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Information Transmission
Chapter 5, Channel coding

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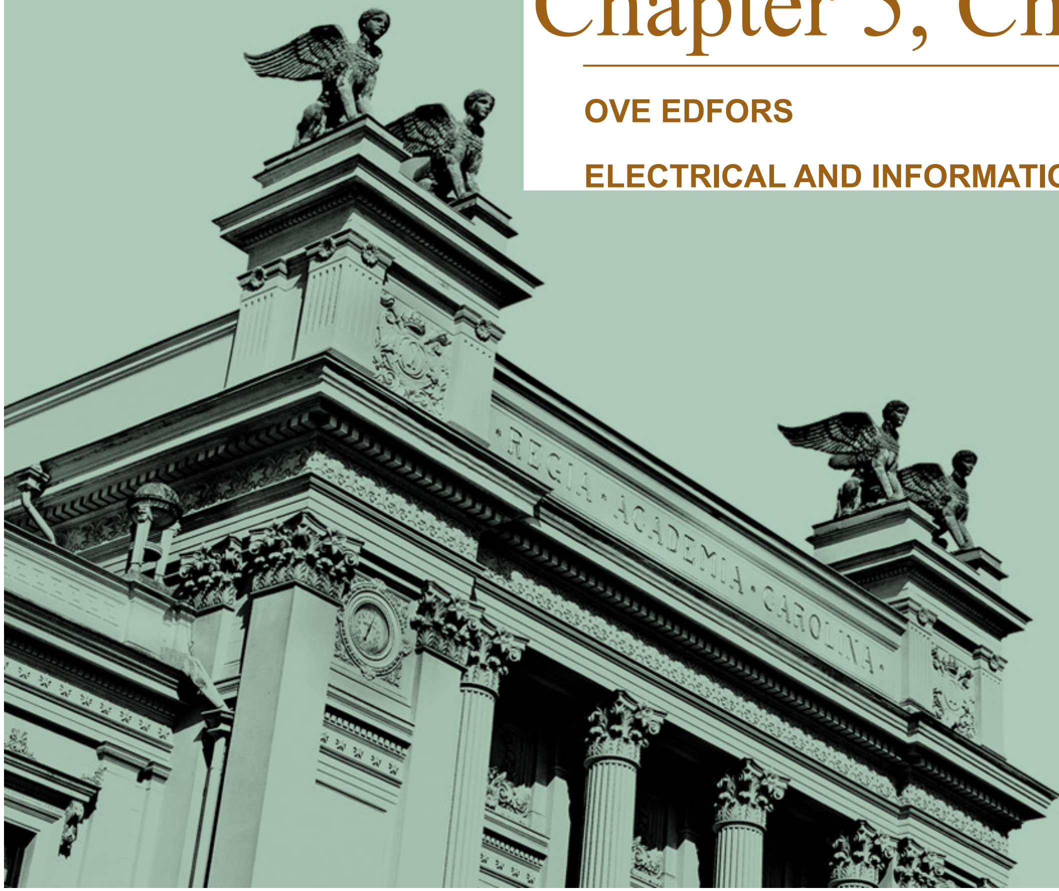
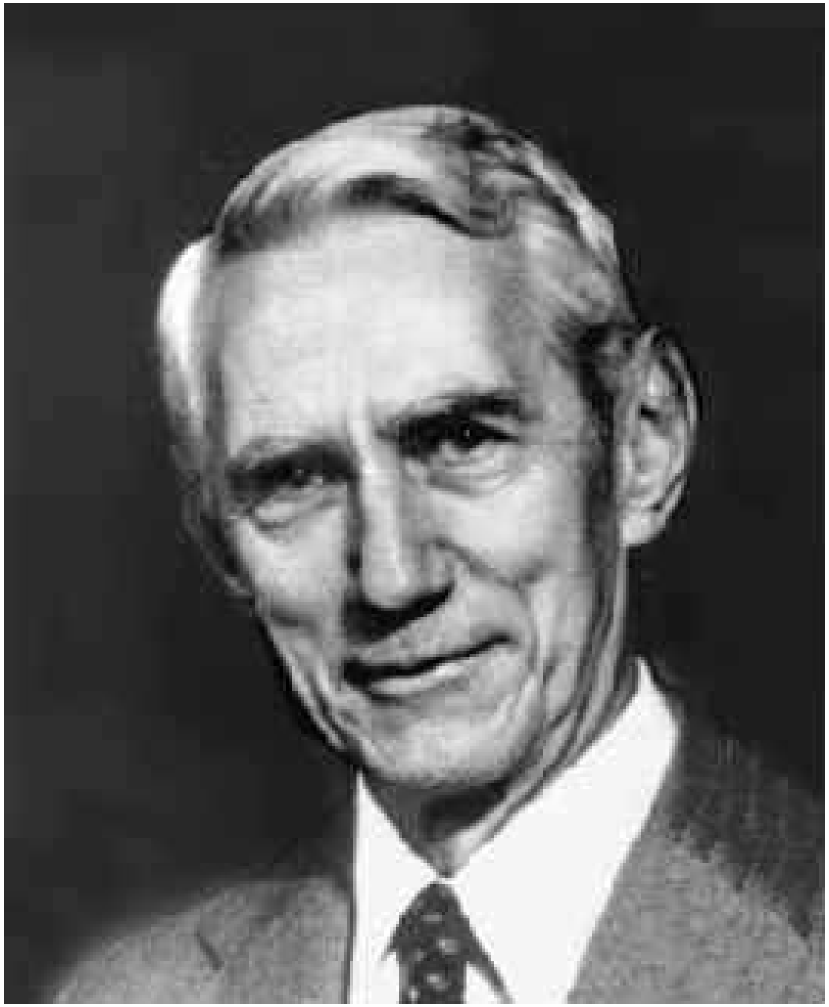
ELECTRICAL AND INFORMATION TECHNOLOGY



