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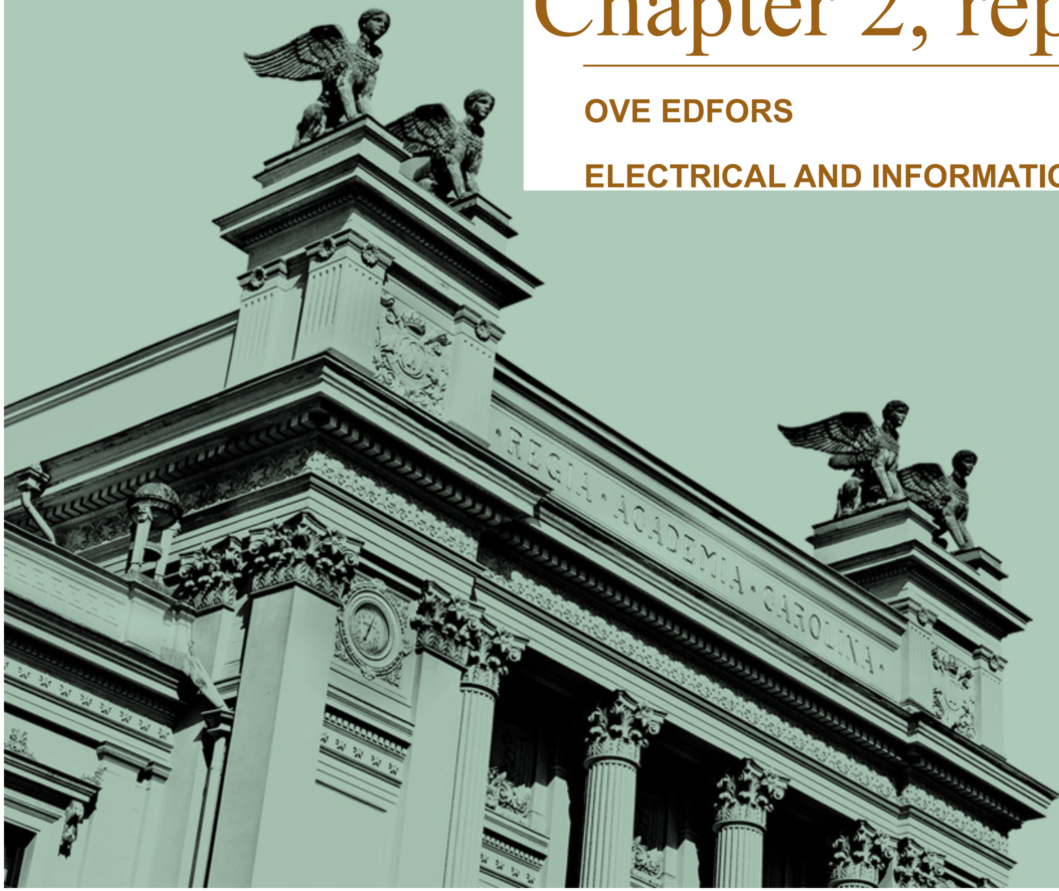
Information Transmission Chapter 2, repetition

