Lecture 1

Digital Signal Processing

Introduction

Introduction to signal processing

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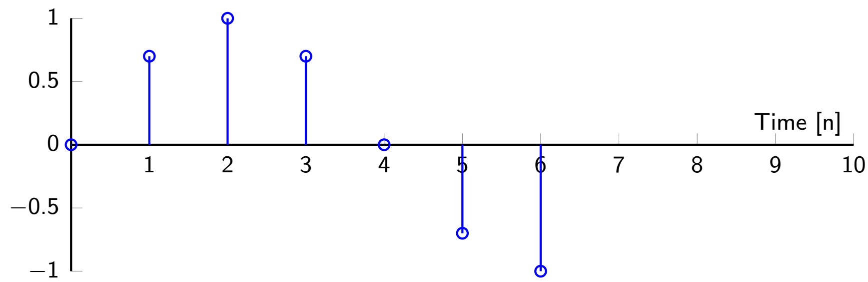
What is a time-discrete signal?

Time-discrete signal

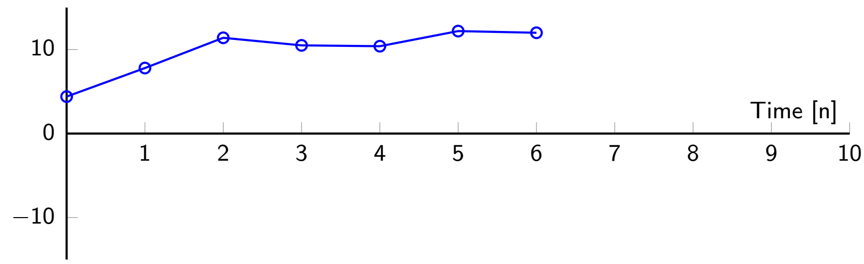
Sine

$$x(n) = \sin\left(2\pi \cdot \frac{1}{8} \cdot n\right) \approx \left\{ \ \dots \quad -1 \quad -0.7 \quad \underline{0} \quad 0.7 \quad 1 \quad 0.7 \quad 0 \quad -0.7 \quad \dots \ \right\} \tag{1}$$



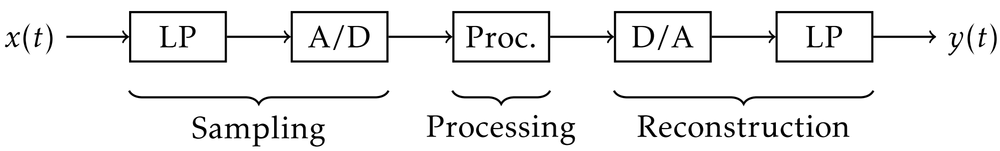
Temperature

$$x(n) = \left\{ \ \underline{4.4} \quad 7.8 \quad 11.4 \quad 10.5 \quad 10.4 \quad 12.2 \quad 12.0 \quad \dots \ \right\} \tag{2}$$



Time-discrete systems

Digital processing of analog signals.



The digital system.

Example of systems

Calculate the average of the five last samples.

$$y(n) = \frac{1}{5} \cdot x(n) + \frac{1}{5} \cdot x(n-1) + \frac{1}{5} \cdot x(n-2) + \frac{1}{5} \cdot x(n-3) + \frac{1}{5} \cdot x(n-4) \tag{3}$$

What does the following variant of the average do?

$$y(n) = \frac{1}{5} \cdot x(n) - \frac{1}{5} \cdot x(n-1) + \frac{1}{5} \cdot x(n-2) - \frac{1}{5} \cdot x(n-3) + \frac{1}{5} \cdot x(n-4) \tag{4}$$

One amplifies low frequencies and the other amplifies the high frequencies, but how? What happens if the output signal of the system is connected back to the input (feedback)?

$$y(n) = 0.9y(n-1) + x(n) \tag{5}$$

$$y(n) = 1.1y(n-1) + x(n) \tag{6}$$

These are the kinds of questions that will be answered during the course.