Learning outcomes

- After this lecture the student should
  - understand the principles of channel coding,
  - understand how typical sequences can be used to find out how ”fast” we can send information over a channel,
  - have a basic knowledge about how channel capacity is related to mutual information and its maximization over the channel input distribution
  - know how to calculate the channel capacity for the binary symmetric channel and the time-discrete additive white Gaussian noise (AWGN) channel

What did Shannon promise?



- As long as the SNR is above -1.6 dB in an AWGN channel we can get reliable communication

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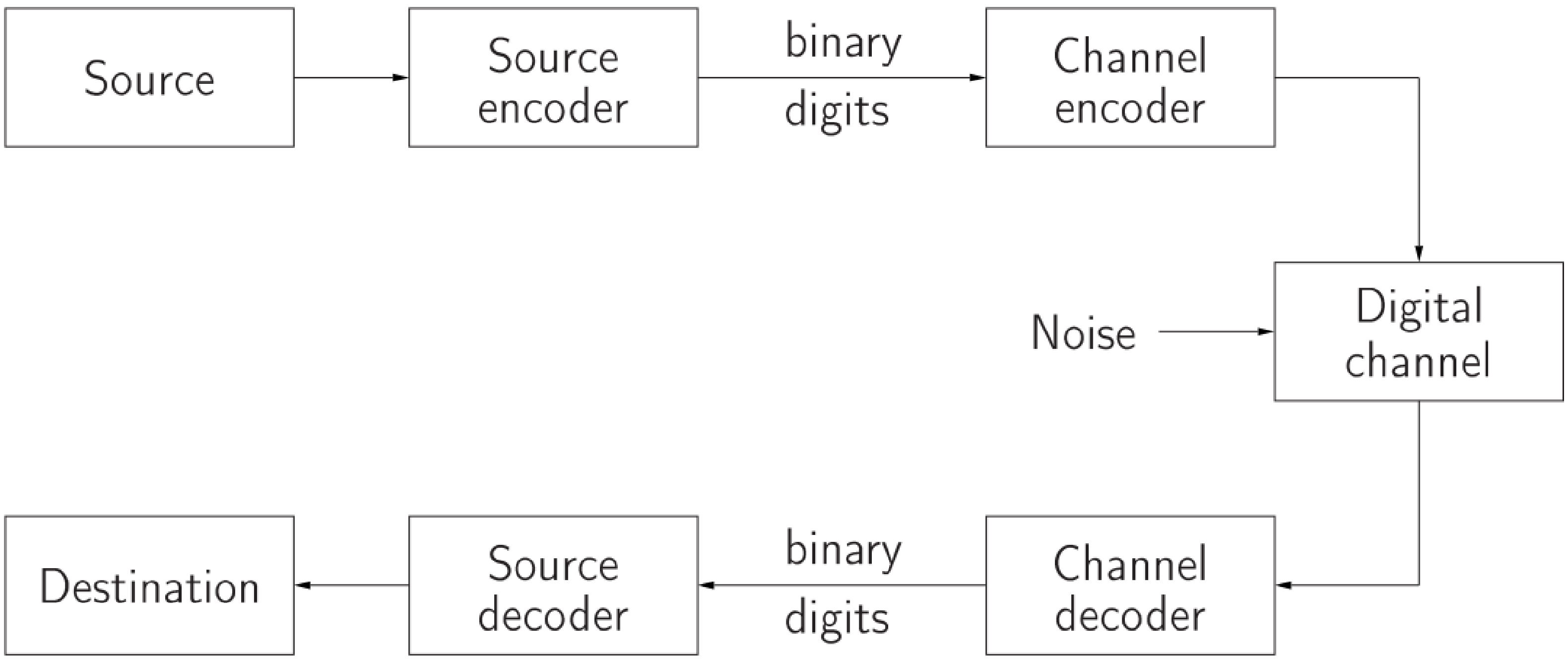
A schematic communication system