OVE EDFORS

ELECTRICAL AND INFORMATION TECHNOLOGY



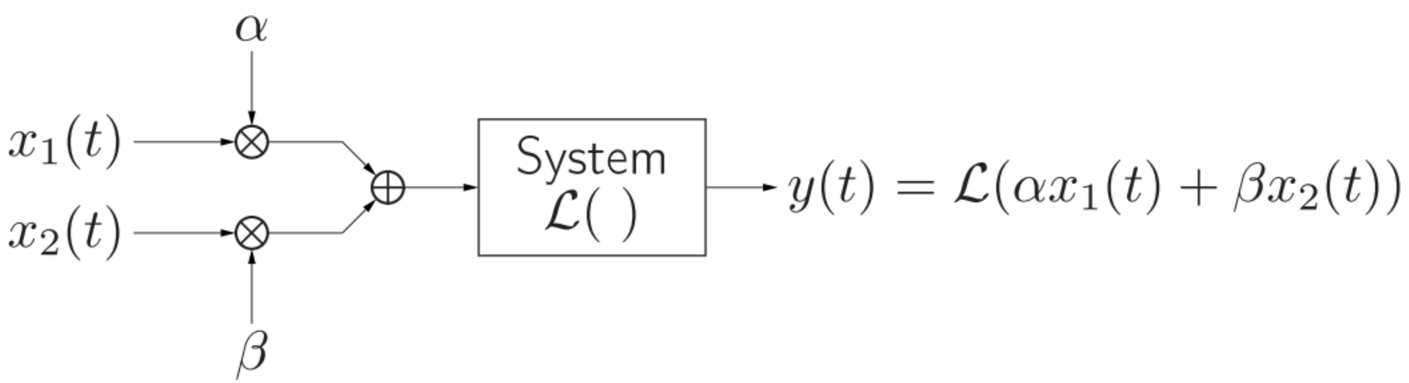
Linear, time-invariant (LTI) systems

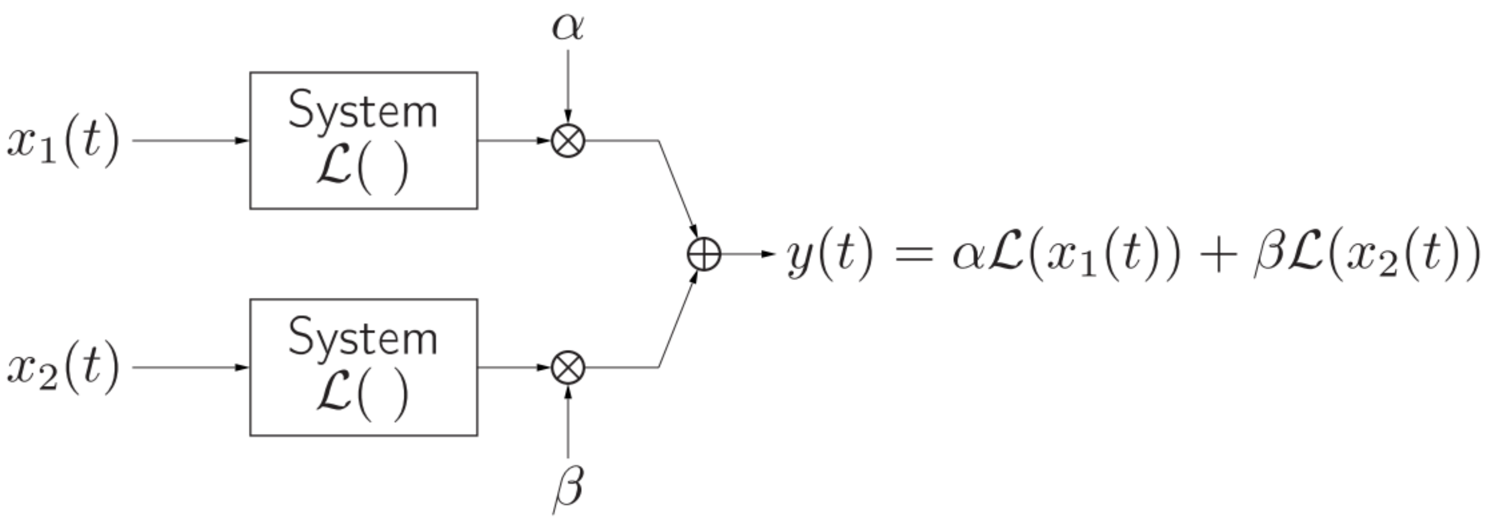
A system $\mathcal{L}$ is said to be *linear* if,
whenever an input $x_1(t)$ yields an output $\mathcal{L}(x_1(t))$
and an input $x_2(t)$ yields an output $\mathcal{L}(x_2(t))$,

We also have $L(\alpha x_1(t) + \beta x_2(t)) = \alpha L(x_1(t)) + \beta \mathcal{L}(x_2(t))$

where α, β are arbitrary real or complex constants.

What does this mean?





$$L(\alpha x_1(t) + \beta x_2(t)) = \alpha L(x_1(t)) + \beta \mathcal{L}(x_2(t))$$

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What does this mean

- Input zero results in output zero for all linear systems!

Superposition:

- the output resulting from an input that is a weighted sum of signals

is the same as

- the weighted sum of the outputs obtained when the input signals are acting separately.

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The delta function

When the duration of our pulse approaches 0, the pulse approaches the *delta function* $\delta(t)$

(also called *Dirac's delta function* or, the *unit impulse*)

$$\delta(t) = 0, \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

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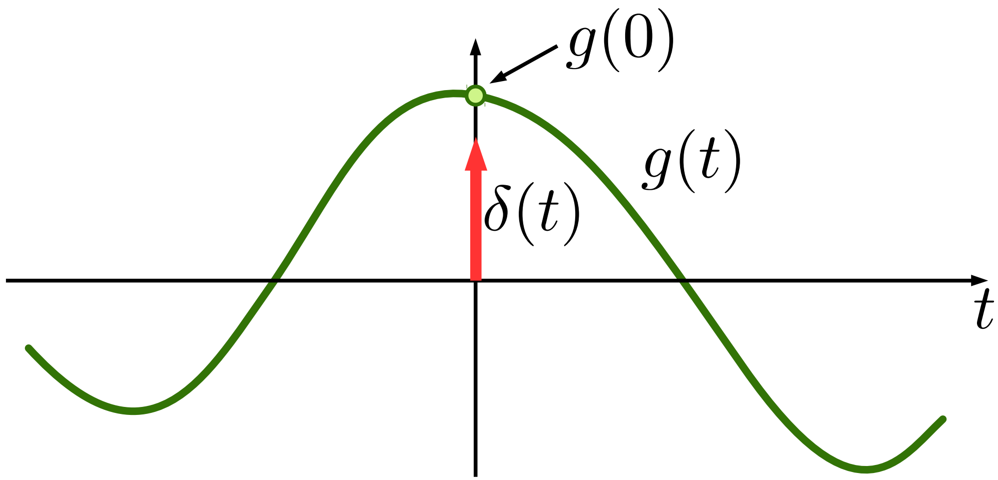
Properties of the delta function