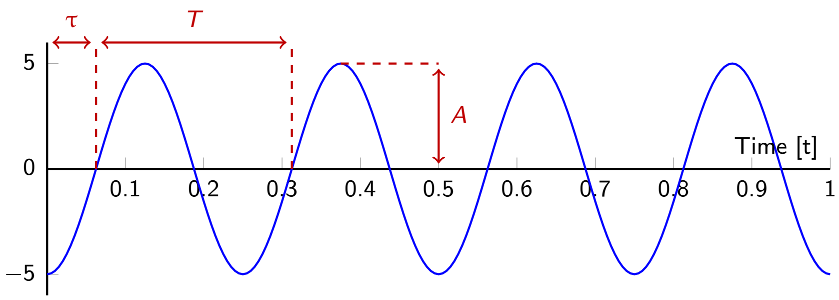
Goal of the course

To understand the connection between systems and their properties, especially frequency properties.

Sinusoids

Time signals

$$x(t) = A \cdot \sin(2\pi F t - \Phi) = A \cdot \sin(\Omega t - \Phi) = A \cdot \sin\left(\Omega\left(t - \frac{\Phi}{\Omega}\right)\right) \tag{7}$$



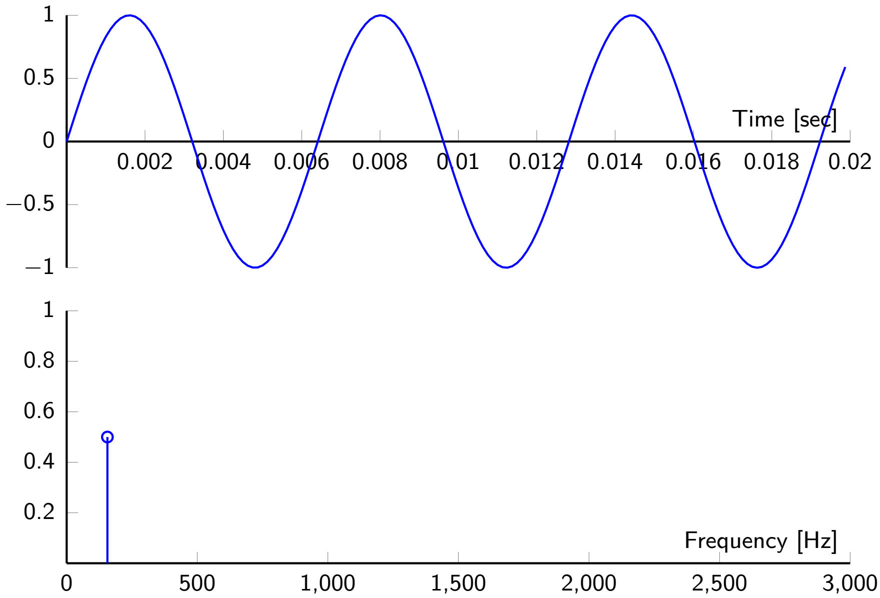
Symbols

A	Amplitude
F	Frequency in Hz
Φ	Phase in rad
$\Omega = 2\pi F$	Frequency in rad/s
$T = \frac{1}{F}$	Time period in second
$\tau = \frac{\Phi}{\Omega}$	Time delay in second

Synthetic sounds

Tones

Sine with frequency $F_0 = 156\,\text{Hz}$.

