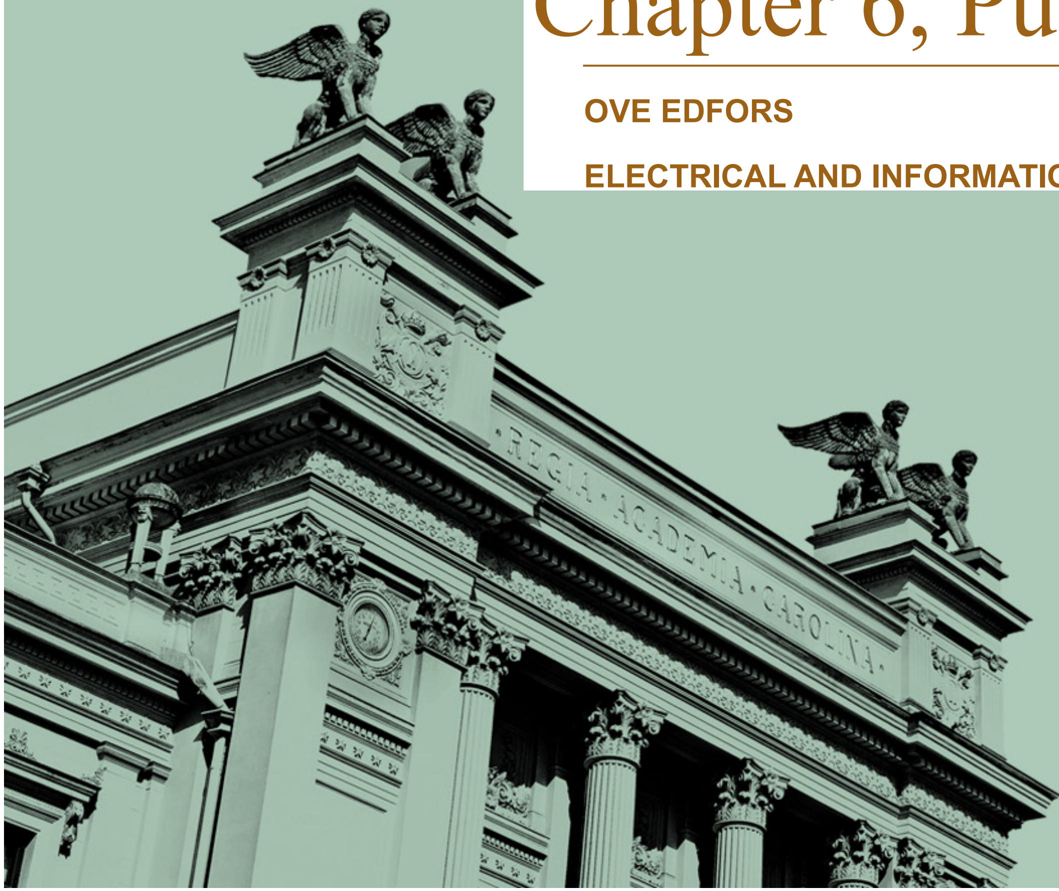
Information Transmission
Chapter 6, Public key crypto

OVE EDFORS

ELECTRICAL AND INFORMATION TECHNOLOGY



Learning outcomes

- After this lecture the student should
  - understand the concept of one-way functions and specifically trapdoor one-way functions,
  - understand what a two-key public-key cryptosystem is and how trapdoor one-way functions can be used to create such a system,
  - understand Euler's totient function and how it can be used to create a trapdoor one-way function,
  - know how to use Euclid's algorithm to calculate the greatest common divisor (gcd) of two natural numbers and understand the relation to Bezout's identity,
  - be able to perform the basic operations of Rivest-Shamir-Adelman (RSA) encryption and decryption, and
  - understand how a digital signature is created using trapdoor one-way functions.

One-way functions

It is possible to exchange secret keys without using a secure channel!

Diffie and Hellman introduced the concepts of *one-way functions* and *trapdoor one-way functions*.

A remarkable idea that dramatically changed the cryptological research.

A one-way function is a function $y=f(x)$ that is "easy" to compute for all x, but it is computationally infeasible to find x if you know only $y=f(x)$.