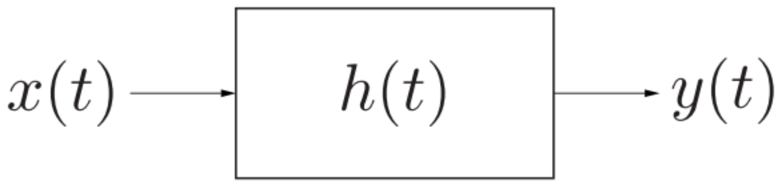
The delta function $\delta(t)$ is defined by the property

$$\int_{-\infty}^{\infty} \delta(t)g(t)dt = g(0)$$

where *g(t)* is an arbitrary function, continuous at the origin.



The output of LTI systems



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

The integral is called convolution and is denoted

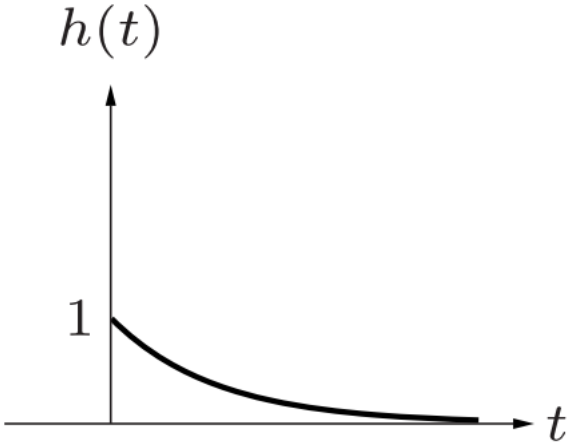
$$y(t) = x(t) * h(t) = h(t) * x(t)$$

The output *y(t)* of a linear, time-invariant system is the convolutional of its input *x(t)* and impulse response *h(t)*.

Example 3

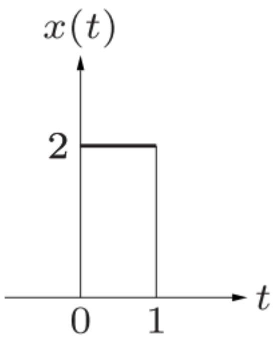
Consider an LTI system with impulse response

$$h(t) = e^{-t}u(t)$$

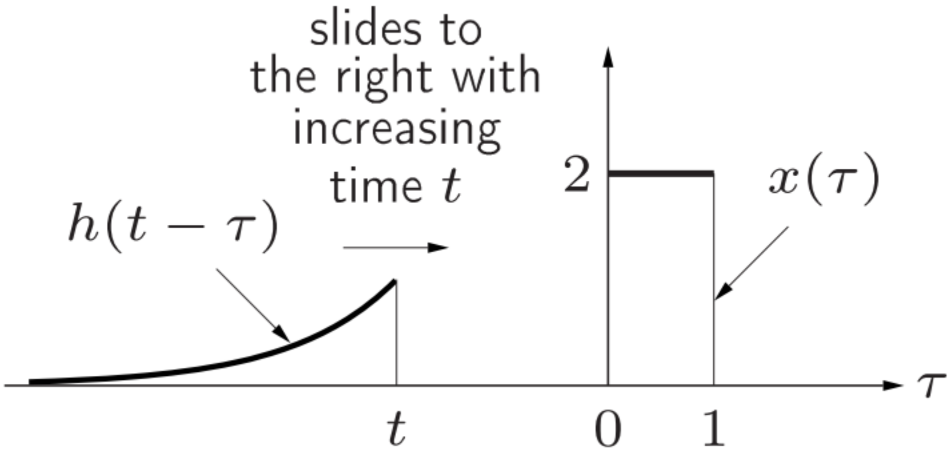


and input

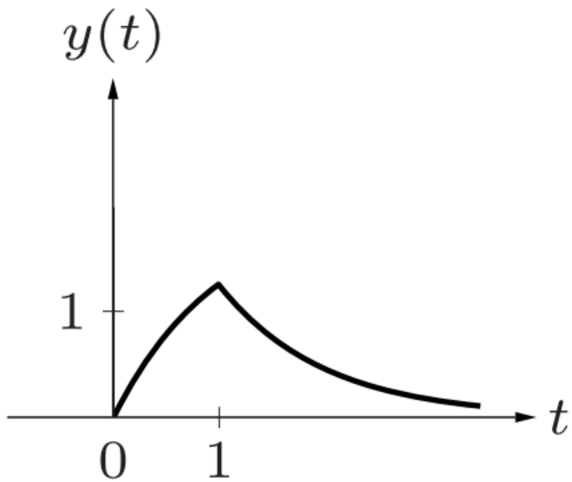
$$x(t) = \begin{cases} 2, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$



