Laboration Exercises in Digital Signal Processing

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Introduction

Introduction

The traditional way to motivate digital signal processing is the wish to process an analog signal digitally using a computer or another form of digital hardware. Today digital signal processing is cheap and fast, can be accomplished on a small area, with great flexibility. Figure 1 shows the basic principle, where a real-world analog signal is sampled to a discrete-time signal consisting of a sequence of values or samples. The number of samples produced per second in this process is determined by the sampling frequency. The discrete-time signal can then be manipulated in different ways using digital signal processing, after which it is interpolated back to an analog signal, and transmitted to a proper analog system, e.g. a loudspeaker or an antenna.

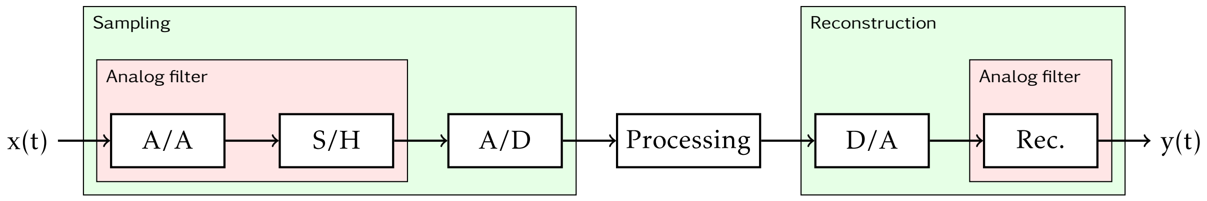


Figure 1: Sampling and reconstruction system.

Sampling

Already at an early stage of this course it is important to understand the principles and purposes of the different parts of this system. Discrete-time signals have a very fundamental property, namely they lack knowledge of at what rate they have been sampled or at what rate they will be reconstructed. Take the example of a CD disc containing a discrete-time signal. This signal does not know what time interval or sampling interval there should be between its samples, instead the CD player provides the information that the signal is to be reconstructed with 44.1 kHz. This was also known in the studio where the disc was recorded. It is often said that a discrete-time signal has distance 1 between its samples, i.e. sampling period 1 without a unit. The sampling frequency is the inverse of the sampling period, and is hence also 1 without a unit. All frequencies of the discrete-time signal are given as a fraction of the sampling frequency. These frequencies are called normalized frequencies and are between 0 and 1. It is only when the signal is reconstructed with a suitable sampling frequency that the normalized frequencies are transformed to real frequencies, since it is then one decides what units are to be used for the time axis.

Another very important property of discrete-time signals is the fact that the largest frequency that can be represented is the frequency of a signal consisting of every second sample positive, and every second sample negative, with equal amplitude. This frequency has a period of 2 samples which is why the normalized frequency is 0.5. One can try to record analog signals of faster variation, but these frequencies will not be represented correctly by the discrete signal. In fact, all frequencies of the analog signal which are above a normalized frequency of 0.5, corresponding to half the sampling frequency, will look like lower frequencies in the discrete-time signal. This phenomenon is called aliasing, and makes it impossible to map the signal back to the analog domain at its original frequency, since the aliased higher frequencies in the reconstruction process are considered to be in the interval from 0 to 0.5. As an example we can consider two analog signals of 100 Hz and 350 Hz which are sampled at 500 Hz. The signal at 350 Hz is above half the sampling frequency and cannot be represented by the discrete-time signal, instead it is considered as a signal of frequency between 0 Hz and 250 Hz, in this case $500 - 350 = 150$ Hz. If these two sampled signals are reconstructed we will get back the original 100 Hz tone, plus a tone of the aliased signal.

The solution of this problem is naturally to only allow recording of signals which can be represented correctly in the discrete-time signal. For this reason one often use an analog anti-aliasing filter, which lets through only frequencies lower than half the sampling frequency, before a signal is transformed from the analog to the digital domain with aid of a sample-and-hold circuit and an analog-to-digital converter. In this case the mapping between the analog and the digital domain is one-to-one; the conditions of the *sampling theorem* are fulfilled.

The fact that the conditions of the sampling theorem are fulfilled also means that we do not lose any information in the A/D converter, i.e. that the original analog signal can be reconstructed (interpolated) exactly from the discrete-time signal. This is accomplished by the digital-to-analog converter holding the value of each sample during a sampling interval at its output. The result is a piecewise constant signal, with stepwise changes at each new sampling interval. These stepwise changes introduce higher frequencies than half the sampling frequency. In fact all discrepancy from the original signal lies in frequencies above half the sampling frequency. For this reason one again applies an analog filter which is called a reconstruction filter in order to eliminate frequencies above half the sampling frequency, which obviously were not sent into the system. Now the analog reconstructed signal looks exactly like the original analog signal, provided that we sampled twice as fast as the largest frequency of the input signal. If this had not been done, the anti-aliasing filter would have eliminated the large frequencies of the input signal, so the condition is fulfilled in any case.

Filters

Finally some words about the concept of *filter*. A filter describes how a certain number of samples of a signal, up to the latest, shall be linearly combined in order to produce a certain output signal. The weights are put in a sequence forming a discrete-time sequence called the *impulse response* of the filter. The impulse response is a representation of an LTI system. Another description of the same system is as a difference equation. It is very important to realize that the concepts of filter and impulse response are not as strange as they first can seem. A filter is nothing but a tool in order to combine input data to obtain a desired output signal.

Let us assume you save money on a bank account each week, where $x[n]$ is the amount of money saved during week number n. We call $x[n]$ a signal, a discrete-time input signal to a system. Let us determine what output signal we desire; this will determine the system. As output signal from the system we choose, for instance, the sum of your savings the last month, i.e. the last four weeks. The difference equation for the system describes the output signal the current week as a sum of the four most recent savings where each of them is given weight 1. In other words $y[n] = 1 \cdot x[n] + 1 \cdot x[n-1] + 1 \cdot x[n-2] + 1 \cdot x[n-3]$. The filter now weights the last four savings and is therefore said to have length 4. The impulse response of the filter is the sequence of weights starting with the latest i.e. $[1 \quad 1 \quad 1 \quad 1]$. This type of filter has a bounded memory and weighs only a limited amount of samples; the filter is therefore called a *finite impulse response* (FIR) filter.