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Typical sequences

All typical long sequences have approximately the same probability and from the law of large numbers it follows that the set of these typical sequences is overwhelmingly probable.

The probability that a long source output sequence is typical is close to one, and, there are approximately

$$2^{nh(p)}$$

typical long sequences.

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Properties of typical sequences

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Let $\mathcal{T}_\epsilon^{(n)}$ be the set of sequences $\boldsymbol{x} = x_1 x_2 \ldots x_n$ such that

$$2^{-n(H(X)+\epsilon)} \leq P_{\boldsymbol{X}}(\boldsymbol{x}) \leq 2^{-n(H(X)-\epsilon)} \tag{5.43}$$

The set $\mathcal{T}_\epsilon^{(n)}$ is called the typical set and it has the following properties:

1. If $\boldsymbol{x} \in \mathcal{T}_\epsilon^{(n)}$, then $P_{\boldsymbol{X}}(\boldsymbol{x}) \approx 2^{-nH(X)}$.
2. $Pr(\mathcal{T}_\epsilon^{(n)}) > 1 - \epsilon$, for n sufficiently large.
3. $|\mathcal{T}_\epsilon^{(n)}| \leq 2^{n(H(X)+\epsilon)}$.

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sequence	probability	
● ● ● ● ●	1/3 1/3 1/3 1/3 1/3 ⇒ 0.0041	
● ● ● ● ○	1/3 1/3 1/3 1/3 2/3 ⇒ 0.0082	
● ● ● ○ ●	1/3 1/3 1/3 2/3 1/3 ⇒ 0.0082	
● ● ● ○ ○	1/3 1/3 1/3 2/3 2/3 ⇒ 0.0165	
● ● ○ ● ●	1/3 1/3 2/3 1/3 1/3 ⇒ 0.0082	
● ● ○ ● ○	1/3 1/3 2/3 1/3 2/3 ⇒ 0.0165	
● ● ○ ○ ●	1/3 1/3 2/3 2/3 1/3 ⇒ 0.0165	
● ● ○ ○ ○	1/3 1/3 2/3 2/3 2/3 ⇒ 0.0329	⋆
● ○ ● ● ●	1/3 2/3 1/3 1/3 1/3 ⇒ 0.0082	
● ○ ● ● ○	1/3 2/3 1/3 1/3 2/3 ⇒ 0.0165	
● ○ ● ○ ●	1/3 2/3 1/3 2/3 1/3 ⇒ 0.0165	
● ○ ● ○ ○	1/3 2/3 1/3 2/3 2/3 ⇒ 0.0329	⋆
● ○ ○ ● ●	1/3 2/3 2/3 1/3 1/3 ⇒ 0.0165	
● ○ ○ ● ○	1/3 2/3 2/3 1/3 2/3 ⇒ 0.0329	⋆
● ○ ○ ○ ●	1/3 2/3 2/3 2/3 1/3 ⇒ 0.0329	⋆
● ○ ○ ○ ○	1/3 2/3 2/3 2/3 2/3 ⇒ 0.0658	⋆
○ ● ● ● ●	2/3 1/3 1/3 1/3 1/3 ⇒ 0.0082	
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○ ● ● ○ ○	2/3 1/3 1/3 2/3 2/3 ⇒ 0.0329	⋆
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○ ● ○ ○ ●	2/3 1/3 2/3 2/3 1/3 ⇒ 0.0329	⋆
○ ● ○ ○ ○	2/3 1/3 2/3 2/3 2/3 ⇒ 0.0658	⋆
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○ ○ ● ● ○	2/3 2/3 1/3 1/3 2/3 ⇒ 0.0329	⋆
○ ○ ● ○ ●	2/3 2/3 1/3 2/3 1/3 ⇒ 0.0329	⋆
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○ ○ ○ ● ○	2/3 2/3 2/3 1/3 2/3 ⇒ 0.0658	⋆
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○ ○ ○ ○ ○	2/3 2/3 2/3 2/3 2/3 ⇒ 0.1317	
	0.9998	

