# Lecture 11

Digital Signal Processing

## Chapter 7

Discrete Fourier Transform
DFT

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rev. 2016

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## The definition of the DFT

$$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n) \mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{k}{N}\cdot n} \tag{1}$$

$$x_{\mathrm{IDFT}}(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) \mathrm{e}^{\mathrm{j}2\pi\cdot\frac{k}{N}\cdot n} \tag{2}$$

Both $x(n)$ and $X(k)$ are periodic and indices are calculated modulo-$N$.

Circular shift:

$$y(n) = x(n - n_0 \mod N) \quad \Rightarrow \quad Y(k) = \mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{k}{N}n_0} \cdot X(k) \tag{3}$$

Circular convolution:

$$y(n) = x_1(n) \otimes_N x_2(n) = \sum_{k=0}^{N-1} x_1(n) x_2(n - k \mod N) \quad \Rightarrow \quad Y(k) = X_1(k) \cdot X_2(k) \tag{4}$$

## Example of DFT

The Fourier transform of an infinite length signal:

$$x(n) = a^n u(n) \quad \Rightarrow \quad X(\omega) = \frac{1}{1 - a\mathrm{e}^{-\mathrm{j}\omega}} \tag{5}$$

The Fourier transform of a finite length signal:

$$x(n) = a^n \quad \text{for } 0 \leq n < N \quad \Rightarrow \quad X(\omega) = \frac{1 - a^N \mathrm{e}^{-\mathrm{j}\omega N}}{1 - a\mathrm{e}^{-\mathrm{j}\omega}} \tag{6}$$

The discrete Fourier transform of a finite length signal:

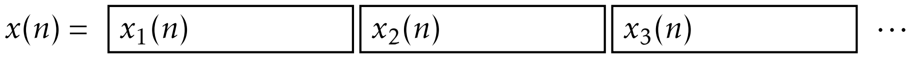
$$x(n) = a^n \quad \text{for } 0 \leq n < N \quad \Rightarrow \quad X(k) = \frac{1 - a^N}{1 - a\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{k}{N}}} \tag{7}$$

## Filtering with the Overlap-Add, (page 487,488)

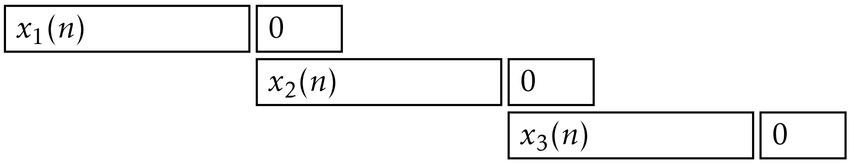
Filtering has been described in time domain as convolving an entire signal with a filter, or in frequency domain by multiplying the DFT of the entire signal with the DFT of the filter. This raises some issues:

- In a real time environment the signal is streaming and is never available as a whole. The signal has no start and no end.
- If the filter is very small in comparison to the length of the signal, the DFT of the filter has to be calculate with high number of DFT-points and waste computational power.

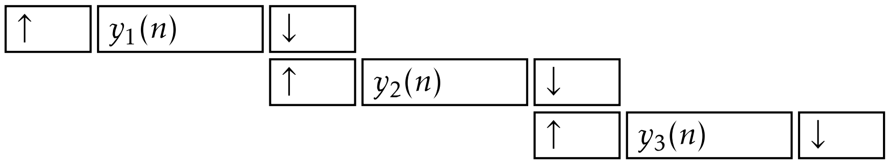
In overlap-add the signal is divided into blocks of length $L$ samples and each block is filtered independently.




The length of the impulse response is $M$ so zero-pad each block to length $N = L+M-1$.




Filter each zero-padded block individually.




The output signal block consists of transient phase (↑), the output signal ($y_k(n)$) and the decay phase (↓). The decay phase of one block overlaps in time with the transient phase of the following block. When the two are added, the signal forms the continuous output signal in the overlapped region.