Let us instead assume you want your filter/system to compute your balance. A simple way to do that is to update the balance at each saving, i.e. the new balance is the old plus the saving amount. The difference equation for this system is $y[n] = 1 \cdot y[n-1] + 1 \cdot x[n]$. It is important that not only the input signal is part of the computation, but also old output signals. This makes the computation of the impulse response more involved since it is, per definition, a rule for how to weigh input signals in order to compute a certain output signal. If old output signals are part of the difference equation the system is called recursive. To solve the problem we see that the same equation can be used for old output signals, e.g. $y[n-1] = 1 \cdot y[n-2] + 1 \cdot x[n-1]$ and $y[n-2] = 1 \cdot y[n-3] + 1 \cdot x[n-2]$ etc. The important insight in this situation is that the output signal depends on all input samples (savings) up until now. The impulse response in this case is the infinitely long sequence $[1 \quad 1 \quad 1 \quad 1 \quad \cdots]$ and is called an *infinite impulse response* (IIR) filter.

In this simple example we have only used weights equal to 1, but with different weights for input signals of different age, the filter structure turns out to be very flexible. Each input signal can be divided into its frequency components, and in the rest of this course we will see how the filters can be used to enhance, attenuate, or delay different parts of these frequencies. In fact, it is exactly the same thing to combine input signals from different time instances to an output signal as it is to amplify, attenuate and delay the frequency components of a signal.

Overview

The detailed theoretical understanding of the different blocks in the system above will grow during the course. The computer exercises has three aims:

1. to illustrate applications of the theoretical concepts,
2. to describe how complicated systems can be built based on the basic components, and
3. to give examples of real-world signals from application areas where digital signal processing is used.

Development in signal processing is often done in the Matlab environment. Therefore the computer exercises also aim at giving experience with Matlab and to teach its different signal processing tools and how to use them. The step from development in Matlab to a real-time algorithm in C, which can be run on a microprocessor or a digital signal processor, is treated in later courses.

Part I

Laboration: A System for Recording

1 Introduction

The next system we will take a look at is shown in figure 1 and describes how a measured analog signal is converted to digital data which is then processed for the purpose of e.g. information extraction. This system is also very common and is used e.g. for recording of sounds for digital storage, to record other types of signals (for instance medical, electrical or mechanical) for analysis, and receiving transmitted data (if the microphone is replaced by an antenna). We call this system a recording/analysis/receiving unit.

An analog signal is measured with aid of a microphone, electrode, antenna or transducer of any kind. The signal has a frequency content up to a certain limit. The sampling theorem says that we must sample the signal (A/D convert) at a frequency which is twice as large as the largest signal frequency in order to get a correct discrete-time description of the analog signal. If we can not or do not want to sample at a such a high rate we can sample at a lower rate, provided the interesting information is in the lower frequency bands, but one must then eliminate frequencies above half the sampling frequency, in order to avoid aliasing and destruction of the signal. This is accomplished with the anti-aliasing filter. Usually noise is received at measurements, and the noise may reside at high frequencies, which is why anti-aliasing filters are almost always used before A/D conversion. After this the signal is read during a sampling period in the S/H circuit so that the A/D converter gets time to read the value of each sample.

In this exercise you will increase your understanding about a signal's frequency representation and how the DFT tool can be used. The theory is illustrated with an ECG signal. Your task is to analyze it in Matlab and extract the meaning of the different frequencies. The analog (electrical) signal was sampled (recorded) at the Department of Cardiology, Lund University Hospital, and stored digitally. Digital signal processing is performed off line by a computer. The analysis can also be made in real-time in order to use the system as an alarm.

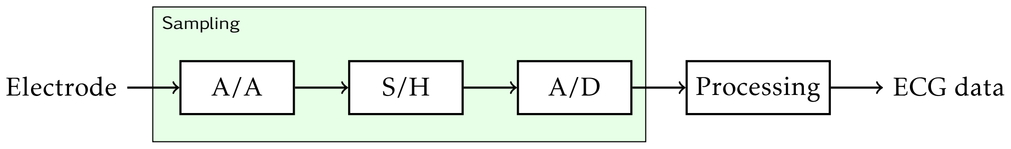


Figure 1: Sampling system.

2 An ECG Signal

The recorded ECG signal is sampled at 1 kHz. Thus, all frequencies over 500 Hz have been eliminated. This is not a problem since the body is not expected to produce faster variations than representable by approximately 250 Hz. Together with this lab is the file `ekg1.mat`.

Exercise 1 Load the EKG file into the Matlab workspace by writing:

```
load('data/ekg1.mat');
```

Investigate the downloaded file:

```
size(ekg1)
```

You find out that the file `ekg1` consists of a discrete-time signal of 10 000 samples that are stored as a row vector (1 row and 10 000 columns). Using the sampling frequency 1 kHz, this means a length of 10 s. Investigate the file further by writing:

```
plot(ekg1)
```

You see a ECG file of 10 s containing 10 heart beats, where the peaks correspond to the electrical impulse that contracts the heart muscle and squeezes the blood out to the body, and the following small peaks correspond return to the original configuration. You also see a slow movement in the base line, i.e. the amplitude level between the beats. This variation is caused by breathing, and also by ground level variation due to the measurement process at the body. The Y-axis unit is micro volts. ....

Exercise 2 As mentioned the signal contains no information a bout how quickly it has been registered. Matlab uses sampling enumeration as X-axis for plots, as in the last exercise. It is always more interesting to have a time units on the X-axis, i.e. to map the sample enumeration to time units. This can be accomplished using the sampling frequency:

```
FT=1000;
N=10000;
n=0:N-1;
t=n/FT;
```

where `FT` is the sampling frequency, `N` is the signal length, `n` is the sample index counted from zero, and `t` is a new time indexation for the 10 000 samples that is given by the total number of samples divided by the number of samples per second. Plot the signal again but now supply the the X-axis unit:

```
plot(t,ekg1)
xlabel('time (s)')
ylabel('amplitude (microV)')
```

It can be seen that the pulse seem to be approximately 1 beat per second or 60 beats per minute. ....