What is the output?



$$y(t) = \begin{cases} 0, & t < 0 \\ 2(1 - e^{-t}), & 0 \le t < 1 \\ 2(e-1)e^{-t}, & t \ge 1 \end{cases}$$



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Euler's formula

In school we all learned about complex numbers and in particular about Euler's remarkable formula for the complex exponential

$$e^{j\phi} = \cos\phi + j\sin\phi$$

Where $j = \sqrt{-1}$

$\cos\phi$ is the real part $\Re\{e^{j\phi}\}$, of $e^{j\phi}$

$\sin\phi$ is the imaginary part $\Im\{e^{j\phi}\}$, of $e^{j\phi}$

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The transfer function

The transfer function

$$H(f_0) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau) e^{-j2\pi f_0 \tau} d\tau$$

is called the *frequency function* or the *transfer function* for the LTI system with impulse response *h(t)*.

Phase and amplitude functions

The frequency function is in general a complex function of the frequency:

$$H(f) = A(f)e^{j\phi(f)}$$

where

$$A(f) = |H(f)|$$

is called the *amplitude function* and

$$\phi(f) = \arctan \frac{\Im\{H(f)\}}{\Re\{H(f)\}}$$

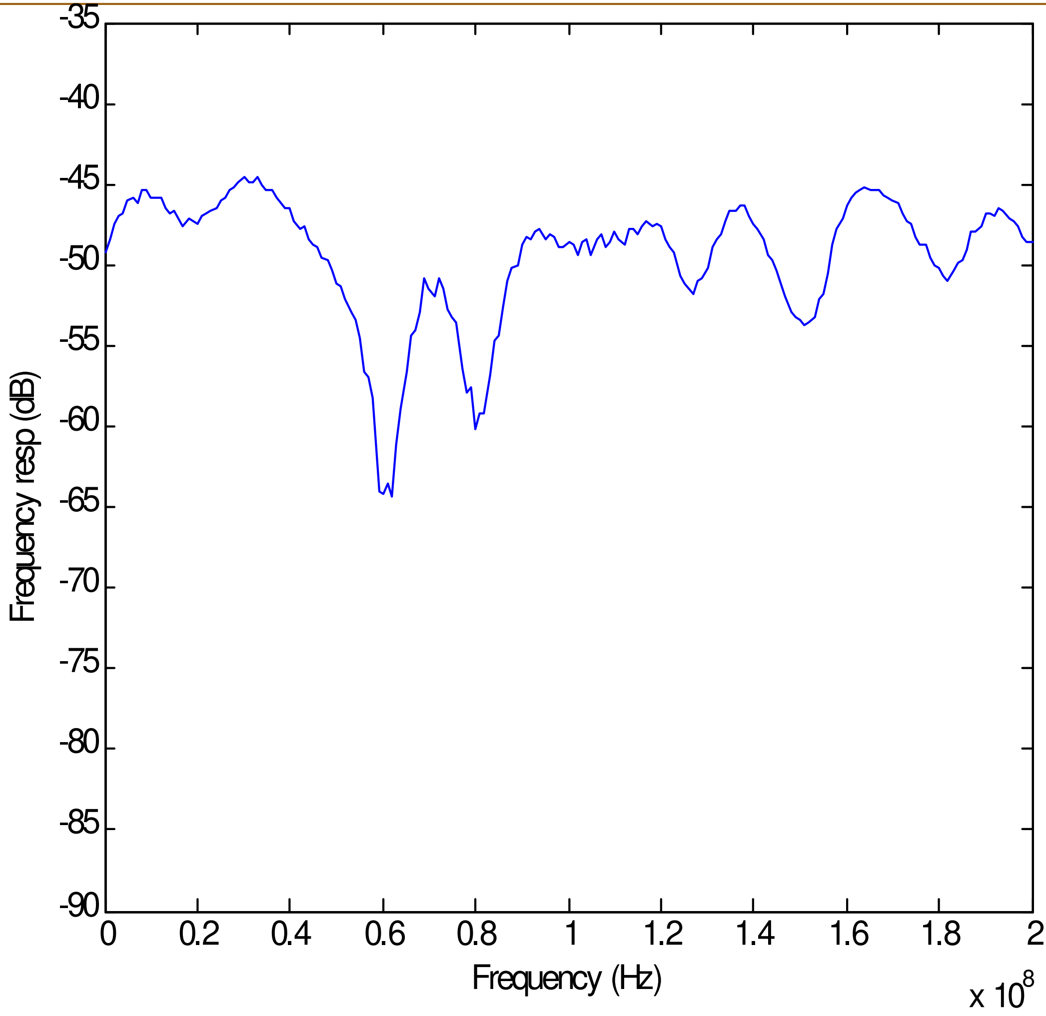
is called the *phase function*.