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Longer typical sequences

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Let us now choose a smaller ϵ namely $\epsilon = 0.046$

(5 % of h(1/3)), and increase the length of the sequences.

Then we obtain the following table:

n	$\lvert\mathcal{T}_\epsilon^{(n)}\rvert$	$Pr(\mathcal{T}_\epsilon^{(n)})$
100	$2^{92.6}$	0.660
500	$2^{474.9}$	0.971
1000	$2^{953.4}$	0.998
2000	$2^{1910.3}$	1.000

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Typical sequences in text

If we have L letters in our alphabet, then we can compose L^n different sequences that are n letters long.

Only approximately $2^{nH(X)}$, where $H(X)$ is the uncertainty of the language, of these are “meaningful”.

What is meant by “meaningful” is determined by the structure of the language; that is, by its grammar, spelling rules etc.

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Typical sequences in text

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Only the fraction

$$\frac{2^{nH(X)}}{L^n} = \frac{2^{nH(X)}}{2^{n\log L}} = 2^{-n(\log L - H(X))}$$

which vanishes when *n* grows provided that $H(X) < \log L$, is ”meaningful”.

For the English language *H(X)* is typically 1.5 bits/letter and $\log_2 26 \approx 4.7$ bits/letter.

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Structure in text

Shannon illustrated how increasing structure between letters will give better approximations of the English language.

Assuming an alphabet with 27 symbols---26 letters and 1 space---he started with an approximation of the first order.

The symbols are chosen *independently* of each other but with the actual probability distribution (12 % E, 2 % W, etc.):

OCRO HLI RGWR NMIELWIS EU LL NBNESEBYA

TH EEI ALHENHTTPA OOBTTVA NAH BRL

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Structure in text

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Then Shannon continued with the approximation of the second order. The symbols are chosen with the actual *bigram* statistics---when a symbol has been chosen, the next symbol is chosen according to the actual conditional probability distribution:

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Structure in text

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The approximation of the third order is based on the *trigram* statistics---when two successive symbols have been chosen, the next symbol is chosen according to the actual conditional probability distribution:

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REP.

The principle of source coding

Consider the set of typical long output sequences of *n* symbols from a source with uncertainty *H(X)* bits per source symbol.

Since there are fewer than $2^{n(H(X)+\epsilon)}$ typical long sequences in this set, they can be represented by $n(H(X)+\epsilon)$ binary digits; that is, by $H(X)+\epsilon$ binary digits per source symbol.