Exercise 3 Use the DFT tool to see what frequencies the signal contains; the heart beat frequency should be the strongest. In Matlab the DFT is computed using the command `fft`. This command needs to know the number of evenly spaced frequencies in the interval from 0 to 1 to compute the spectrum. If you for instance choose to compute FFT for M points, the FFT will computed the spectrum at the normalized frequencies $[\frac{0}{M} \quad \frac{1}{M} \quad \cdots \quad \frac{M-1}{M}]$ If you want to know the physical frequencies this correspond to you must multiply them with the sampling frequency. We choose to compute the FFT at $M = 10\,000$ points.

```
M=10000;
ekgspek=fft(ekg1,M);
```

The vector `ekgspek` contain both the amplitude and the phase function and is therefore complex valued. We are here interested only in which frequencies are in the signal, which is why we plot only the amplitude function which corresponds to the absolute value of the complex spectrum.

```
plot(abs(ekgspek))
```

Since we did not supply any X-axis data the spectrum was plotted only against the enumeration of its samples. This information is difficult to interpret, but we do know the normalized frequencies they correspond to, and since we know the sampling frequency we can transform the normalized frequencies to the physical frequencies.

```
f=(0:M-1)/M*FT;
plot(f,abs(ekgspek))
xlabel('frequency (Hz)')
```

An important observation: Frequencies above half the sampling frequency (500 Hz) are only a mirrored image of the frequencies below 500 Hz. It is hence below 500 Hz we are to study the figure. We zoom in at the left part (0 Hz to 500 Hz in frequency and 0 to 800 000 in amplitude):

```
axis([0 500 0 800000])
```

It seems this signal is over-sampled, i.e. the prerequisites for the sampling theorem were filled with a great margin. Not much energy is above 50 Hz in the spectrum. We zoom further, this time in both variables:

```
axis([0 20 0 150000])
```

We see the frequencies constituting the ECG signal, meaning that the signal mathematically can be broken down to these frequencies, not that the signal has been created by addition of a number of sinusoids with these frequencies. This is of course not the case; the signal has been created by many cells simultaneously changing their surface potential which give rise to an electrical field in the chest, measured with an electrode. We have now designed a computer interface to the physicians, and we would like to interpret the spectral plot. We see that the frequency zero has the largest energy. The reason is the varying base line which albeit not varying with a large amplitude compared to the beats, but yet is spread all over the signal. This spectral peak we are not interested in. The next top is almost exactly at 1 Hz. This frequency with harmonics at 2 Hz, 3 Hz and 4 Hz, and so on, represents the peak waveform of the heart beats. The fundamental, i.e. the 1 Hz tone, corresponds to the repetition frequency of the signal. This frequency is thus a measure of the heart beat frequency. The computer program should therefore look for the first peak, overlooking the possible peak at 0 Hz. . . . .

Exercise 4 Finally we shall investigate the way a simple filter may affect the ECG signal. We start by plotting the signal but this time at the upper half of a figure.

```
subplot(2,1,1)
plot(t,ekg1)
```

We now try to decrease the noise of this signal by specifying that the output signal at time n is a weighted average of the fifteen last samples. The filter is then fifteen coefficients with equal values, set for example to 0.2, for the input samples with index n to $n-14$. The filtering is done using convolution (the `conv` function in Matlab) of the input signal and the filter coefficients.

```
h=0.2*ones(1,15);
y=conv(ekg1,h);
y=y(15:end);
```

The length of the convolution is the input signal length plus the filter length minus one. The last row takes away the transient at the beginning (the filter doesn't have data at all its coefficients at the beginning), and makes the length of the output signal equal the length of the input signal.

```
subplot(2,1,2)
plot(t,y)
```

We see a signal of considerably smaller noise. The signal is so clean that we can see a small peak before each of the great peaks. The small peaks correspond to atrial contraction which fill the ventricles with blood, so they are full when the great contraction takes place at the great peak. . . . .

Part II

Laboration: IIR Filter Design

1 Introduction

A time-discrete system $H(z)$ can be described by the location of its poles and zeros in the z-plane. From the pole-zero plot, we can estimate the frequency response $H(\omega)$ by looking at the value of $H(z)$ on the unit circle. In general, the value of $|H(\omega)|$ is high when close to a pole, and low when close to a zero.

2 Preparation Tasks

Preparation task 1 Match the pole-zero plots with their respective amplitude response function in figure 1. . . . .

Preparation task 2 Sketch the approximate amplitude response functions of the pole-zero plots in figure 2. . . . .

3 Laboration Tasks

The program `mkiir` allows you to visually design an IIR filter by placing poles and zeros in the z-plane. A short introduction to the program is at the end of this laboration part.

Exercise 1 Using the filter design program, place a single pole or zero and move it around the z-plane (note that a single pole or zero means the *pair* of complex conjugated poles or zeros that automatically appear in the pole-zero plot). What happens to the frequency response as you move the pole or the zero around the z-plane? What is the relationship between the frequency response and the angle and radius of the pole or zero? . . . .

Exercise 2 Select the prototype filter *Low pass* and a filter specification appears in the frequency response window. Place poles and zeros in the z-plane and try to fulfill the filter specification by having the amplitude response inside the unshaded region. Use as few poles and zeros as possible. Don't forget to use the gain parameter to shift the amplitude response vertically to fit within the design constraints. . . . .

Optional Select another prototype filter and try to fulfil another design. Try to design the high-pass filter or the band-pass filter. . . . .

Optional You can also use Matlab to design a filter. The following code designs a Chebychev band-pass filter with stop-band ripple.

```
[b, a] = cheby2(4, 40, [.15, .35]*2);
```

The filter is a 4:th order IIR filter, the stop-band ripple (stop-band attenuation) is 40 dB, and the pass-band is between the normalized frequencies 0.15 and 0.35. Save the filter:

```
save('myfilter.mat', 'b', 'a');
```

Load it into the IIR filter design program to see where the poles and zeros are located.

Note: The factor 2 on the frequency range in `cheby2` is due to Matlab having different notation of normalized frequency, where 1 corresponds to half the sampling frequency instead of the whole sampling frequency. . . . .

The filters you have designed thus far have been described by stop-bands and pass-bands. Sometimes it is necessary to not block a range of frequencies but rather a single frequency. You can design a filter blocking a single frequency by placing a zero on the unit circle at the desired frequency.

One major drawback with placing only a single zero, corresponding to an FIR-filter where all the poles are located at the origin of the z-plane, is that a large area around the zero is significantly attenuated. To achieve a near flat response except close to the frequency we wish to cancel, the poles can be moved out from the origin towards the zero. The closer the poles are to the zeros, the narrower the notch. These filters are called *notch filters*. The equation for a single-frequency notch filter is

$$H(z) = \frac{\left(1 - e^{j\omega_0} z^{-1}\right)\left(1 - e^{-j\omega_0} z^{-1}\right)}{\left(1 - r \cdot e^{j\omega_0} z^{-1}\right)\left(1 - r \cdot e^{-j\omega_0} z^{-1}\right)} \tag{1}$$

where ω_0 is the normalized frequency to be canceled and r is the radius of the pole. The two numerator terms are the two complex conjugate zeros, and the two denominator terms are the two complex conjugate poles. Filters with multiple notches is designed by multiplying more complex conjugate poles and zeros at the desired frequencies to the numerator and denominator. The radius r controls the width of the notch, and is typically close to and less than 1 so that is is placed just inside the unit circle. The zeros are placed on the same frequencies, but *on* the unit circle.

