Chapter 2: Linear systems & sinusoids

Ove Edfors
Dept. of EIT, Lund University

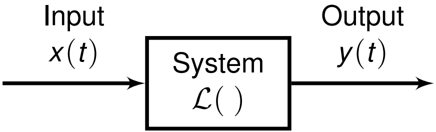
Learning outcomes

After this lecture, the student should

- understand what a linear system is, including linearity conditions, their consequences, and the superposition principle,
- understand what time-invariance is, and its connection to linear systems – linear and time-invariant (LTI) systems,
- understand what impulses and step functions are and how they are defined,
- have a basic understanding of what generalized functions are,
- understand what the impulse response of an LTI system is and how it can be used to calculate output signals,
- be able to calculate the output signal of an LTI system, when given an input signal and the system's impulse response,
- understand causality and why it is an important property of real, physical, systems, and
- be able to find out if a certain given LTI system is causal or not.

Linear systems, what's that?

Linear systems – Definition



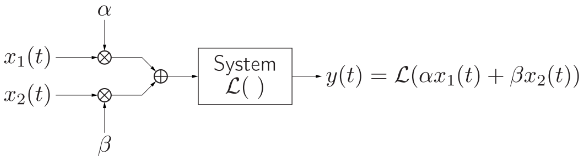
Definition (Linear system)

A system $\mathcal{L}$ is said to be **linear** if, whenever an input $x(t) = x_1(t)$ yields an output $y(t) = \mathcal{L}(x_1(t))$ and an input $x(t) = x_2(t)$ yields an output $y(t) = \mathcal{L}(x_2(t))$ this also leads to that we have $\mathcal{L}(\alpha x_1(t) + \beta x_2(t)) = \alpha\mathcal{L}(x_1(t)) + \beta\mathcal{L}(x_2(t))$, where α and β are real or complex valued constants.

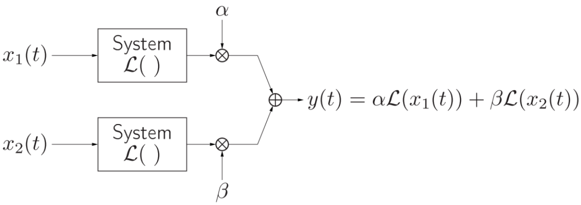
Short version for memory: Weighted sum of inputs leads to same weighted sum of outputs.

Linear systems – Property illustration

This system with two superpositioned input signals



is **equivalent** to these two parallel systems, each with its own input, but outputs superpositioned the same way (same α and β)



The linearity condition – infinite sums

In order to extend the linearity condition

$$\mathcal{L}\left(\alpha x_1(t) + \beta x_2(t)\right) = \alpha \mathcal{L}\left(x_1(t)\right) + \beta \mathcal{L}\left(x_2(t)\right)$$

to infinite sums (integrals – think Riemann sum) we must assume that the system $\mathcal{L}$ is "sufficiently smooth or continuous".

In other words: "Small" changes in input must give "small" changes in output.

The linearity condition – equivalent formulation

The linearity condition

$$\mathcal{L}(\alpha x_1(t) + \beta x_2(t)) = \alpha \mathcal{L}(x_1(t)) + \beta \mathcal{L}(x_2(t))$$

can be replaced by two simpler conditions

$$\mathcal{L}(x_1(t) + x_2(t)) = \mathcal{L}(x_1(t)) + \mathcal{L}(x_2(t)) \quad \text{[Additivity]}$$
$$\mathcal{L}(\alpha x(t)) = \alpha \mathcal{L}(x(t)) \quad \text{[Homogeneity]}$$

Important note: Both simplified conditions must be true for the first (linearity) condition to be true!

Linear system properties

A consequence of the linearity condition

Assuming that both input $x_1(t)$ and $x_2(t)$ are zero, i.e.,

$$x_1(t) = x_2(t) = 0$$

we obtain [additivity]

$$\mathcal{L}(0 + 0) = \mathcal{L}(0) + \mathcal{L}(0),$$

which can **only** be true if $\mathcal{L}(0) = 0$.

An important consequence

Zero input results in **zero output** for ALL linear systems!

Note: This property is not true for nonlinear systems, which may e.g. "oscillate" despite a zero-input.

Superposition principle

A system is said to be linear if *the output resulting from an input that is a weighted sum of signals* is the same as *the weighted sum of the outputs obtained when the input signals are acting separately*. This means

$$\mathcal{L}(\underbrace{\alpha x_1(t) + \beta x_2(t)}_{\text{Weighted sum of input signals}}) = \underbrace{\alpha \mathcal{L}(x_1(t)) + \beta \mathcal{L}(x_2(t))}_{\text{Corresponding weighted sum of output signals}}$$

In other words: Per definition, linear systems obey the superposition principle. (See definition of *linear system*.)