An example with a real measured radio channel

Measurement example, the radio channel

- Measurement in the lab
- Center frequency 3.2 GHz
- Measurement bandwidth 200 MHz, 201 frequency points
- 60 measurement positions, spaced 1 cm apart
- Measured with a vector network analyzer

Transfer function



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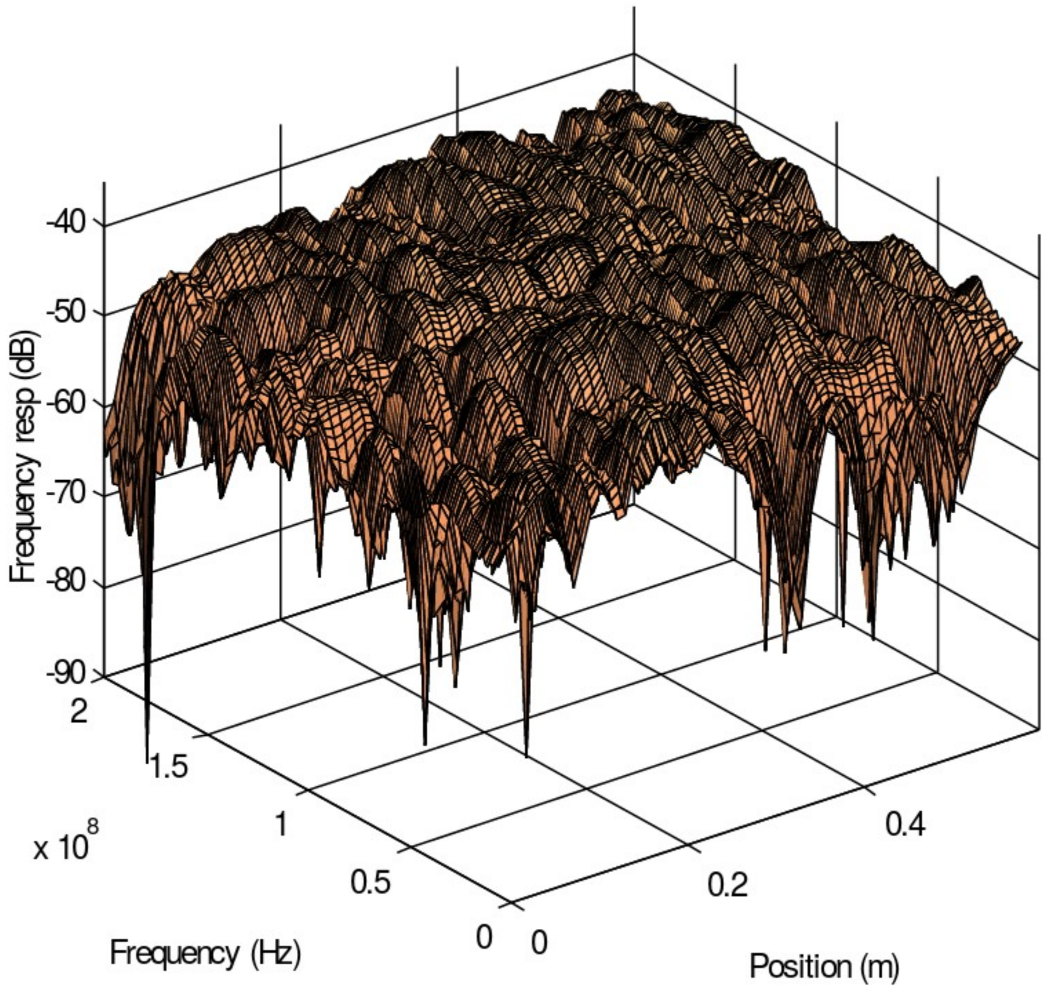
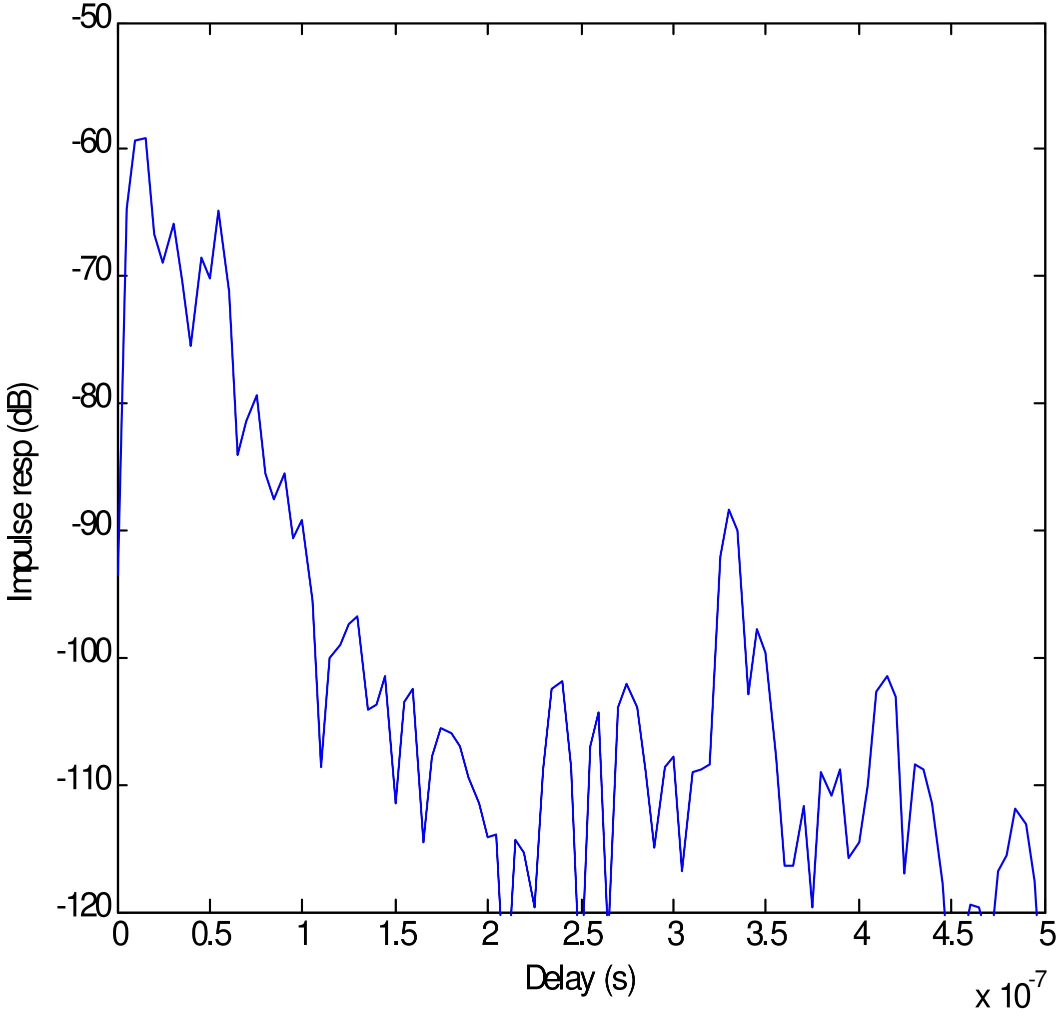
Transfer function, all positions