Exercise 3 Use the program `jukebox` to load a small sequence of a music signal into Matlab. Pick any signal from the list.

```
[x, fs] = jukebox;
```

The music files have been disturbed by sinusoidal signals with unknown frequencies. Plot the power spectral density of the signal and examine the frequency contents.

```
pwelch(x, [], [], [], fs);
```

Also listen to the signal.

```
sound(x, fs);
```

The graph that appears will have a few narrow peaks indicating some disturbing sinusoidal signals. Which frequencies are disturbing the signal you selected? List these frequencies in a vector.

```
freq = [...];
```

. . . .

Exercise 4 From the list of frequencies you wish to cancel, you can create a list of the poles (located near the unit circle at radius `r`) and the zeros (located on the unit circle):

```
z = exp(-1j*2*pi*[freq, -freq]/fs);
p = exp(-1j*2*pi*[freq, -freq]/fs)*r;
```

Set the radius of the poles, `r`, to a suitable value. Note than the frequency spectrum is symmetric, and this is why the negative frequencies ae also considered in constructing the poles and zeros. The numerator and the denominator polynomials of the filter in (1) can now be created:

```
b = poly(z);
a = poly(p);
```

Filter your signal and plot the power spectral density:

```
y = filter(b, a, x);
pwelch(y, [], [], [], fs);
```

If the sinusoidal disturbances are not canceled, go back and redesign the filter with new or adjusted notch frequencies. If the disturbances have been removed, listen to the filtered signal:

```
sound(y, fs);
```

The disturbances should be completely removed. . . . .

Optional Save your notch filter and import it into the `mkiir` program.

```
save('mynotch.mat', 'b', 'a');
```

Do the location of the poles and zeros match your expectations? . . . .

4 IIR Filter Design Application

The program `mkiir`, shown in figure 4, allows you to visually design an IIR filter by placing poles and zeros in the z-plane. The program shows a complex pole-zero plane (top) and the filter amplitude response (bottom) on the left hand side, and program options on the right hand side.

Pole-Zero Plot

The pole-zero plot shows the unit circle and the real and imaginary axes. This area is where you place, move and delete your poles and zeros. Poles and zeros are automatically placed in complex conjugate pairs. Therefore, only IIR filters with odd-ordered numerator and denominator polynomials can be designed by and imported to the program.

Amplitude Response

The amplitude response updates continuously with the current filter response as you add, move or delete poles or zeros from the filter. This area can also display a prototype filter specification you can attempt to design using poles and zeros.

Program Options

The settings panel on the right hand side allows you to control the behaviour and actions of the filter design application.

Modify Poles or Zeros

Select whether to add, move and delete poles or zeros. Poles or zeros, depending on this option, are added to the filter by left clicking on the pole-zero plot. Poles or zeros are moved by left clicking and dragging an existing pole or zero, and are removed by left clicking near an existing pole or zero.

Poles and zeros can only be placed on or inside the unit circle on the z-plane. Poles or zeros placed outside the unit circle will be projected onto the unit circle so that you can place a point exactly on the unit circle.

Filter Prototype

Show a filter prototype in the amplitude response area. Selecting a filter prototype overlays a pre-defined design constraint on the amplitude response for you to design a filter against.

Filter Gain

Set a global filter gain or attenuation to vertically shift the amplitude response. This option allows you to scale the frequency response to fit within the design constraint.

Mirror and Switch Poles and Zeros

Mirror poles and zeros across the imaginary axis, or switch poles to zeros and zeros to poles.

Import and Export Filter

Import pre-designed filter coefficients from Matlab, or export your filter to Matlab. This allows you to design a filter in Matlab and import it into the design application to see how different filter designs places the poles and zeros, or to export your filter coefficients for use in Matlab.