## Example

**Given:** The input signal

$$x(n) = \left\{ \begin{array}{cccccccccccc} 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{array} \right\} \tag{8}$$

and the filter

$$h(n) = \left\{ \begin{array}{cccc} 1 & 1 & 0 & 0 \end{array} \right\} \tag{9}$$

**Find:** The block-filtering $y(n) = x(n) * h(n)$ with $L = 4$, $M = 4$ and $N = L + M - 1 = 7$.

**Solution:**

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_1 * h$ | = | 1 | 1 | 1 | 2 | 1 | 0 | 0 | | | | | | | | |
| $x_2 * h$ | = | | | | | 1 | 1 | 1 | 1 | 0 | 0 | 0 | | | | |
| $x_3 * h$ | = | | | | | | | | | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| $x * h$ | = | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |

# DFT of a sine (whole number of periods)

**Given:**

$$x(n) = \cos\left(2\pi \cdot \frac{k_0}{N} \cdot n\right) \tag{10}$$

**Find:** The discrete Fourier transform $X(k)$ of $x(n)$.

**Solution:**

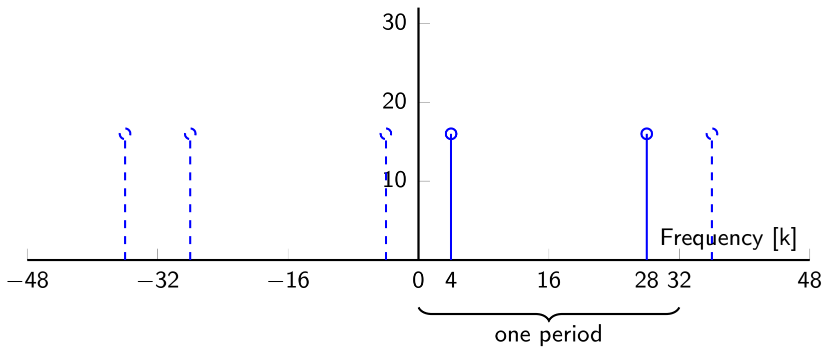
$$X(k) = \sum_{n=0}^{N-1} x(n) \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k}{N} \cdot n} \tag{11}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2} \cdot \left[\mathrm{e}^{\mathrm{j}2\pi \cdot \frac{k_0}{N} \cdot n} + \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k_0}{N} \cdot n}\right] \cdot \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k}{N} \cdot n} \tag{12}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2} \cdot \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k-k_0}{N} \cdot n} + \sum_{n=0}^{N-1} \frac{1}{2} \cdot \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k+k_0}{N} \cdot n} \tag{13}$$

$$= \frac{N}{2} \cdot [\delta(k - k_0 \mod N) + \delta(k + k_0 \mod N)] \tag{14}$$

The spectrum for $k_0 = 4$ and $N = 32$.




The signal

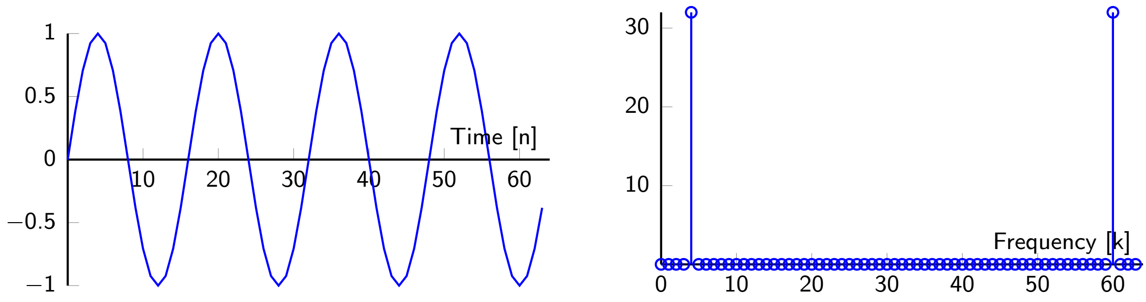
$$x(n) = \sin\left(2\pi \cdot \frac{k_0}{N} \cdot n\right) \tag{15}$$