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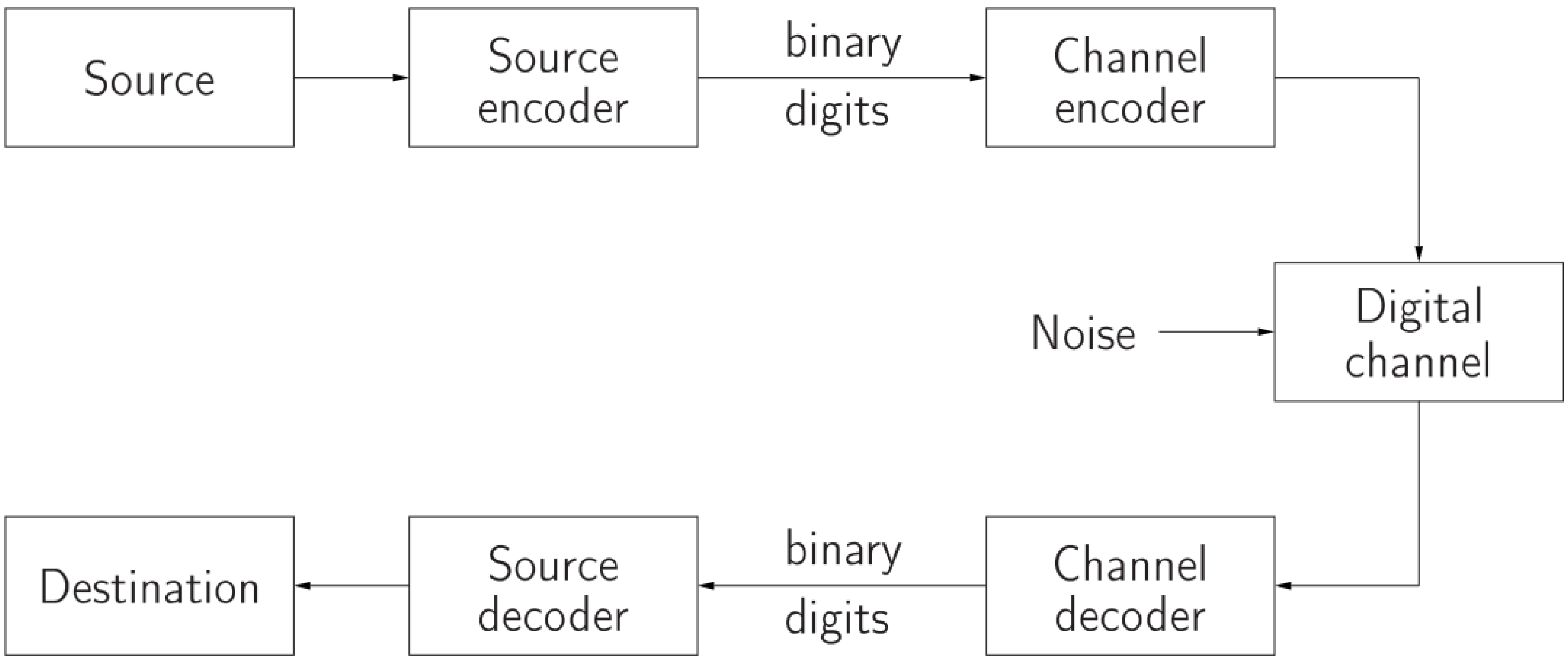
Channel coding

A schematic communication system



Fans (of a typical input sequence and its typical output sequences)

Consider a channel with input *X* and output *Y*.

Then we have approximately $2^{NH(X)}$ and $2^{NH(Y)}$ typical input and output sequences of length *N*, respectively.

Furthermore, for each typical long input sequence we have approximately $2^{NH(Y|X)}$ typical long output sequences that are jointly typical with the given input sequence, we call such an input sequence together with its jointly typical output sequences a *fan*.