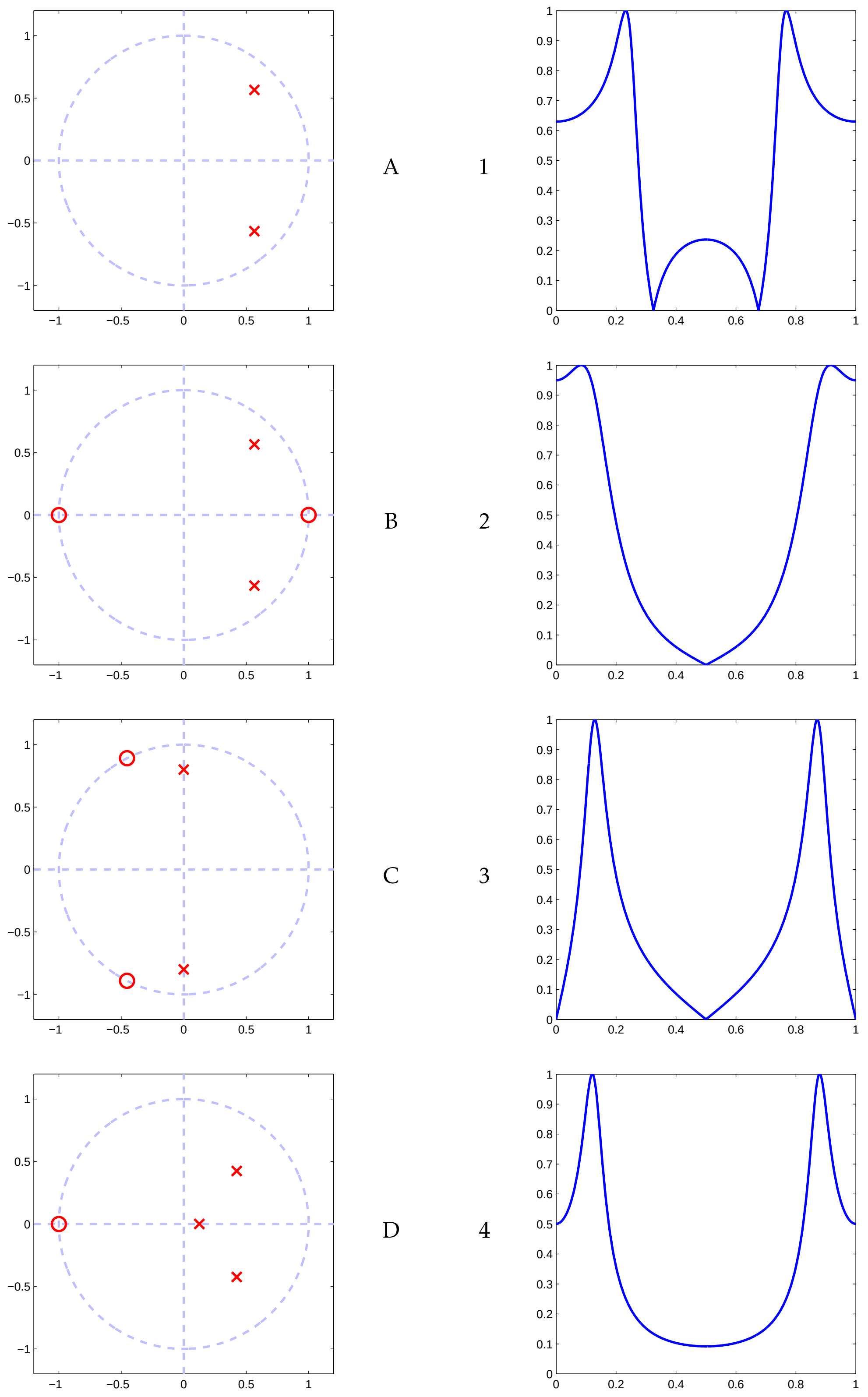


Figure 1: Match pole-zero plot with amplitude response.

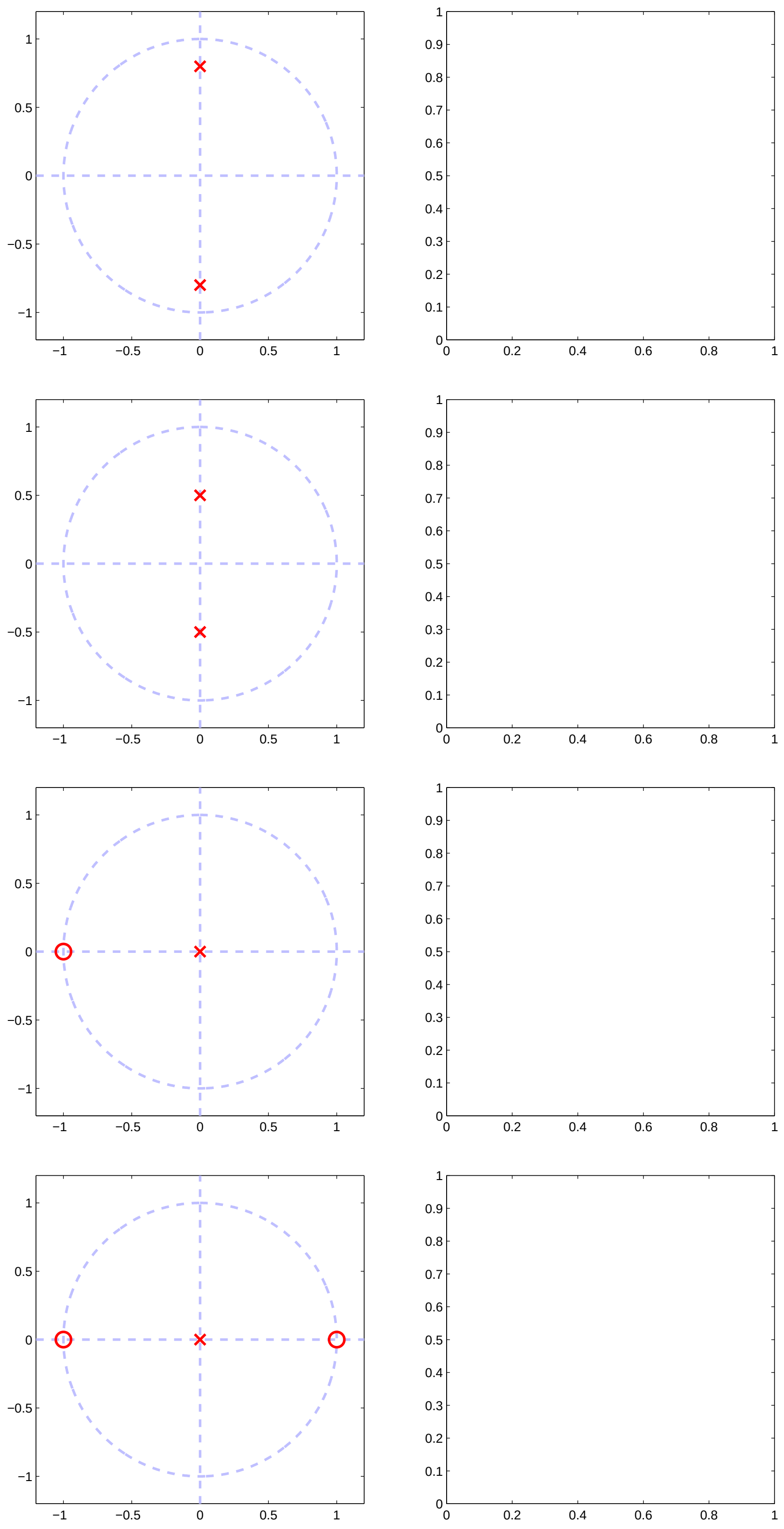
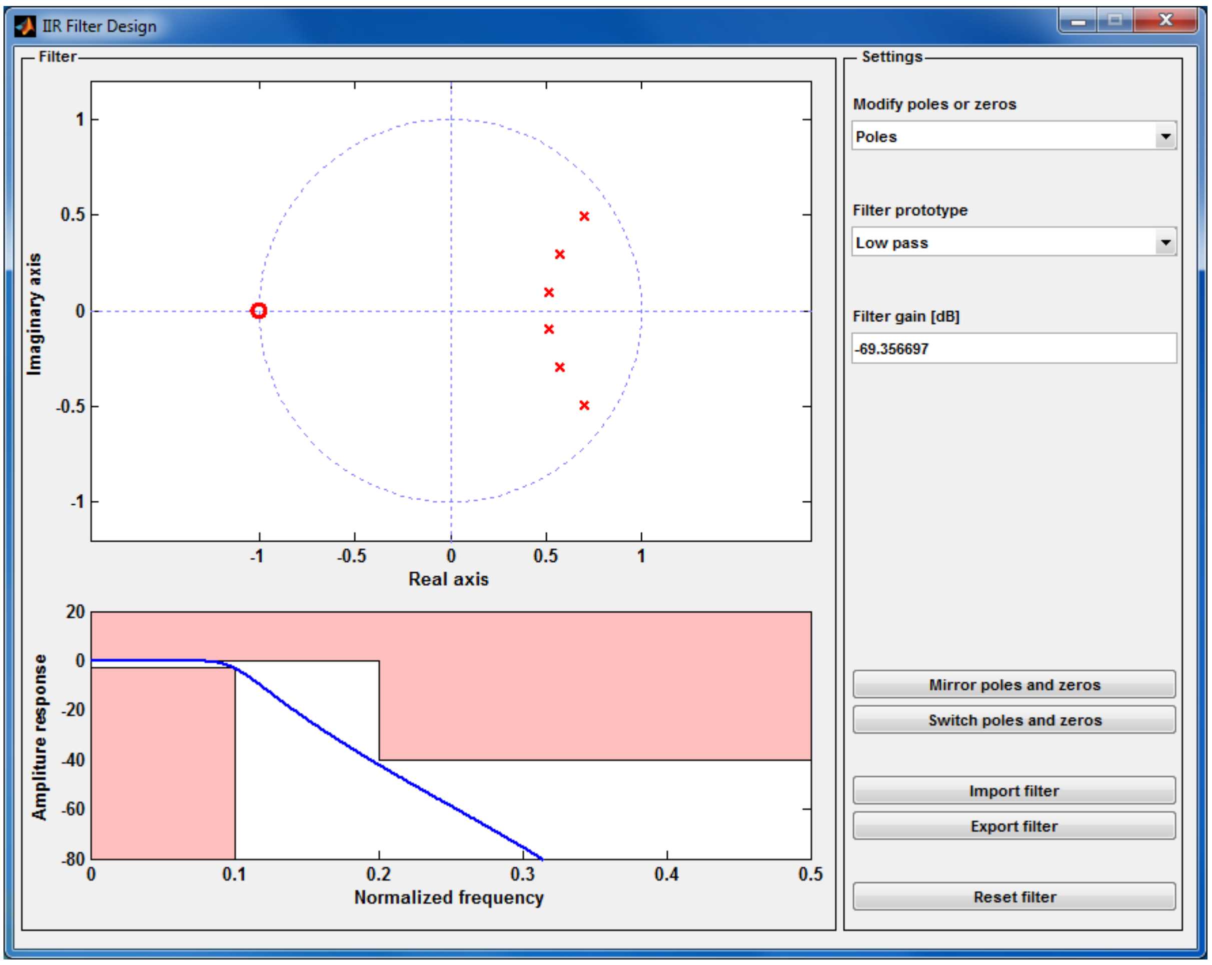