instead yields

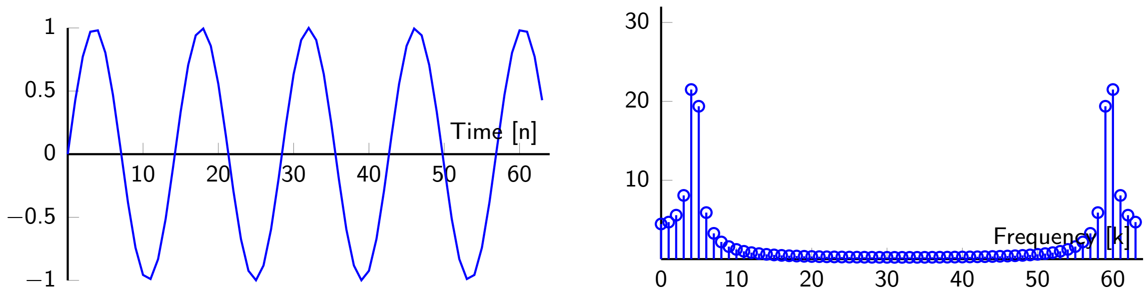
$$X(k) = \frac{N}{2\mathrm{j}} \cdot [\delta(k - k_0 \mod N) + \delta(k + k_0 \mod N)] \tag{16}$$

# Examples of the effect that $x(n)$ is periodic for the DFT

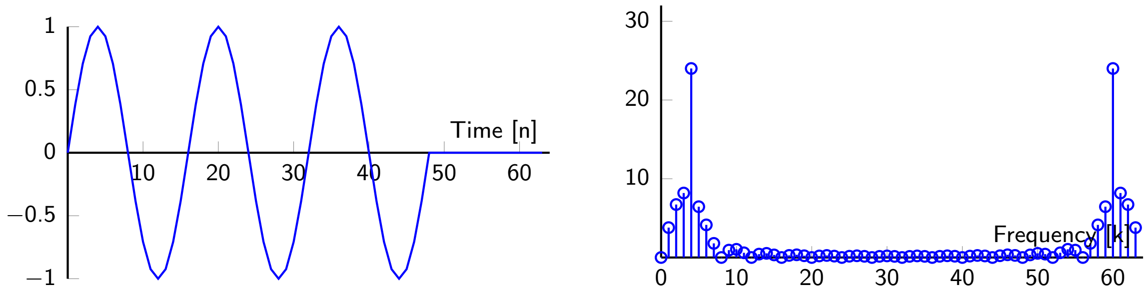
The DFT of a sine with a whole number of periods:




The DFT of a sine with a fractional number of periods:




The DFT of a sine with zero padding:



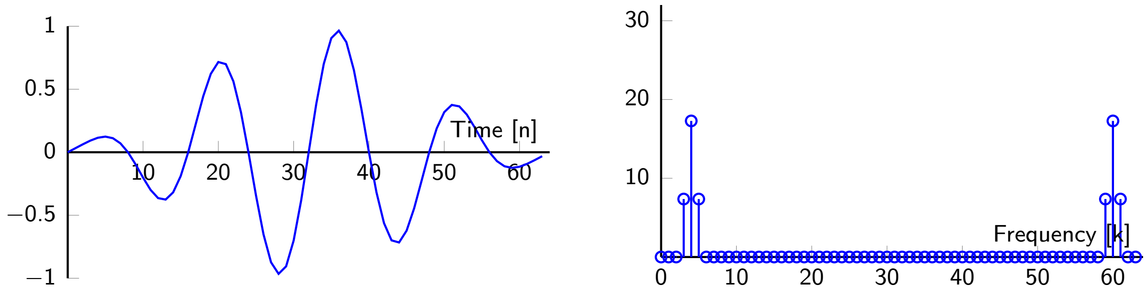

By multiplying the signal with a window that attenuates the signal near the ends the effect of the truncation can be reduced:

$$x_w(n) = x(n) \cdot w(n) \tag{17}$$

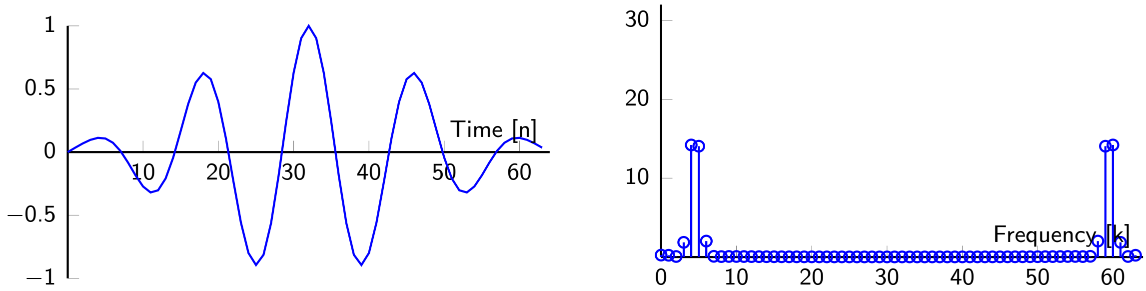
where the window $w(n)$ is, for example, a Hamming windows defined as

$$w_{\text{hamming}}(n) = 0.54 + 0.46 \cos\left(2\pi \cdot \frac{1}{N-1} \cdot \left(n - \frac{N-1}{2}\right)\right) \tag{18}$$

The DFT of a windowed sine with a whole number of periods:

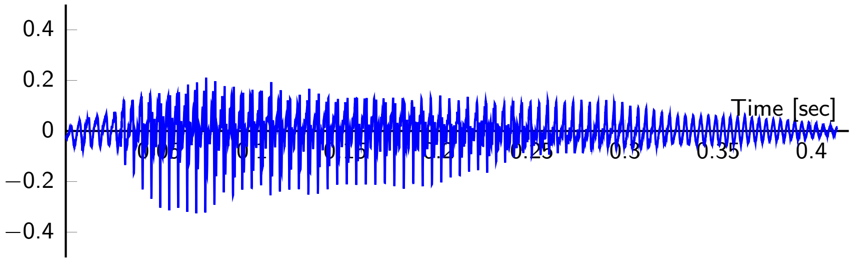


The DFT of a windowed sine with a fractional number of periods:

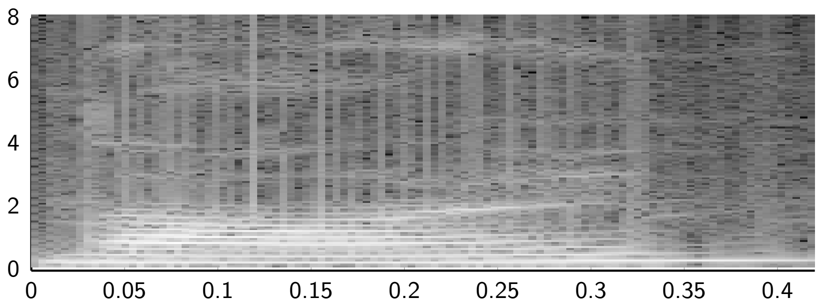



## Spectrogram: spectrum as a function of time

Select $N$ consecutive sample from the signal $x(n)$ with a sliding window and calculate an $N$-point DFT. Plot the spectrum as a function of both frequency and as a function of time (the location of the sliding window).




A long window means high frequency resolution but low time resolution (horizontal bands).




A short window means high time resolution but low frequency resolution (vertical bands).