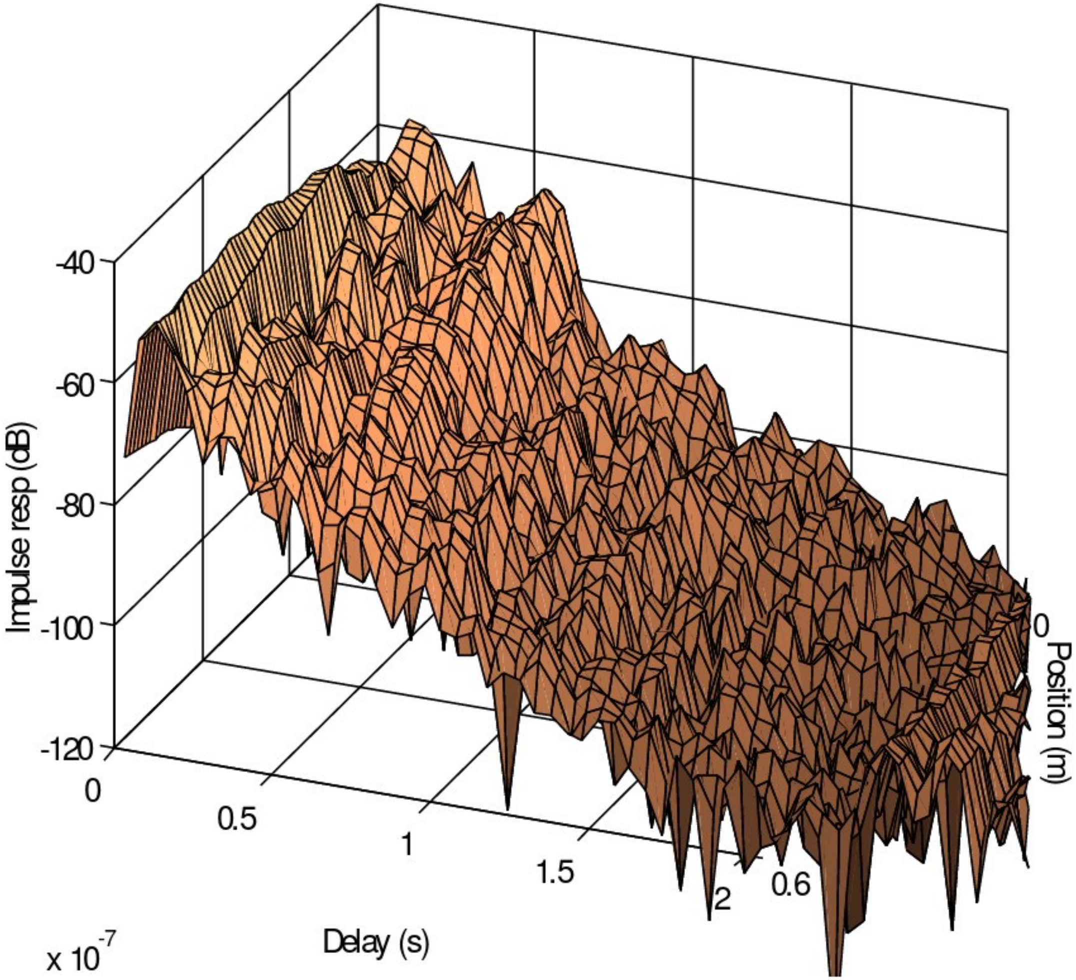


Impulse response



What are the delays?
How is the signal affected for different delays?
How does it change with time?