Figure 2: Sketch amplitude response.



Modify poles or zeros: Poles

Filter prototype: Low pass

Filter gain [dB]: -69.356697

Figure 3: Pole-zero based IIR filter design application.

Part III

Laboration: Image Filtering

1 Introduction

This laboration aims to give an understanding about the concept of *discrete frequency*. Both one-dimensional time signals and two-dimensional images will be used as to illustrate the concept. The discrete Fourier transform will be introduced and the concept of frequency response will be demonstrated. Additionally, the connection between frequency content and visual appearance will be introduced.

2 Filtering of One-dimensional Signals

Exercise 1 The following code creates a square pulse signal and a moving average filter, and then filters the signal with the filter. Start by creating the signal.

```
N = 256;
n = (0:N-1);
f = 12/N;                 % normalized frequency
x = square(2*pi*n*f);     % square pulse signal with frequency f
```

Next, create the filter and apply it to the signal.

```
L = 7;
h = 1/L * ones(1,L);    % 7-tap moving average filter
y_h = filter(h, 1, x);
```

Finally, plot the result to compare the input signal to the moving average filter and the output signal from the filter.

```
subplot(2, 1, 1); plot(n, x, '.-');
subplot(2, 1, 2); plot(n, y_h, '.-');
```

Try different frequencies of the square signal and see what happens to the filtered signal. .

Exercise 2 The filter in task 1 is a moving average filter with the filter vector defined as:

$$h(n) = \frac{1}{7} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{2}$$

Repeat task 1 but with the following filter vector:

$$\begin{aligned} g(n) &= \delta(n) - h(n) && (3)\\ &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} - \frac{1}{7} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} && (4)\\ &= \frac{1}{7} \cdot \begin{bmatrix} -1 & -1 & -1 & 6 & -1 & -1 & -1 \end{bmatrix} && (5) \end{aligned}$$

This filter can be constructed using the code below.

```
g = [zeros(1, floor(L/2)) 1 zeros(1, floor(L/2))] - h;
y_g = filter(g, 1, x);
subplot(2, 1, 1); plot(n, x, '.-');
subplot(2, 1, 2); plot(n, y_g, '.-');
```

Explore the differences between the filtered signal when using the filters $h(n)$ and $g(n)$ at different frequencies of the input signal. What happens to the input signal at sharp edges, and what happens on areas with constant amplitude? ....

Exercise 3 It is interesting to observe the frequency response of these two filter. This can be done using the code below.

```
H = fft(h,128);
G = fft(g,128);

subplot(2, 2, 1); stem(h);
```

```
axis([0 L (-1/L)*2 1]);
title('h(n)'); xlabel('Time index n'); ylabel('Amplitude');

subplot(2, 2, 2); stem(g);
axis([0 L (-1/L)*2 1]);
title('g(n)'); xlabel('Time index n'); ylabel('Amplitude');

subplot(2, 2, 3); plot(0:(2)/128:(1-1/128)*2,abs(H));
axis([0 1 0 max([abs(H), abs(G)])]);
title('H(\omega)'); xlabel('\omega / \pi'); ylabel('Amplitude');

subplot(2, 2, 4); plot(0:(2)/128:(1-1/128)*2,abs(G));
axis([0 1 0 max([abs(H), abs(G)])])
title('G(\omega)'); xlabel('\omega / \pi'); ylabel('Amplitude');
```

What is the frequency characteristic of these two filters? (i.e. which one is lowpass and which one is highpass) Change the filter length L. Do you observe any change in the frequency response of the filters? ....

3 Filtering of Two-dimensional Signals

A filter operation can be described as a filter mask sliding over a signal. For every sample of the input signal, the filter mask is placed on the current sample and its neighbouring samples. The signal and the mask is multiplied, and the sum of all products is the filtered output sample value for the current input sample. The same idea of a filter mask is applies to two-dimensional signals as well, for example images.

Exercise 4 Download the file grace-hopper.tif. The following code loads and displays the image:

```
I = imread('data/grace-hopper.tif');
x2 = double(I)/255;
imshow(x2);
```

Apply a two-dimensional 5×5 moving average filter:

```
L = 5;
h2 = 1/(L*L) * ones(L);
y_h2 = filter2(h2, x2);
imshow(y_h2);
```

Experiment with different filter sizes and observe the effects. ....

