1}{1 - az^{-1}} \qquad \Leftrightarrow \qquad h(n) = a^n u(n) \tag{35}$$

Pole in $p_1 = a$.



Stable if $|a| < 1$, or equivalently if the pole lies inside the unit circle.

Second order IIR-filter

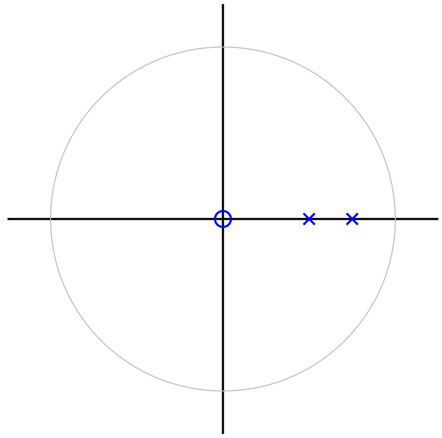
$$\begin{aligned} H(z) &= \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} & (36) \\ &= \frac{z^2}{(z - p_1)(z - p_2)} & (37) \\ &= \frac{1}{1 - (p_1 + p_2)z^{-1} + p_1 p_2 z^{-2}} & (38) \end{aligned}$$

We split the problem into two cases.

Real poles:

$$\begin{aligned} H(z) &= \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} & (39) \\ &= \frac{A}{1 - p_1 z^{-1}} + \frac{B}{1 - p_2 z^{-1}} & (40) \end{aligned}$$

$$h(n) = (A p_1^n + B p_2^n) \cdot u(n) \tag{41}$$



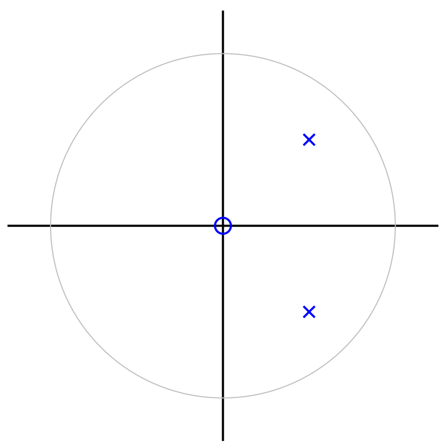
Stable if $|p_1| < 1$ and $|p_2| < 1$.

Complex conjugated poles: Two poles where $p_{1,2} = re^{\pm j\omega_0}$:

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} \tag{42}$$

$$= \frac{1}{1 - (p_1 + p_2)z^{-1} + p_1 p_2 z^{-2}} \tag{43}$$

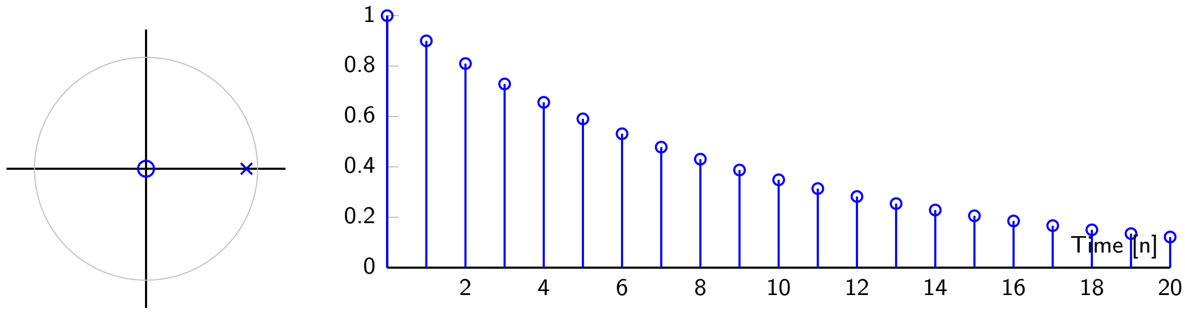
$$= \frac{1}{1 - 2r\cos(\omega_0)z^{-1} + r^2 z^{-2}} \tag{44}$$



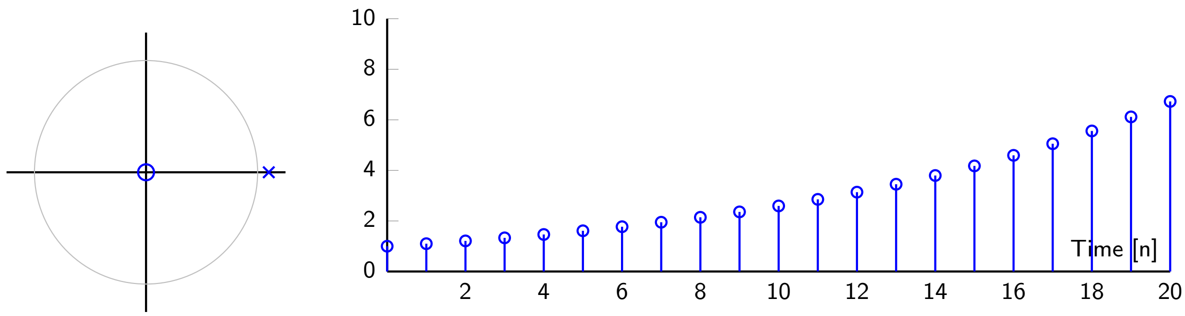
Stable if $|p_1| < 1$ and $|p_2| < 1$, or equivalently if $|r| < 1$.