LUND UNIVERSITY

Linear and time invariant systems

Time invariance

A system $\mathcal{I}$ is said to be *time-invariant* if a delay of the input by any amount of time only causes the same delay of the output. More precisely:

Definition (Time invariance)

A system $\mathcal{I}$ with output

$$y(t) = \mathcal{I}(x(t))$$

is time-invariant if

$$y(t-\tau) = \mathcal{I}(x(t-\tau))$$

holds for all delays (shifts) τ.

What it means: The system as such does not change its behavior with time. Its components do not "grow old".

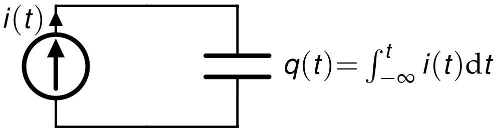
Linear and time-invariant (LTI) systems

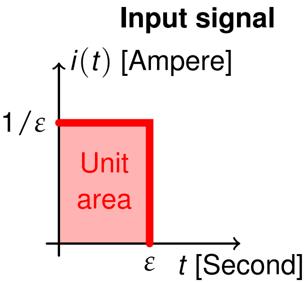
A system that is both **linear** and **time-invariant** is called an **LTI system**.

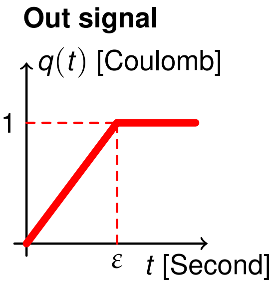
LTI systems are an important class of systems which have many applications in real life!

A circuit example: Impulses and step functions

Consider the output when charging a capacitor with a current pulse:

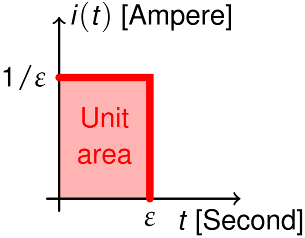


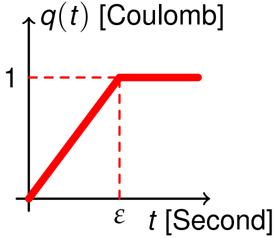




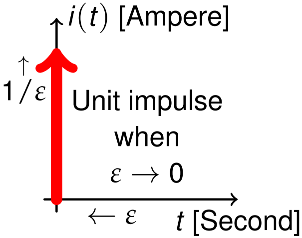
LUND UNIVERSITY

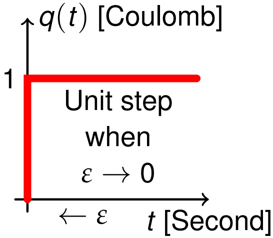
A circuit example: Impulses and step functions





Now, let the current pulse become shorter and shorter, approaching zero-length in time.





LUND UNIVERSITY

Impulses, steps and impulse responses

The delta/unit-impulse function

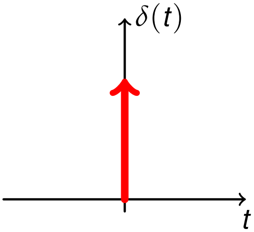
When the length of our pulse (with unit area) in the circuit example approaches zero, the pulse approaches the *delta-function* $\delta(t)$ – also called the *unit impulse* or *Dirac's delta-function*[1].

Definition (Delta function)

The delta function $\delta(t)$ (unit impulse) is a function that has the following two properties:

$$\delta(t) = 0 \text{ for all } t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1$$



This is a very important function. We will use it throughout the course!

[1] Named after Paul Dirac (1902-1984), one of the founders of quantum mechanics.

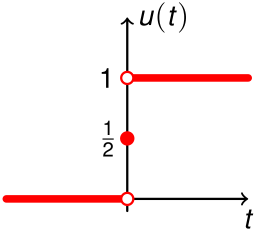
The unit-step function

If a delta/unit-impulse function acts as input in our circuit example, the charge in the capacitor will rise immediately from 0 to 1 Coulomb, at $t = 0$, and stay there forever. We say that when the pulse length on the input approaches zero, the output (charge) approaches the *unit-step function* $u(t)$ – also called Heaviside's step-function[2].

Definition (Unit-step function)

The unit-step function $u(t)$ is defined as

$$u(t) = \begin{cases} 0 & , \quad t < 0 \\ 1/2 & , \quad t = 0 \\ 1 & , \quad t > 0 \end{cases}$$



This is another very important function. We will use it throughout the course!