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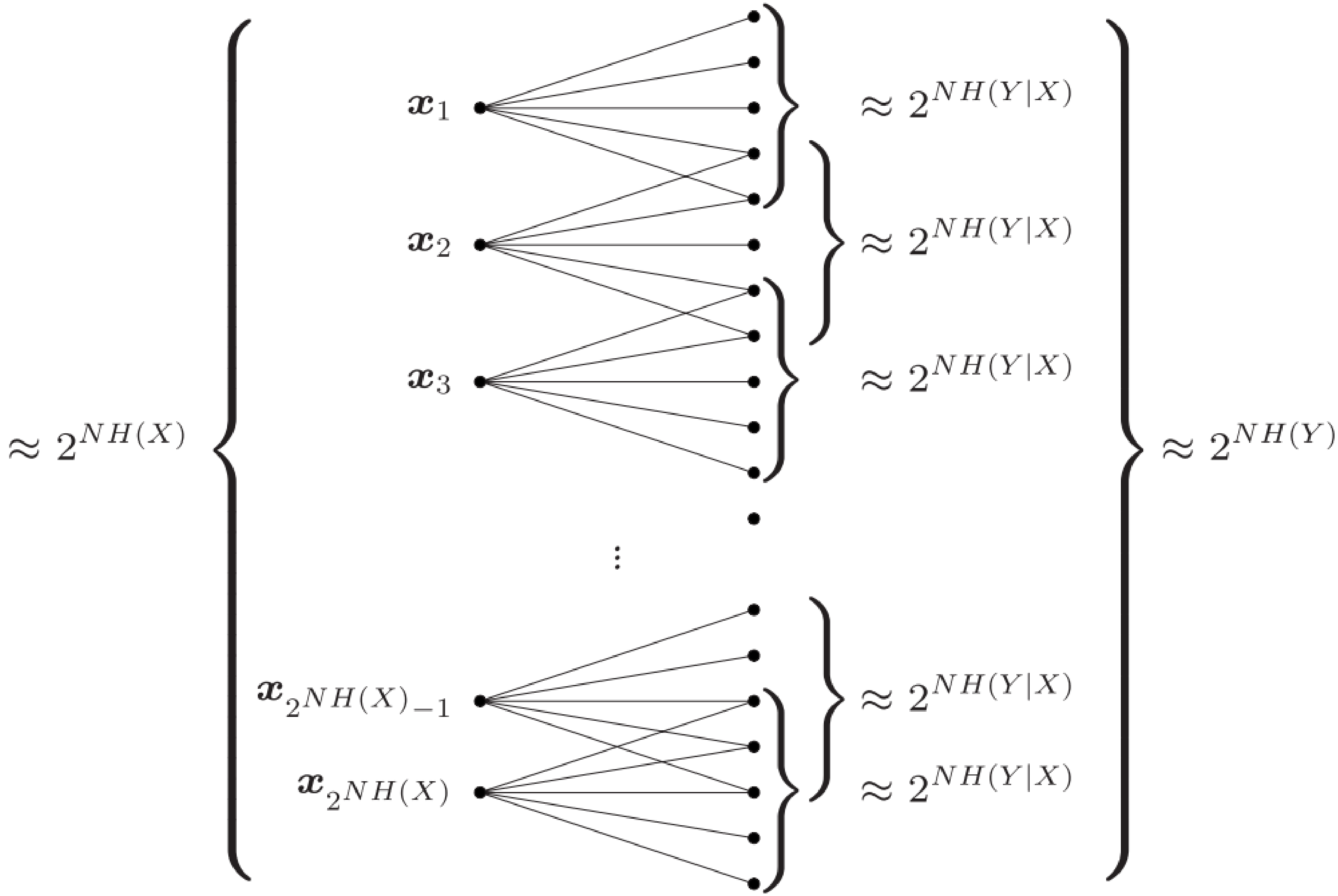

We can have at most

$$\frac{2^{NH(Y)}}{2^{NH(Y|X)}} = 2^{N(H(Y)-H(Y|X))} = 2^{NI(X;Y)}$$

non-overlapping fans

Maximum rate

Each fan can represent a message. Hence, the number of distinguishable messages, $M = 2^K = 2^{RN}$, can be at most, $2^{NI(X;Y)}$, that is $2^{RN} = 2^{NI(X;Y)}$

Equivalently, the largest value of the rate *R* for non-overlapping fans is

$$R = I(X;Y) \ \text{bits/channel use}$$

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Channel capacity

Since we would like to communicate with as high code rate *R* as possible we choose the input symbols according to the probability distribution $P_X(x)$ that maximizes the mutual information *I(X;Y)*. This maximum value is called the *capacity* of the channel,

$$C \stackrel{\text{def}}{=} \max_{P_X(x)} \{I(X;Y)\} \text{ bits/channel use}$$

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Channel capacity

Let the encoder map the messages to the typical long input sequences that represent non-overlappling fans, which requires that the code rate R is at most equal to the capacity of the channel, that is,

$$R \leq C$$

Then the received typical long output sequence is used to identify the corresponding fan and, hence, the corresponding typical long input sequence, or, equivalently, the message, and this can be done correctly with a probability arbitrarily close to 1.