Trapdoor one-way functions

- A trapdoor one-way function is a family of invertible functions f_K such that
  - when K is known, we can easily find algorithms E_K and D_K that compute $f_K(x)$ and its inverse $f_K^{-1}(y)$, respectively, for all x and y,
  - when K is not known, it is computationally infeasible to compute $f_K^{-1}(y)$, even if we do know E_K.
- The algorithm E_K depends on a secret *trapdoor parameter* T such that D_K and, hence, $f_K^{-1}(y)$ is easy to find when we know T but it is computationally infeasible when we do not know T.

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A public-key cryptosystem

Using a trapdoor one-way function we can design a so-called *two-key or public-key cryptosystem.*

Such a system can be arranged by the intended ***receiver*** (!) of encrypted information as follows.

The receiver selects his trapdoor one-way algorithm E_K, keeps the trapdoor parameter T secret, but publishes openly the encryption algorithm E_K.

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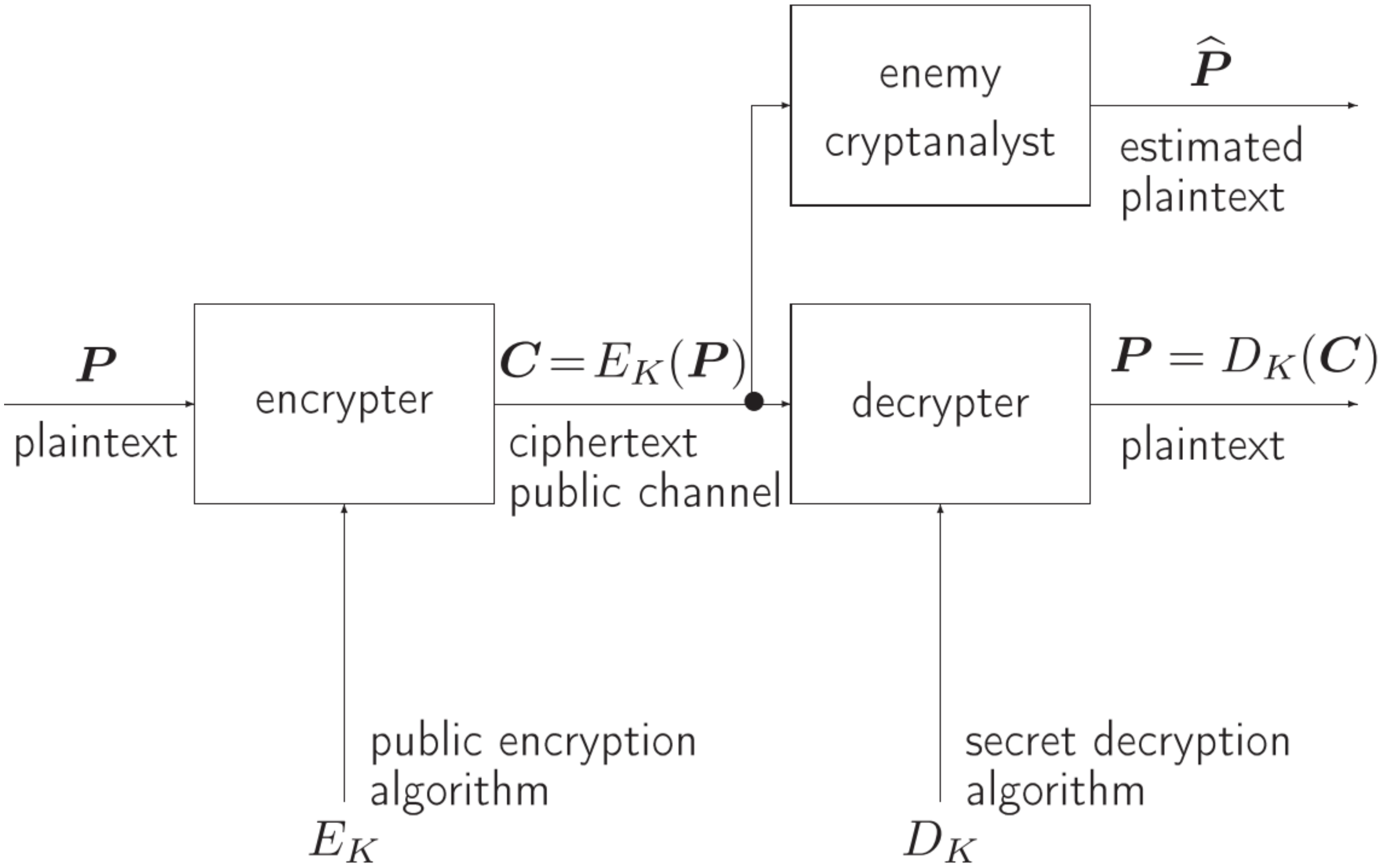
A public-key cryptosystem