Exercise 5 The filter in task 4 is a two-dimensional moving average filter with the filter vector defined as:

$$h(m,n) = \frac{1}{5^2} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{6}$$

Repeat task 4 but with the following filter vector:

$$g(m,n) = \delta(n,n) - h(n,n) \tag{7}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \frac{1}{5^2} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{8}$$

$$= \frac{1}{5^2} \cdot \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 24 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix} \tag{9}$$

This filter can be constructed using the code below.

```
delta = zeros(L);
delta(ceil(L/2),ceil(L/2)) = 1;
g2 = delta - h2;
y_g2 = filter2(g2, x2);
imshow(y_g2);
```

Explore the differences between the filtered image when using the filters $h(m,n)$ and $g(m,n)$ at different frequencies of the input signal. What happens to the input signal at sharp edges, and what happens on areas with constant amplitude? ....

Exercise 6 Similarly as in the 1D case, it is interesting to observe the frequency response of these 2D filters. This can be done using the code below.

```
N = 32;
H2 = fftshift(fft2(h2, N, N));
G2 = fftshift(fft2(g2, N, N));
M = max(max([abs(H2), abs(G2)]));

n = -(L-1)/2:(L-1)/2;
w = -(N-1)/2:(N-1)/2;

subplot(2, 2, 1);
stem3(n, n, h2);
set(gca, 'ZLim', [-0.5, 1]);

subplot(2, 2, 2)
stem3(n, n, g2);
set(gca, 'ZLim', [-0.5, 1]);

subplot(2, 2, 3);
surf(w, w, abs(H2));

subplot(2, 2, 4);
surf(w, w, abs(G2));
```

What is the frequency characteristic of these two filters? (i.e. which one is lowpass and which one is highpass) Change the filter length L. Do you observe any change in the frequency response of the filters? ....

Part IV

Laboration: Spectrogram

1 Introduction

Two ways to visualize a signal is by plotting either the time-domain sequence, for example $x(n)$, or the corresponding frequency-domain sequence, for example $X(\omega)$, in a graph. Often neither of these graphs can reveal the structure of the signal. The time-domain graph can show when the signal is active but not its frequency content, and the frequency-domain can show the frequency content but not when the frequencies are active.

A compromise is to visualize a signal using a *spectrogram*. The spectrogram is a combination of the time-domain and the frequency-domain graphs that shows both the temporal and the frequency structure of a signal. The spectrogram is made by splitting the signal into a number of shorter consecutive frames. Each frame is then treated as individual shorter signals and the discrete Fourier transform (DFT) is used to calculate the frequency content of each frame.

2 Making the Spectrogram

Exercise 1 Start by making a function since we will use this spectrogram in following tasks. The function should take the signal $x(n)$, the size of the DFT N and a window parameter. Save this script as `myspectrogram.m` so it can be called from other scripts later.

```
function P = myspectrogram(x, N, window)
        % rest of code goes here
end
```

The frequency transform, the DFT or the equivalent function `fft` in Matlab, gives us the full spectrum from 0 to 2π. Since we are only using real-valued signals, we also know that the spectrum is symmetric; that is

$$X(\omega) = X(-\omega) = X(2\pi - \omega) \tag{10}$$

so we are only interested in the unique frequencies from 0 to π. For an N-point DFT we need to determine the M first unique frequency bins.

```
if mod(N, 2) == 0
    M = N/2+1;
else
    M = (N-1)/2;
end
```

Since we want to use an N-point DFT we also need to split the signal into frames of N samples. The function `buffer` splits the signal vector into an $N \times S$ matrix. Each consecutive column of the matrix is a frame of the contiguous sequence. The size N is the desired frame length and S is the number of frames as a result of splitting the signal. The value of S depends on the length of the signal; the longer the signal, the more frames we get.

```
S = ceil(length(x)/N);
xb = buffer(x, N);
```

Create a buffer to store the spectrogram. It is an $M \times S$ matrix; M is the desired number of unique frequency bins and S is the number of frames from splitting the input signal.

```
Xb = zeros(M, S);
```

The `window` parameter is a function that we wish to use to create the window. Call the window function to create the window vector $w(n)$ of a desired length. The window sequence $w(n)$ will be multiplied by the signal sequence $x_i(n)$. Since multiplication in the time-domain is equivalent to convolution in the frequency-domain, this can be seen as filtering of the frequency-domain sequence. Different windows makes different filters in the frequency-domain with different properties and tradeoffs.

```
w = window(N);
```

The signal $x(n)$ has been split into frames $x_i(n)$. The spectrogram is created by calculating the DFT of each frame in sequence; that is

$$X_i(\omega) = \mathrm{DFT}\{x_i(n) \cdot w(n)\} \tag{11}$$

The resulting spectrum is stored in the spectrogram buffer.

```
for i=1:S
    xi = xb(:,i);
    Xi = fft(xi.*w);
    Xb(:, i) = Xi(1:M);
end
```

Only the amplitude of the spectrogram buffer is of interest. Furthermore, the amplitude is transformed into a dB-scale. This is the final spectrogram returned by our function.

```
P = mag2db(abs(Xb));
```

. . . .

3 Using the Spectrogram

Exercise 2 Load a signal to analyze.

```
x = audioread('data/tetris.wav');
```

Calculate the spectrogram using the function from the previous task. The number 64 is the number of frequency bins, and consequently also the number of sampler per frame when splitting the signal, and `@rectwin` is the window function.

```
P = myspectrogram(x, 64, @rectwin);
```

Plot and compare the three ways described in the introduction to visualize the signal. The first graph shows the time-domain sequence, the second graph shows the frequency-domain sequence, and the third graph is the spectrogram.

```
subplot(3, 1, 1); plot(x);
subplot(3, 1, 2); plot(abs(fft(x)));
subplot(3, 1, 3); imagesc(P);
```

. . . .

Exercise 3 Change the number of frequency bins and window function and describe what tradeoffs are made by changing the parameters. Two common window functions are the rectangular window (`@rectwin`) and the Hamming window (`@hamming`).

- What happens when a different number of frequency bins are used?
- What happens when a different window function is used?
- What are the tradeoffs when changing these parameters?

. . . .

Part V

Laboration: Frequency-Domain Equalization

1 Introduction

A sequence $x(n)$ that goes through a system $h(n)$ will produce the output sequence

$$y(n) = x(n) * h(n) \tag{12}$$

where $y(n)$ corresponds to the received signal and $h(n)$ the propagation channel. In wireless communications, the multipath propagation channel (filter) $h(n)$ will cause *inter-symbol interference* when transmitting a sequence of symbols from a transmitter to a receiver. A multipath channel is a channel where multiple copies, delayed and attenuated "randomly" due to interfering objects between the transmitter and the receiver, are added at the receiver. Since some propagation paths to the receiver is longer than others, a transmitted symbol may end up overlapping (interfering) other symbols at the receiver. To suppress the inter-symbol interference in the receiver, an equalizer filter $h^{-1}(n)$ will be used to reverse the channel; that is

$$x(n) = y(n) * h^{-1}(n) \tag{13}$$

where $h^{-1}(n)$ is the inverse of the filter $h(n)$.