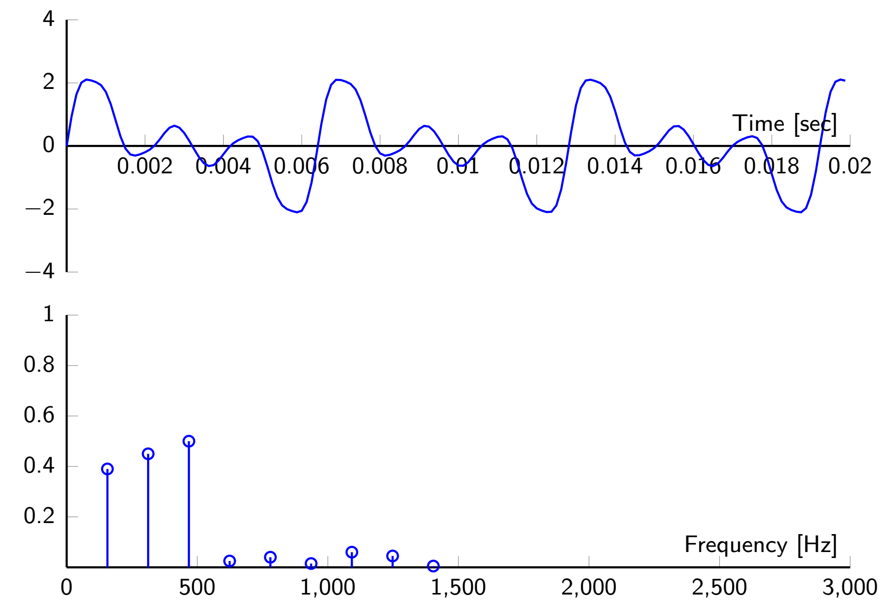


Additive synthesis

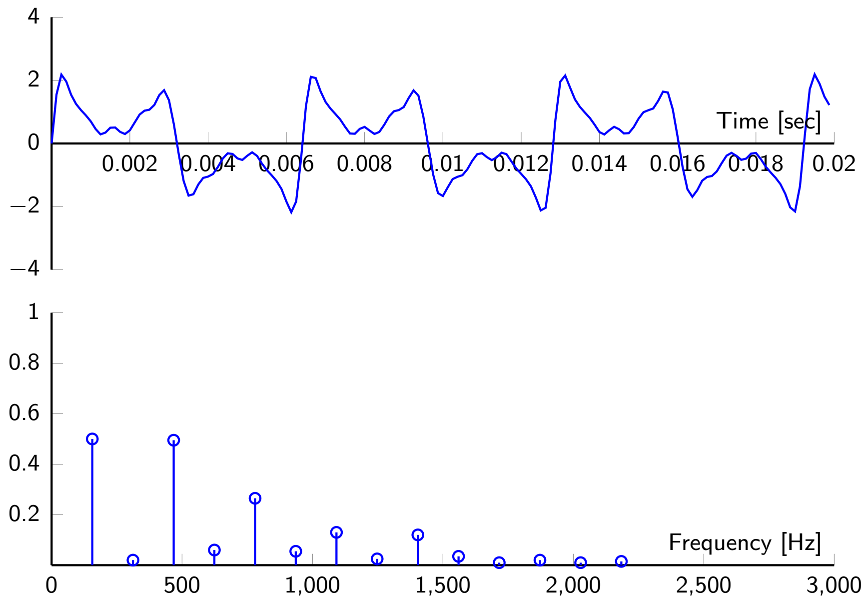
Sum of sinusoids of different frequencies, harmonic signal.

$$x(t) = \sum_k a_k \sin(2\pi k F_0 t) \tag{8}$$

Trombone with fundamental frequency $F_0 = 156\,\text{Hz}$ and 8 harmonics.



Clarinet with fundamental frequency $F_0 = 156\,\text{Hz}$ and 13 harmonics.

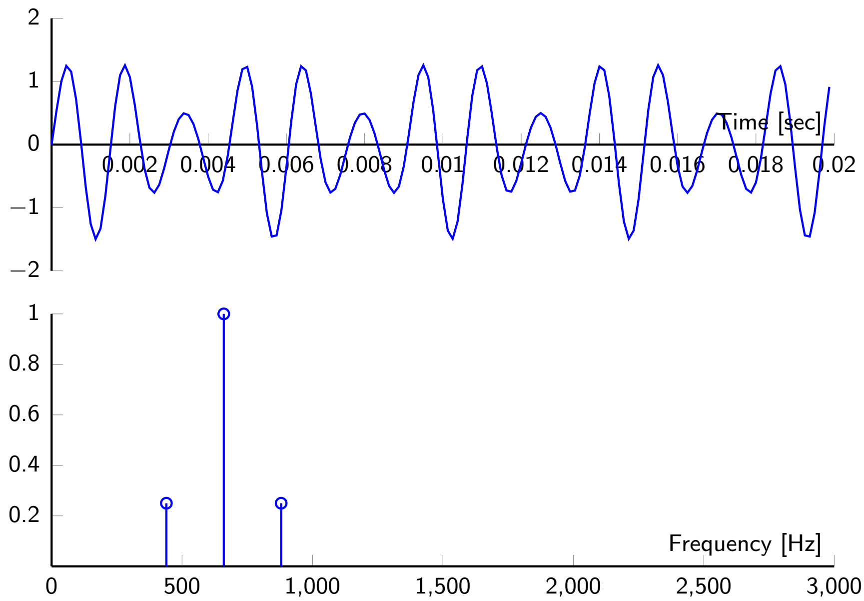


AM-synthesis

Modulate a sinusoid in amplitude.

$$x(t) = (1 + 0.5\sin(2\pi F_m t)) \cdot \sin(2\pi F_0 t) \tag{9}$$

Modulation with $F_0 = 660\,\text{Hz}$ and $F_m = 220\,\text{Hz}$.