Some background theory

Modular arithmetic

- Modular arithmetic is an arithmetic system for integers, where numbers "wrap around" upon reaching a certain value — the modulus.
- ***Congruence***: For a positive integer n, two numbers a and b are said to be congruent modulo n if their difference $a - b$ is an integer multiple of n. We denote this as

$$a \equiv b \pmod{n}$$

which implies, for some integer k,

$$a = kn + b$$

Modular arithmentic (examples)

$a \equiv b \pmod{n}$	$a = k \cdot n + b$
$2 \equiv 0 \pmod{2}$	$2 = 1 \cdot 2 + 0$
$12 \equiv 0 \pmod{4}$	$12 = 3 \cdot 4 + 0$
$13 \equiv 1 \pmod{4}$	$13 = 3 \cdot 4 + 1$
$13 \equiv 2 \pmod{11}$	$13 = 1 \cdot 11 + 2$
$-8 \equiv 7 \pmod{5}$	$-8 = -3 \cdot 5 + 7$

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Greatest common divisor (gcd)

- The greatest common divisor (gcd) of two or more integers, which are not all zero, is the largest positive integer that divides each of the integers.

- Example: gcd(24,54) = 6, since

Integers dividing $24 = 2^3 \cdot 3$ are 1, 2, 3, 4, **6**, 8, 12

Integers dividing $54 = 2 \cdot 3^3$ are 1, 2, 3, **6**, 9, 18, 27

Euclid's algorithm to calculate gcd

Given two natural numbers n_1 and n_2, where $n_1 > n_2$. Divide continually the larger by the smaller as follows:

$$\begin{aligned}
n_1 &= q_0 n_2 + r_0 && \text{(dividing } n_2 \text{ into } n_1\text{)} \\
n_2 &= q_1 r_0 + r_1 && \text{(dividing } r_0 \text{ into } n_2\text{)} \\
r_0 &= q_2 r_1 + r_2 && \text{(dividing } r_1 \text{ into } r_0\text{)} \\
r_1 &= q_3 r_2 + r_3 && \text{(etc.)} \\
&\vdots \\
r_{i-2} &= q_i r_{i-1} + r_i \\
r_{i-1} &= q_{i+1} r_i
\end{aligned}$$

Then r_i is the greatest common divisor of n_1 and n_2, denoted

$$r_i = \gcd(n_1, n_2).$$

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Relatively prime (coprime) numbers

- Two integers *a* and *b* are said to be relatively prime (coprime) if the only positive integer that divides both of them is 1, i.e., if their gcd(a,b) = 1.

  Consequently, any prime number that divides one does not divide the other.

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Euler's totient function

Euler's totient function, denoted $\Phi(n)$, is the number of integers between 1 and n that are relatively prime with n, that is, they have no common factors with n.

General case: Given the (unique) prime factorization of n, we can calculate Euler's totient function as

$$\Phi(n) = \Phi(p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_m}\right)$$

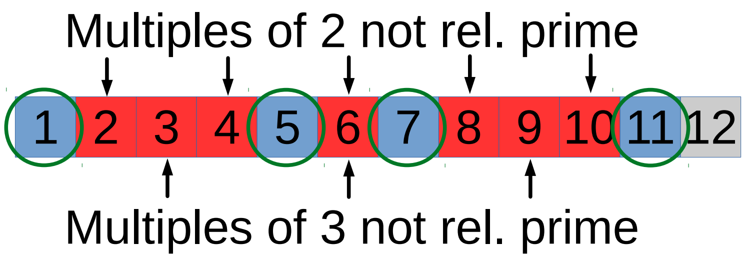
Important special case: When n is a product of two primes $n = p \cdot q$ then

$$\Phi(n) = \Phi(p \cdot q) = \Phi(p)\Phi(q) = (p-1)(q-1)$$

Euler's totient function (examples)