In high-speed wireless communications, or in underwater acoustic communications, the length of the inverse filter is usually too large to be efficiently implemented. The computational complexity becomes too high for time-domain equalization. Resorting to the property of the discrete Fourier transform, that convolution in time-domain is equivalent to multiplication in frequency-domain, we can implement the equalization in frequency-domain efficiently with the help of the DFT.

Since circular convolution is actually required in the time-domain, we need some kind of modification on the transmit signal. The signal $x(n)$ is divided into multiple frames where each frame has N samples. Copy the last P samples to the front, where these samples are called the *cyclic prefix* (CP). Then combine these segments as a new sequence $t(n)$ and transmit.

$$x(n) = \underbrace{\boxed{\quad x_i(n) \quad}}_{N \text{ samples}} \; \boxed{\quad x_{i+1}(n) \quad} \; \boxed{\quad x_{i+2}(n) \quad} \tag{14}$$

$$t(n) = \overset{\text{copy}}{\underbrace{\boxed{\text{CP}}}_{P}\underbrace{\boxed{\quad x_i(n) \quad}}_{N \text{ samples}}} \; \boxed{\text{CP}}\boxed{\quad x_{i+1}(n) \quad} \; \boxed{\text{CP}}\boxed{\quad x_{i+2}(n) \quad} \tag{15}$$

The received signal $r(n)$ will be the linear convolution between the signal $t(n)$ and the multipath channel $h(n)$. If we remove the cyclic prefix from the received signal frames, each received frame $r_i(n)$ will correspond to the circular convolution between the signal $x_i(n)$ and the channel $h(n)$; that is

$$r_i(n) = x_i(n) \otimes_N h(n) \tag{16}$$

or equivalent

$$R_i(\omega) = X_i(\omega) \cdot H(\omega) \tag{17}$$

The original transmit signal can be recovered by frequency-domain equalization and IFFT as

$$X_i(\omega) = \frac{H^*(\omega)}{|H(\omega)|^2} \cdot Y_i(\omega) \tag{18}$$

2 Filtering the Transmit Signal

Exercise 1 Produce or load a piece of music (or voice) signal.

```
[x, Fs] = audioread('data/tetris.wav');
```

Listen to the clean input signal.

```
sound(x,Fs);
```

. . . .

Exercise 2 Define the size of the frames and the cyclic prefixes. The total transmit frame size is the sum of the signal frame size and the cyclic prefix size. In this example, the frame length is 500 ms and the cyclic prefix is 125 ms, so the total transmit frame size becomes 625 ms.

```
P = Fs*0.125;
N = Fs*0.5;
L = P+N;
```

Generate a multipath channel and filter the produced signal. The cyclic prefix has to be at least as long as the multipath channel. For simplicity, the multipath channel has the same length of the cyclic prefix. The multipath channel is simulated by a filter with random coefficients.

```
h = randn(P,1);
h = h/norm(h);
```

Filter the input signal with the multipath channel.

```
z = filter(h,1,x);
```

Listen to distorted signal. This is what the receiver would have received if no additional processing was performed on the signal. A cyclic prefix is added before transmitting and inverse filtering is performed when receiving in order to reverse the effect of the multipath channel.

```
sound(z,Fs);
```

. . . .

Exercise 3 Construct the transmit signal with cyclic prefix and filter it through the multipath channel. Split the signal into consecutive frames using the function `buffer` to an $N \times S$ matrix. The transmit buffer is an $L \times S$ matrix so that it can contain both the cyclic prefix and the signal frame.

```
S = ceil(length(x) / N);
xb = buffer(x, N);
tb = zeros(L, S);
```

Loop over all frames and concatenate the cyclic prefix and the signal frame. The cyclic prefix is copied from the end of the signal frame and placed at the beginning of the transmit buffer.

```
for i=1:S
    tb(:, i) = vertcat(xb(N-P+1:N, i), xb(:,i));
end
```

The transmit signal is then constructed by flattening the transmit buffer matrix in to a single vector.

```
t = tb(:);
```

. . . .

3 Equalize the Receive Signal

Exercise 4 The received signal is produced by filtering the transmit signal with the multipath channel.

```
r = filter(h,1,t);
```

The received signal is actually filtered by linear convolution. However, due to the cyclic prefixes being introduced, the effect of linear filtering on the actual signal frames becomes equivalent to circular convolution. Therefore, we can use frequency-domain filtering, which is equivalent to circular convolution, to undo the actual linear convolution in the time-domain. Transform the time-domain multipath channel into a frequency domain multipath channel and invert the filter.

```
H = fft(h,N);
Hinv = conj(H)./abs(H).^2;
```

Split the received signal into an $L \times S$ matrix for easier processing. Likewise, create an $N \times S$ received buffer that will contain the equalized signal.

```
rb = buffer(r, L);
yb = zeros(N, S);
```

We know that the received signal $r(n)$ is

$$r_i(n) = x_i(n) \otimes_N h(n) \tag{19}$$

which is equivalent to

$$R_i(\omega) = X_i(n) \cdot H(n) \tag{20}$$

We can therefore undo the filtering by filtering with the corresponding inverse filter.

$$Y_i(\omega) = R_i(\omega) \cdot \frac{1}{H(\omega)} = X_i(n) \cdot H(n) \cdot \frac{1}{H(\omega)} = X_i(\omega) \tag{21}$$

Loop over all frames and perform frequency domain filtering with the inverse filter. Note that only the signal part of the received frame is filtered; the cyclic prefix is only appended during transmission to make the linear convolution behave like circular convolution. Fill in the necessary code to reconstruct the transmitted signal.

```
for i=1:S
    ri = rb(P+1 : L, i);
    yi = ... % your code here
    yb(:, i) = yi;
end
```

Flatten the received buffer to reconstruct the received signal.

```
y = yb(:);
```

Listen to the received signal.

```
sound(y,Fs);
```

Show the spectrograms of the original signal, the received signal as if no cyclic prefixes were added, and the reconstructed signal. Compare the received signal with the original signal and with the effect of multipath channels. Also use your own spectrogram functions from the previous task and compare the results (using suitable parameters).

```
subplot(3, 1, 1); specgram(x);
subplot(3, 1, 2); specgram(z);
subplot(3, 1, 3); specgram(y);
```

....