Pole-zero diagrams and impulse responses

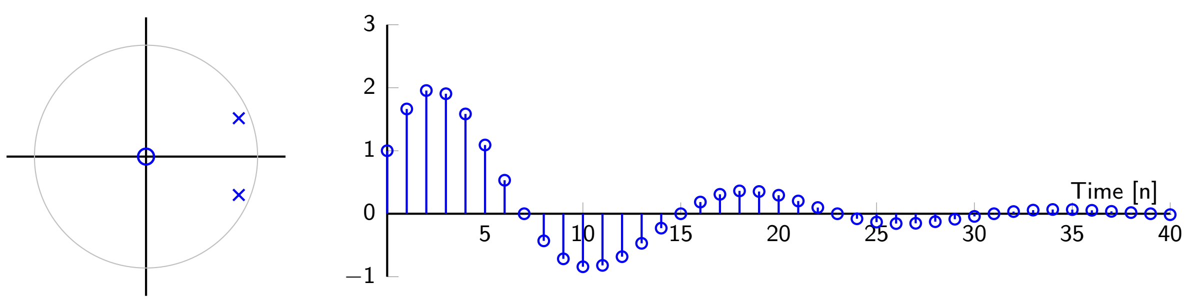
Exponentially decaying step



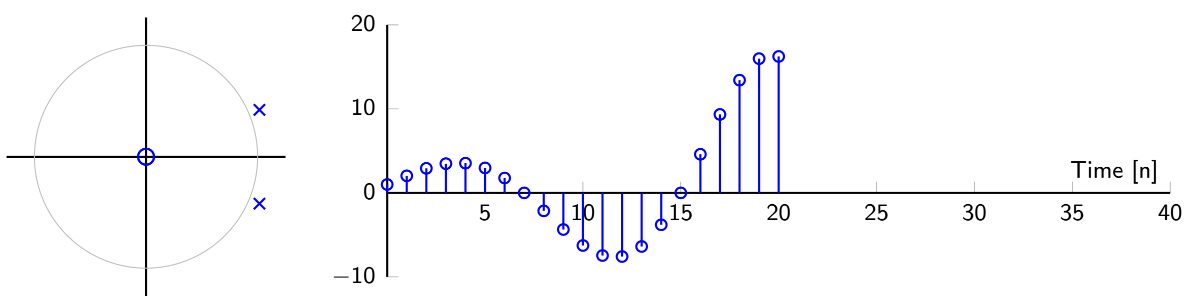
Exponentially increasing step



Damped oscillator



Unstable oscillator



Some comments on stability

- FIR-filter have all their poles at the origin.
- FIR-filter are always stable in the sense that a limited input signal can never yield an output signal that is unlimited in amplitude.

- IIR-filter does not have all their poles at the origin.
- IIR-filter are stable only if all their poles lie within the unit circle.

Analog $H(s)$ and discrete $H(z)$

Analog: Stable if all poles are in the left half-plane.

$$H(s) = \int_t h(t)\mathrm{e}^{-st}\,\mathrm{d}t \tag{45}$$

Discrete: Stable if all poles lie within the unit circle.

$$H(z) = \sum_n h(n)z^{-n} \tag{46}$$

where $z \sim \mathrm{e}^s$.

Example

$$p_{\text{analog}} = -0.5 \pm 0.5\mathrm{j} \tag{47}$$

$$\begin{aligned} p_{\text{digital}} &\sim \mathrm{e}^{-0.5\pm\mathrm{j}0.5} && (48)\\ &= \mathrm{e}^{-0.5}\mathrm{e}^{\pm\mathrm{j}0.5} && (49)\\ &= 0.6\mathrm{e}^{\pm\mathrm{j}0.16\pi} && (50)\end{aligned}$$

$$s = \sigma + \mathrm{j}\Omega \qquad \Leftrightarrow \qquad z = r\cdot\mathrm{e}^{\mathrm{j}\omega} \tag{51}$$

Solving general differential equations

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \tag{52}$$

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \tag{53}$$

$$\begin{aligned} Y(z) &= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \cdot X(z) && (54)\\ &= b_0 \cdot \frac{z^{-M}}{z^{-N}} \cdot \frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)} \cdot X(z) = H(z)X(z) && (55)\end{aligned}$$

where z_i are the zeros (roots to the numerator polynomial) and p_i are the poles (roots to the denominator polynomial). Inverse transform $Y(z)$ by partial fraction expansion and the solution is given by the resulting $y(n)$.