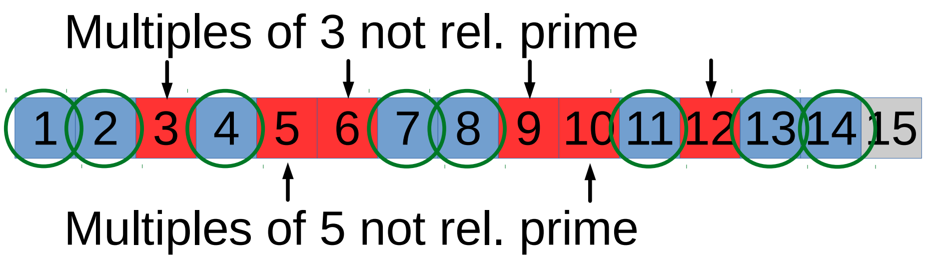
“General” case

$$\Phi(12) = \Phi(2^2 3) = 12\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 4$$



Special case with two primes

$$\Phi(15) = \Phi(3 \cdot 5) = (3 - 1) \cdot (5 - 1) = 8$$



A useful trapdoor one-way function

Theorem 6.2 (Euler) Let a and n be two integers that are relatively prime. Then

$$a^{\phi(n)} \equiv 1 \pmod{n} \tag{6.17}$$

or, equivalently,

$$R_n(a^{\phi(n)}) = 1 \tag{6.18}$$

where $R_d(i)$ denotes the remainder r when the integer i is divided by the divisor d, that is, $i = qd + r$, $0 \leq r < |d|$.

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Bézout's identity

Given integers n_1 and n_2 not both zero, there exist integers s and t such that

$$\gcd(n_1, n_2) = sn_1 + tn_2$$

Use Euclid's algorithm backwards to find s and t

Étienne Bézout
(1730-1783)

A public-key encryption algorithm

The RSA algorithm

1. Choose two distinct prime numbers p and q.
2. Compute $n = pq$. n is used as the modulus for both the public and private keys. Its length, usually expressed in bits, is the key length.
3. Compute $\Phi(n) = \Phi(p)\,\Phi(q) = (p - 1)(q - 1)$ where Φ is Euler's totient function.
4. Choose an integer e such that $1 < e < \Phi(n)$ and $\gcd(e, \Phi(n)) = 1$; i.e., e and $\Phi(n)$ are coprime.

   The *public key* consists of the modulus n and the public encryption exponent e

Source: Wikipedia

The RSA algorithm cont.

5. Determine d as $d \equiv e^{-1} \pmod{\Phi(n)}$; i.e., d is the multiplicative inverse of e (modulo $\varphi(n)$), i.e. solve for d given $d \cdot e \equiv 1 \pmod{\Phi(n)}$. d is kept as the private key exponent.

The *private key* consists of the private decryption exponent d, which must be kept secret and is used together with the modulus n to calculate the clear text.
The parameters p, q, and $\Phi(n)$ must also be kept secret because they can be used to calculate d.

Source: Wikipedia

RSA encryption

- Alice transmits her public key *(n, e)* to Bob and keeps the private key secret.
- Bob wishes to send a message *p* to Alice. He first turns *p* into an integer *P*, such that $0 \leq P < n$. He then computes the ciphertext *C* corresponding to $C = P^e \pmod{n}$
- Bob then transmits *C* to Alice.

RSA decryption

- Alice can recover *P* from *C* by using her private key exponent *d* via computing $P \equiv C^d \pmod{n}$
- Given *P*, Alice can recover the original message *p.*

Since an essentially larger amount of computation is involved in a two-key cryptosystem than in a comparably secure single-key cryptosystem, two-key cryptosystems are mainly used in hybrid systems.

Conclusion

Everybody can look up the public parameters n and e, but only those who know at least one of the secret parameters p,q, and d that are included in the trapdoor parameter T can decrypt.

If the enemy cryptanalyst, however, can factor n, then he can easily compute $\Phi(n)$ and obtain the secret decryption exponent d, and, hence, obtain the plaintext.