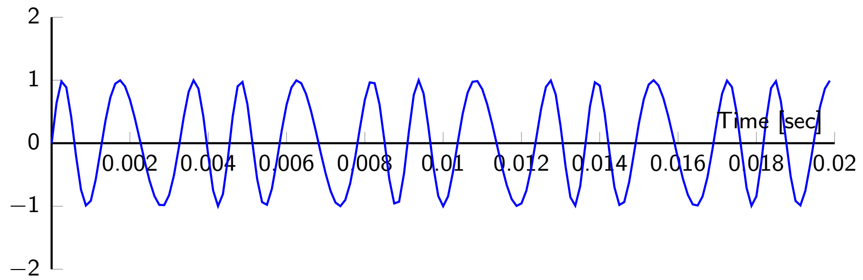


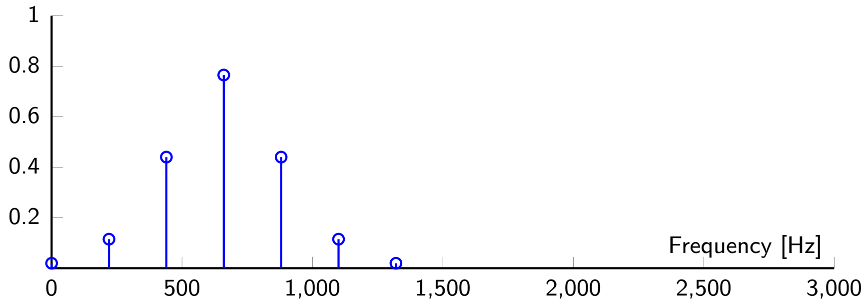
FM-synthesis

Modulate a sinusoid in frequency.

$$x(t) = \sin(2\pi F_0 t + 3\sin(2\pi F_m t)) \tag{10}$$

Modulation with $F_0 = 660\,\text{Hz}$ and $F_m = 220\,\text{Hz}$.





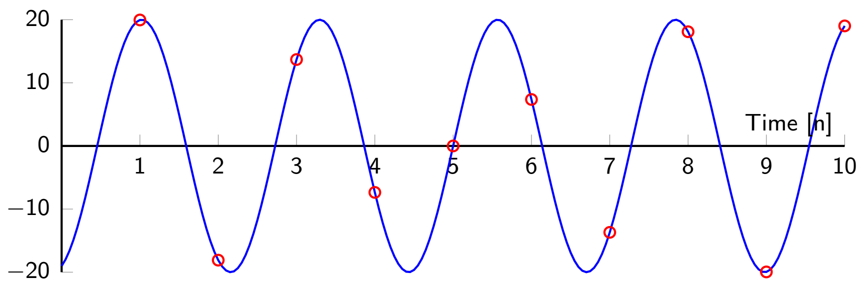
Sampling (page 21, 23)

The signal

$$x(t) = 20\cos(2\pi 440t - 0.4\pi) \tag{11}$$

is read with a frequency of $F_s = 1000\,\text{Hz}$, or equivalently with $T_s = \frac{1}{F_s} = \frac{1}{1000} = 0.001\,\text{s}$ between each read.

$$x(n) = x(t \mid t = nT_s = \frac{n}{F_s}) = 20\cos\left(2\pi \cdot \frac{440}{1000} \cdot n - 0.4\pi\right) \tag{12}$$



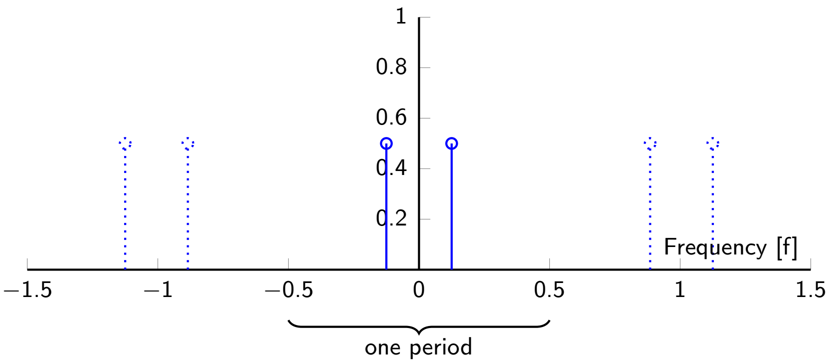
Notations:

$\Omega = 2\pi F$	Frequency and phase for *continuous* signals (real frequency).
$\omega = 2\pi f$	Frequency and phase for *discrete* signaler (digital frequency).