[2] Named after Oliver Heaviside (1850–1925), a famous English mathematician.

Generalized functions and LTI systems

Generalized functions/distributions

The delta function is not a function in the ordinary sense – it a *generalized function* or a *distribution*. These need to be treated in a special way.

The derivative of a generalized function/distribution is indirectly defined:

Definition (Derivative of generalized function/distribution)

The derivative $g'(t)$ of a generalized function/distribution $g(t)$ is given by

$$\int_{-\infty}^{\infty} g'(t)\varphi(t)\mathrm{d}t \stackrel{\text{def}}{=} -\int_{-\infty}^{\infty} g(t)\varphi'(t)\mathrm{d}t$$

where $\varphi(t)$ is a test function which is *infinitely differentiable* (smooth) and has *compact support* (zero outside some bounded interval).

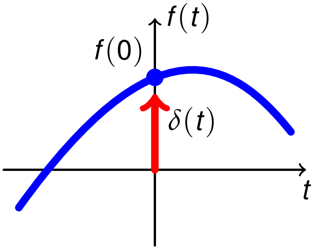
Important property of the delta function

Using the definition of the delta function $\delta(t)$ we can conclude that

$$\int_{-\infty}^{\infty} \delta(t) f(t) \mathrm{d}t = f(0)$$

if $f(t)$ is a function which is continuous at $t = 0$.

We can see this as the above integral *sampling* (picking up the value of) the function $f(t)$ at $t = 0$.



QUIZ: Sampling at a certain time

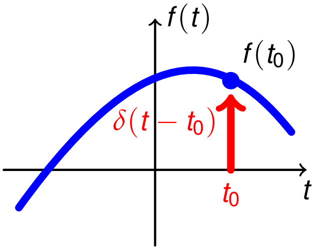
As we have seen, this expression

$$\int_{-\infty}^{\infty} \delta(t) f(t) \mathrm{d}t = f(0)$$

accomplishes a sampling of $f(t)$ at $t = 0$.
What would the corresponding expression sampling $f(t)$ at an arbitrary time $t = t_0$?

Solution: By moving (delaying) the $\delta(t)$ function by t_0 units along the t-axis, we get

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t) \mathrm{d}t = f(t_0)$$



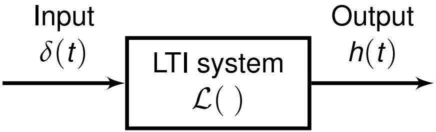
Derivatives and integrals

Between delta and step functions, $\delta(t)$ and $u(t)$, we have the following relationships (derivative defined for generalized functions):

$$u'(t) = \delta(t)$$

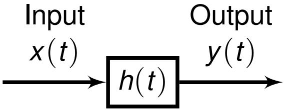
$$\int_{-\infty}^{t} \delta(t)\mathrm{d}t = u(t)$$

LTI system impulse response

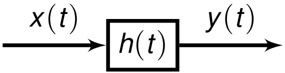


The **impulse response** $h(t)$ of an LTI system is defined as the output when an (Dirac/unit) impulse at $t = 0$ acts as input.

We often draw the block diagram of such as system:



LTI system input-output relation



The input-output relation for an LTI system with impulse response $h(t)$ is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\mathrm{d}\tau$$

where $x(t)$ is the input and $y(t)$ the corresponding output.

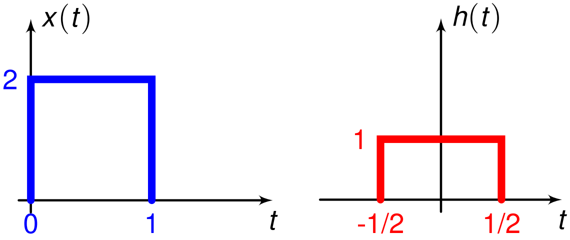
This particular integral is called a **convolution** and is *one of the most important mathematical concepts in the course*. Convolution is commutative and denoted by $*$. We write

$$\boxed{y(t) = x(t) * h(t) = h(t) * x(t).}$$

LUND UNIVERSITY

Example 1

Assume we have an LTI system with input $x(t)$ and impulse response $h(t)$, according to the figures below.



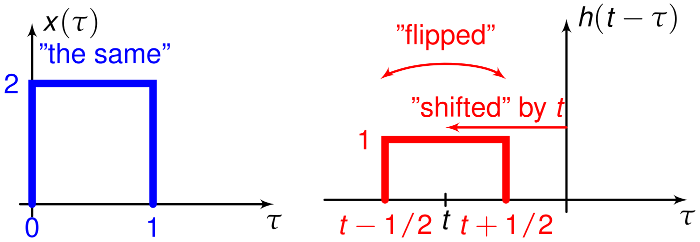
What will the output $y(t)$ be?