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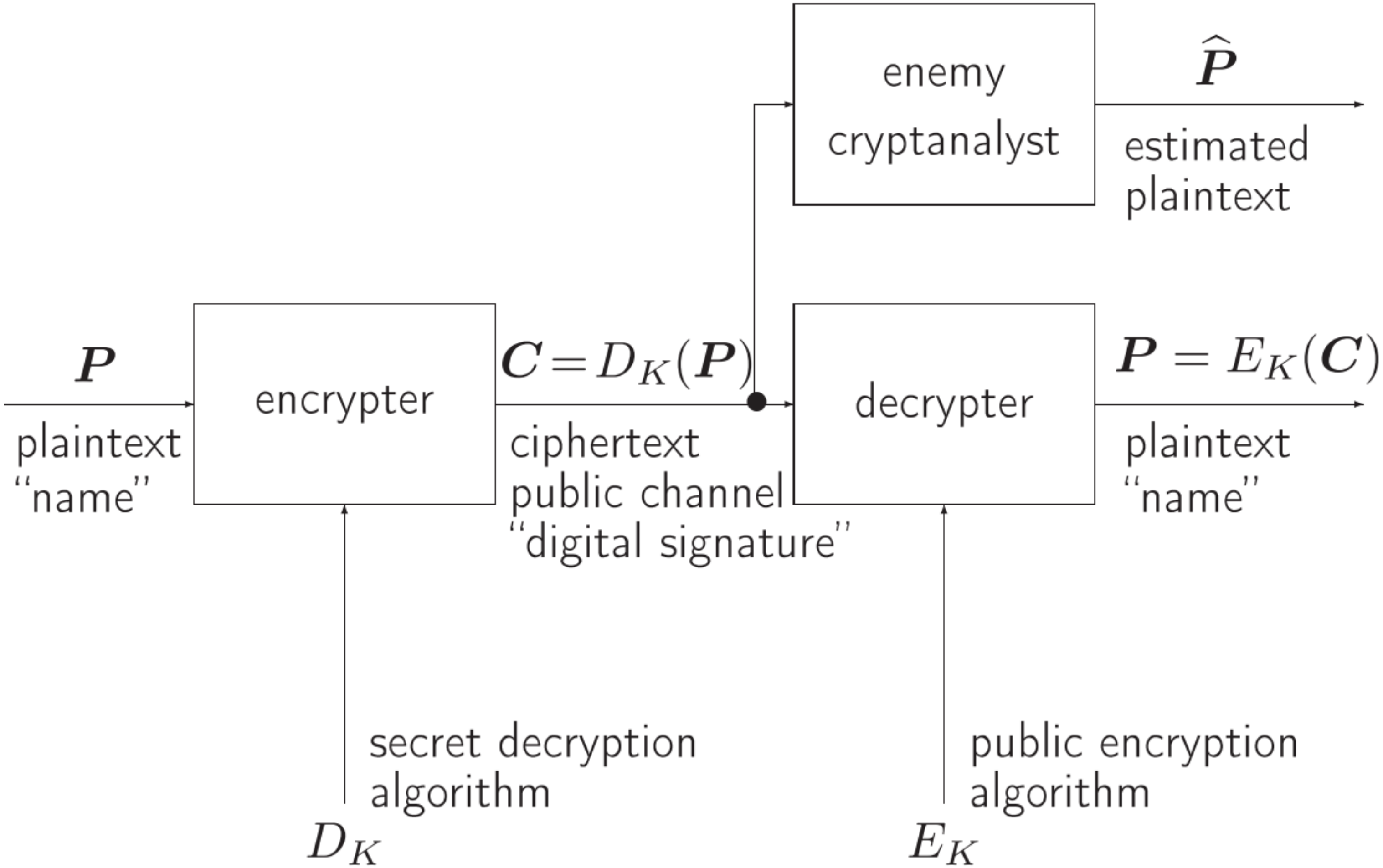
Digital signatures

A trapdoor one-way function can be used to identify a sender - to obtain a *digital signature* - but at the expense of giving up secrecy.

The *sender* who would like to create an unforgeable digital signture uses his *secret* algorithm D_K and creates a ciphertext by using, for example, his name as plaintext.

Anybody can use the senders *public* algorithm E_K to decrypt the ciphertext, and, hence, recover the sender's plaintext.

Digital signatures



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