The spectrum for a discrete signal is periodic.

$$x(n) = \cos(2\pi f_0 n) = \cos(2\pi(f_0 + k)n) \qquad \text{for } k \text{ integer} \tag{13}$$

How do we draw the frequency content of a signal?



An example of frequencies and the periodic spectrum. The sampled frequency f' in the table is always $-0.5 \leq f' < 0.5$.

F [Hz]	F_s [Hz]	$\rightarrow$	$f = \frac{F}{F_s}$	$f' \pm k$
1	4	$\rightarrow$	0.25	0.25 ± 0
6.3	1	$\rightarrow$	6.3	$0.3 + 6$
5.8	1	$\rightarrow$	5.8	$-0.2 + 6$
48000	4000	$\rightarrow$	12	$0 + 12$

The digital frequencies f and f' have no physical unit.

Discrete signals (page 43)

Discrete signals are denoted $x(n)$ (sometimes also $x[n]$).

$$x(n) = \begin{cases} 1 & 0 \leq n < 3 \\ 4 & n = 3 \\ 0 & \text{otherwise} \end{cases} = \left\{ \begin{matrix} \ldots & 0 & \underline{1} & 1 & 1 & 4 & 0 & \ldots \end{matrix} \right\} = \left\{ \begin{matrix} \underline{1} & 1 & 1 & 4 \end{matrix} \right\} \tag{14}$$

Impulse

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} = \left\{ \begin{matrix} \ldots & 0 & \underline{1} & 0 & 0 & 0 & \ldots \end{matrix} \right\} \tag{15}$$

Step

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} = \left\{ \begin{matrix} \ldots & 0 & \underline{1} & 1 & 1 & 1 & \ldots \end{matrix} \right\} \tag{16}$$

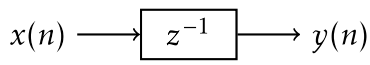
A signal is *causal* if the values are zero for all negative indices.

Using the impulse we can write

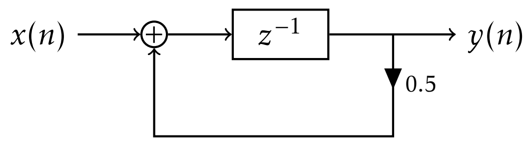
$$x(n) = \left\{ \begin{matrix} \underline{1} & 4 & 1 \end{matrix} \right\} = 1 \cdot \delta(n) + 4 \cdot \delta(n-1) + 1 \cdot \delta(n-2) = \sum_k x(k)\delta(n-k) \tag{17}$$

Example of systems (page 58, 59)

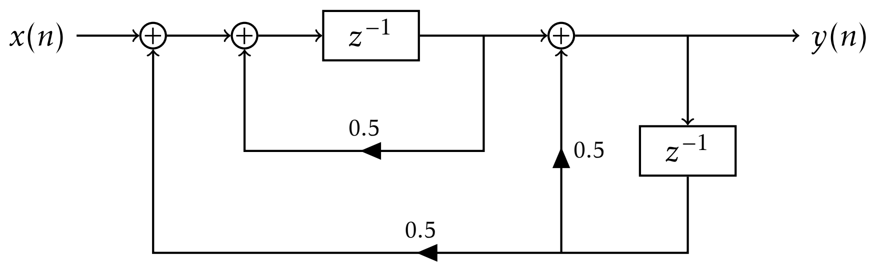
Delay: $y(n) = x(n-1)$



First order system: $y(n) = 0.5 \cdot y(n-1) + x(n-1)$



Second order system



Now we need to use the z-transform in chapter 3.

More about structures in chapter 9.

Definitions (page 45)

Energy: A signal is called an *energy signal* if $E < \infty$.