Impulse response, all positions



The Fourier transform

The Fourier transform

The Fourier transform of the signal *x(t)* is given by the formula

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

This function is in general complex:

$$X(f) = A(f)e^{j\phi(f)}$$

where $A(f) = |X(f)|$ is called the spectrum of *x(t)*

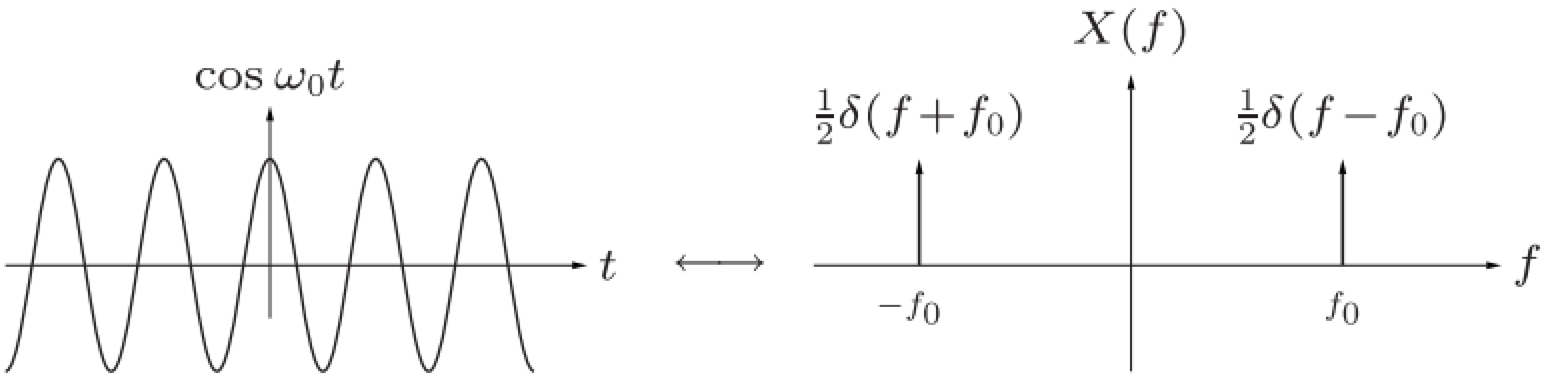
and $\phi(f)$ its phase angle.

Spectrum of a cosine

$$X(f) = \int_{-\infty}^{\infty} \cos \omega_0 t \; e^{-j\omega_0 t} dt = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

Hence we have a Fourier transform pair

$$\cos \omega_0 t \leftrightarrow \frac{1}{2}\delta(f - f_0) + \frac{1}{2}(f + f_0)$$



Properties of the Fourier transform

1. Linearity

$$ax_1(t) + bx_2(t) \leftrightarrow aX_1(f) + bX_2(f)$$

2. Inverse

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j\omega t} df$$

3. Translation (time shifting)

$$x(t - t_0) \leftrightarrow X(f)\, e^{-j\omega t_0}$$

4. Modulation (frequency shifting)

$$x(t)\, e^{j\omega_0 t} \leftrightarrow X(f - f_0)$$

Properties of the Fourier transform

5. Time scaling

$$x(at) \leftrightarrow \frac{1}{|a|} X(f/a)$$

6. Differentiation in the time domain

$$\frac{d}{dt} x(t) \leftrightarrow j\omega X(f)$$

7. Integration in the time domain

$$\int_{-\infty}^{t} x(\tau) d\tau \leftrightarrow \frac{1}{j\omega} X(f)$$

8. Duality

$$X(t) \leftrightarrow x(-f)$$

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Properties of the Fourier transform

9. Conjugate functions

$$x^*(t) \leftrightarrow X^*(-f)$$

10. Convolution in the time domain

$$x_1(t) * x_2(t) \leftrightarrow X_1(f)X_2(f)$$

11. Multiplication in the time domain

$$x_1(t)x_2(t) \leftrightarrow X_1(f) * X_2(f)$$

12. Parseval's formulas

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = \int_{-\infty}^{\infty} X_1(f)X_2^*(f)df$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

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Fourier transform of a convolution