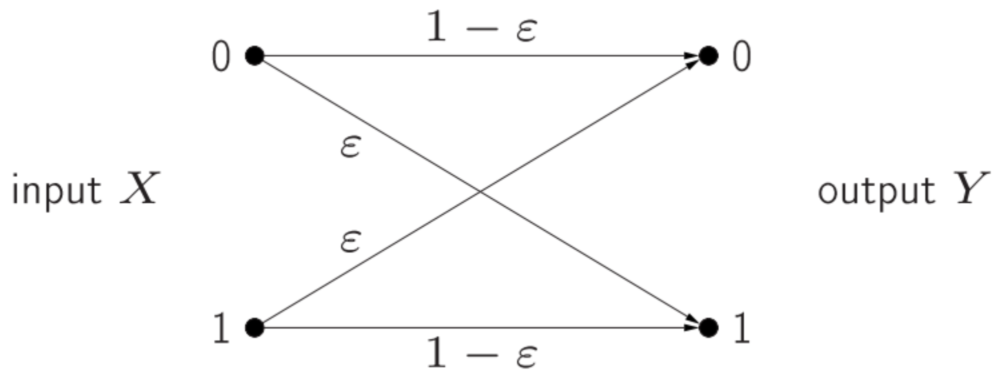
Channel coding theorem

Suppose we transmit information symbols at rate *R=K/N* bits per channel using a block code via a channel with capacity *C*.

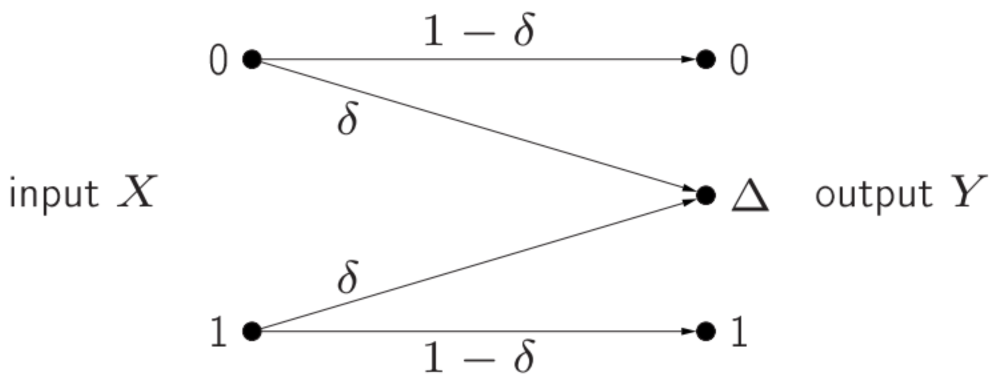
Provided that *R<C* we can achieve arbitrary reliability, that is, we can transmit the symbols virtually error-free, by choosing *N* sufficiently large. Conversely, if *R>C*, then significant distortion must occur.

Binary symmetric channel
Binary erasure channel

BSC



BEC



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Channel capacity of the BSC

$$
\begin{aligned}
C_{BSC} &= \max\{I(X;Y)\} \\
&= \max\{(H(Y) - H(Y \mid X))\} \\
&= 1 - \left(- \sum_{i=1}^{2} \sum_{j=1}^{2} P_{XY}(x_i, y_j) \log P_{Y|X}(y_j \mid x_i) \right) \\
&= 1 + \sum_{i=1}^{2} \sum_{j=1}^{2} P_{Y|X}(y_j \mid x_i) P_X(x_i) \log P_{Y|X}(y_j \mid x_i) \\
&= 1 + 2(1-\varepsilon)\frac{1}{2}\log(1-\varepsilon) + 2\varepsilon\frac{1}{2}\log\varepsilon \\
&= 1 + (1-\varepsilon)\log(1-\varepsilon) + \varepsilon\log\varepsilon \\
&= 1 - h(\varepsilon) \ \text{ bits/channel use}
\end{aligned}
$$