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{18}$$

Power: A signal is called a *power signal* if $P < \infty$.

$$P = \frac{1}{N} \cdot \sum_{n=0}^{N-1} |x(n)|^2 \tag{19}$$

Even symmetry

$$x(n) = x(-n) \tag{20}$$

Odd symmetry

$$x(n) = -x(-n) \tag{21}$$

System with finite memory: FIR; for example

$$y(n) = x(n) + x(n-1) \tag{22}$$

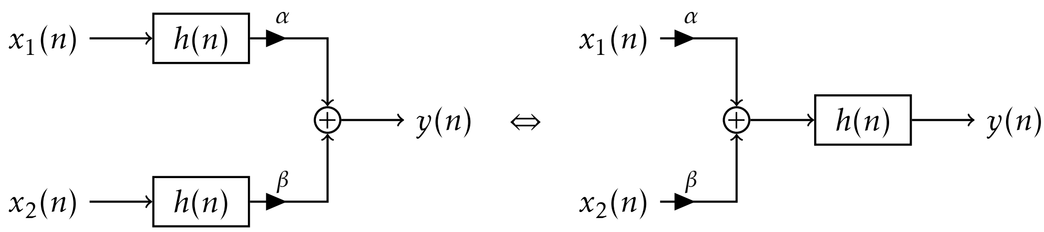
System with infinite memory: IIR; for example

$$y(n) = 0.5 \cdot y(n-1) + x(n) \tag{23}$$

Linearity

$$x(n) = \alpha x_1(n) + \beta x_2(n) \quad \Leftrightarrow \quad y(n) = \alpha y_1(n) + \beta y_2(n) \tag{24}$$

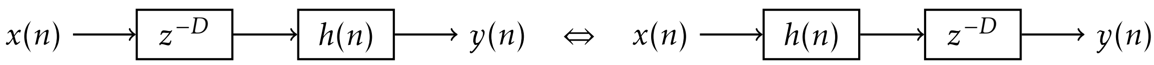
Equivalent block diagram:



Time invariant or Shift invariant

$$x(n) \to x(n-D) \quad \Leftrightarrow \quad y(n) \to y(n-D) \tag{25}$$

Equivalent block diagram:



BIBO-stability: Bounded Input Bounded Output.

$$|x(n)| \leq M_x \quad \Leftrightarrow \quad \left|y(n)\right| \leq M_y < \infty \tag{26}$$

Mathematics in the course

Complex numbers

$$z = a + \mathrm{j}b = r \cdot \mathrm{e}^{\mathrm{j}\Phi} = r \cdot \cos(\Phi) + \mathrm{j}r \cdot \sin(\Phi) \tag{27}$$

where $r = \sqrt{a^2 + b^2}$ and $\Phi = \arctan(b/a)$ if $a \neq 0$.

Euler's formula

$$\cos(\omega) = \frac{\mathrm{e}^{\mathrm{j}\omega} + \mathrm{e}^{-\mathrm{j}\omega}}{2} \tag{28}$$

$$\sin(\omega) = \frac{\mathrm{e}^{\mathrm{j}\omega} - \mathrm{e}^{-\mathrm{j}\omega}}{2\mathrm{j}} \tag{29}$$

Integrals

$$I = \int_0^T e^{-j\omega t}\, dt \tag{30}$$

$$= \frac{e^{-j\omega T} - e^{-j\omega 0}}{-j\omega} \tag{31}$$

$$= \frac{1 - e^{-j\omega T}}{j\omega} \tag{32}$$

$$= \frac{e^{-j\omega \cdot \frac{T}{2}} \cdot \left(e^{j\omega \cdot \frac{T}{2}} - e^{-j\omega \cdot \frac{T}{2}}\right)}{j\omega} \tag{33}$$

$$= T \cdot \frac{\sin\left(\omega \cdot \frac{T}{2}\right)}{\omega \cdot \frac{T}{2}} \cdot e^{-j\omega \cdot \frac{T}{2}} \tag{34}$$

Finite geometric sum

$$S_1 = \sum_{n=0}^{N} a^n = 1 + a + \cdots + a^N = \frac{1 - a^{N+1}}{1 - a} \tag{35}$$

Proof:

$$S = \sum_{n=0}^{N} a^n = 1 + a + a^2 + \cdots + a^N \tag{36}$$

$$a \cdot S = a + a^2 + a^3 + \cdots + a^{N+1} \tag{37}$$

$$S - a \cdot S = 1 - a^{N-1} \quad \Rightarrow \quad S = \frac{1 - a^{N+1}}{1 - a} \tag{38}$$

Infinite geometric sum

$$S_2 = \sum_{n=0}^{\infty} a^n = 1 + a + a^2 + \cdots = \frac{1}{1 - a} \qquad |a| < 1 \tag{39}$$

Proof:

$$\lim_{N \to \infty} \frac{1 - a^{N+1}}{1 - a} = \frac{1}{1 - a} \qquad \text{if } |a| < 1 \tag{40}$$