Since the output *y(t)* of an LTI system is the convolution of its input *x(t)* and impulse response h(t) it follows from Property 10 (Convolution in the time domain) that the Fourier transform of its output *Y(f)* is simply the product of the Fourier transform of its input *X(f)* and its frequency function *H(f),* that is,

$$Y(f) = X(f)H(f) = H(f)X(f)$$

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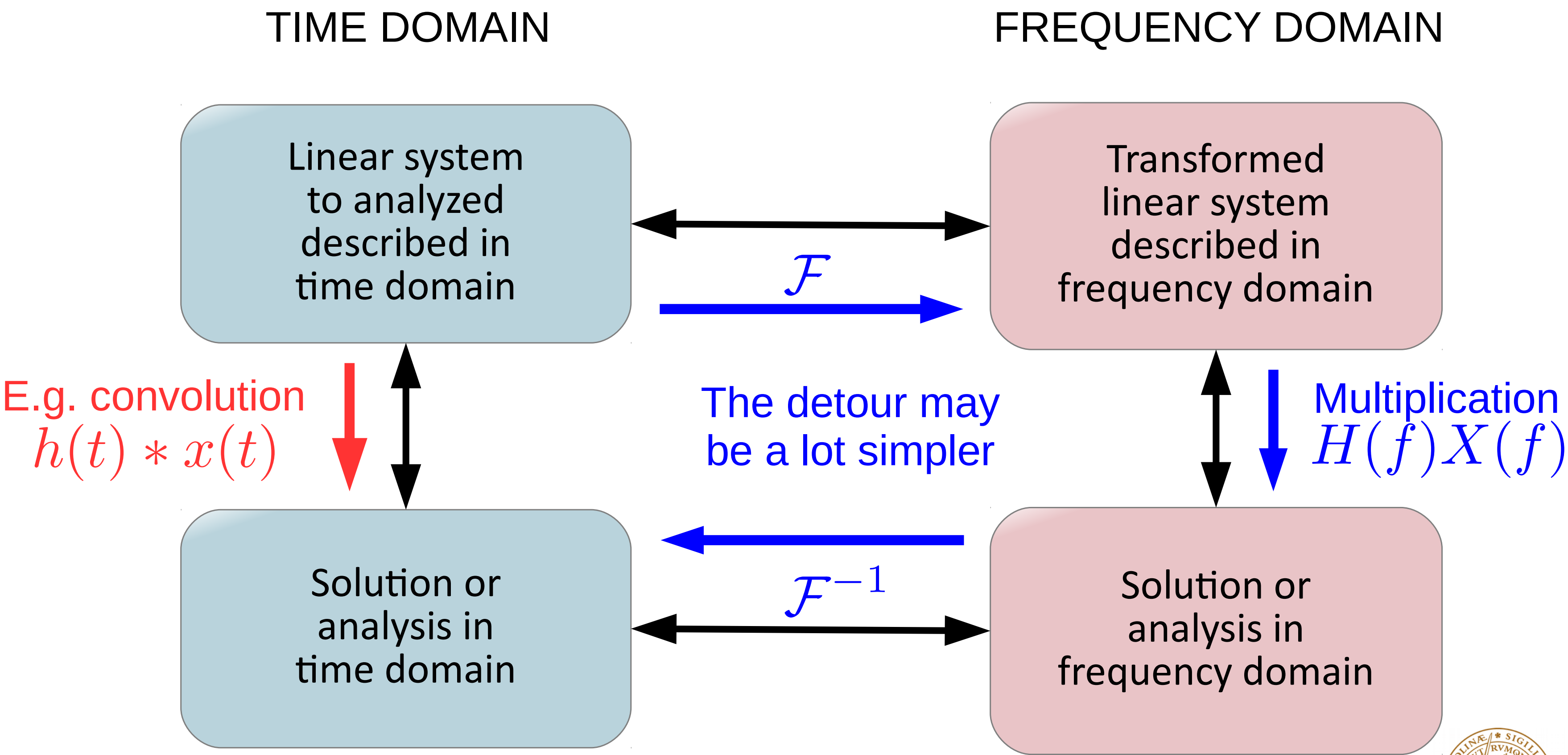
Why use Fourier transforms?



Some useful Fourier transform pairs

(a) *Impulse in the time domain*

$$\delta(t) \leftrightarrow 1$$

(b) *Impulse in the frequency domain*

$$1 \leftrightarrow \delta(f)$$

(c) *Sign function*

$$sgn(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases} \quad \leftrightarrow \quad \frac{2}{j\omega}$$

(d) *Unit step function*

$$u(t) = \begin{cases} 1, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0 \end{cases} \quad \leftrightarrow \quad \frac{1}{j\omega} + \frac{1}{2}\delta(f)$$

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Some useful Fourier transform pairs

(e) *Complex exponential*

$$e^{j\omega_0 t} \leftrightarrow \delta(f - f_0)$$

(f) *Cosine function*

$$\cos \omega_0 t \leftrightarrow \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

(g) *Sine function*

$$\sin \omega_0 t \leftrightarrow \frac{1}{2j}\delta(f - f_0) + \frac{1}{2j}\delta(f + f_0)$$

(h) *Rectangular pulse*

$$rect(t) \leftrightarrow sinc(f) = \frac{\sin \pi f}{\pi f}$$

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