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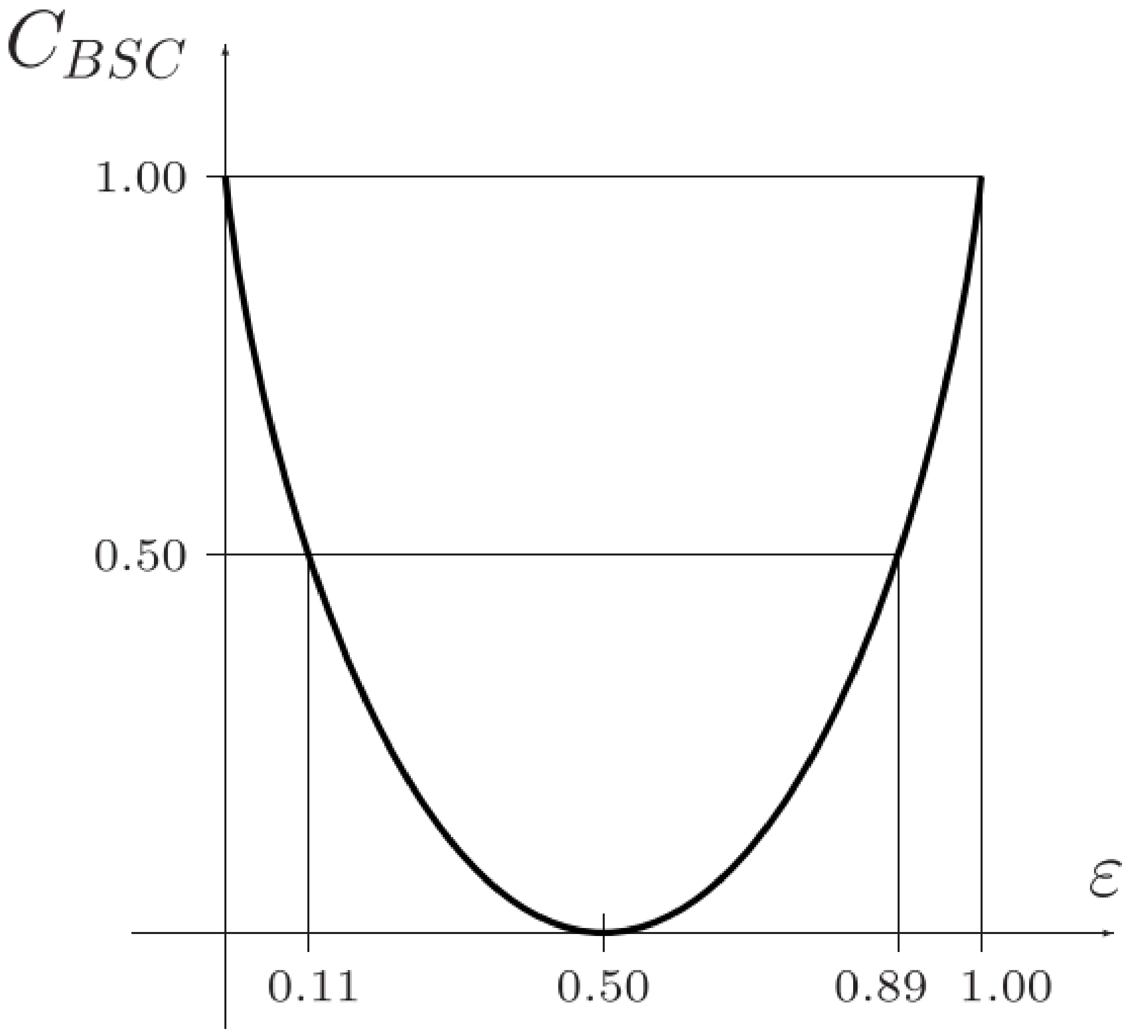
Channel capacity for the BSC



The Gaussian channel

So far we have considered only channels with binary inputs. Now we shall introduce the time-discrete Gaussian channel whose output Y_i at time i is the sum of the input X_i and the noise Z_i

$$Y_i = X_i + Z_i$$

where X_i and Y_i are real numbers and Z_i is a Gaussian random variable with mean 0 and variance $N_0/2$.

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Capacity of the Gaussian channel

A natural limitation on the inputs is an average energy constraint; assuming a codeword of N symbols being transmitted, we require that $x = x_1 \, x_2 \, \ldots \, x_N$

$$\frac{1}{N} \sum_{i=1}^{N} x_i^2 \leq E$$

where *E* is the signaling energy per symbol.

It can be shown that the capacity of a Gaussian channel with energy constraint *E* and noise variance $N_0/2$ is

$$C = \frac{1}{2} \log \left(1 + \frac{2E}{N_0} \right) \quad \text{bits/channel use}$$

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Capacity of band limited Gaussian channel

The channel capacity of the bandwidth limited Gaussian channel with two-sided noise spectral density $N_0/2$

$$C_t^W = W \log\left(1 + \frac{P_s}{N_0 W}\right) \text{ bits/s}$$

where W denotes the bandwidth in Hz and P_s is the signaling power in Watts.

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Shannon's channel coding theorem

In any system that provides reliable communication over a Gaussian channel the signal-to-noise ratio E_b/N_0 must exceed the Shannon limit, -1.6 dB!

So long as $E_b/N_0 > -1.6$ dB, Shannon's channel coding theorem guarantees the existence of a system---although it might be very complex---for reliable communication over the channel.

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