Solution 1

The output of an LTI system with input $x(t)$ and impulse response $h(t)$ is determined by the convolution

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau.$$

Let's start with performing the convolution by using the integral.
First, what does the factors of the integrand, $x(\tau)$ and $h(t-\tau)$, look like on the τ-axis?

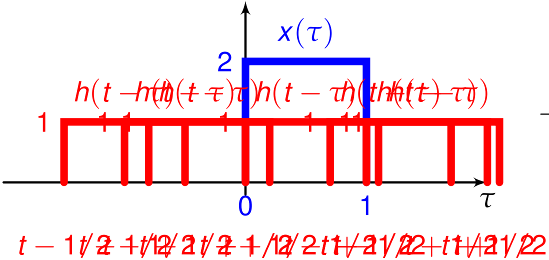


Solution 1 (cont.)

Result of

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \mathrm{d}\tau$$

Let's perform the convolution:



$t < -1/2:$

$$y(t) = 0$$

$-1/2 \leq t < 1/2:$

$$y(t) = 2(t + 1/2)$$

$1/2 \leq t < 3/2:$

$$y(t) = 2(3/2 - t)$$

$t \geq 3/2:$

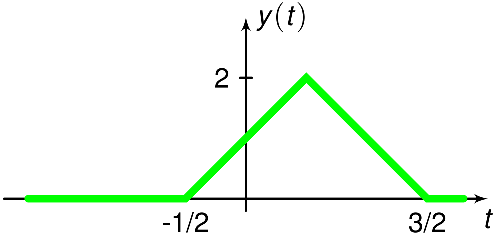
$$y(t) = 0$$

Solution 1 (cont.)

Plotting the graph of the output signal

$$y(t) = \begin{cases} 0, & t < -1/2 \\ 2(t + 1/2), & -1/2 \leq t < 1/2 \\ 2(3/2 - t), & 1/2 \leq t < 3/2 \\ 0, & t \geq 3/2 \end{cases}$$

we get



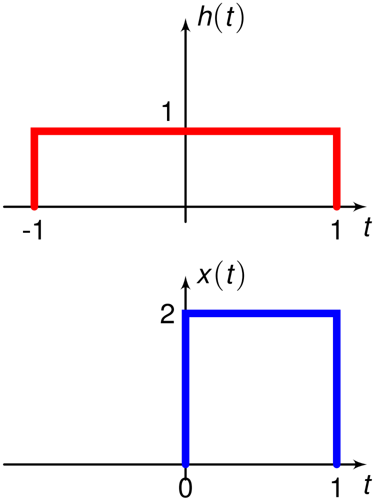
Example 2

Consider an LTI system with impulse response

$$h(t) = \begin{cases} 1, & -1 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

and input signal

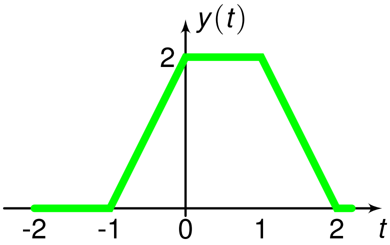
$$x(t) = \begin{cases} 2, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$



Solution 2

Following the same type of reasoning as in Example 1, we get a piece-wise linear output (since $x(t)$ and $h(t)$ are piece-wise constant), but now with four corners rather than three (since $x(t)$ and $h(t)$ are of different length):

$$y(t) = \begin{cases} 0, & t < -1 \\ 2t + 2, & -1 \leq t < 0 \\ 2, & 0 \leq t < 1 \\ -2t + 4, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases}$$



Lund University

Example 3

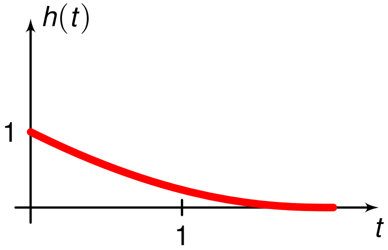
Consider an LTI system with impulse response

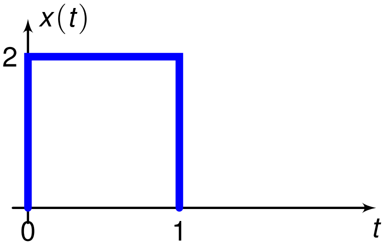
$$h(t) = e^{-t}u(t)$$

and the same input signal

$$x(t) = \begin{cases} 2, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

as in Example 2.

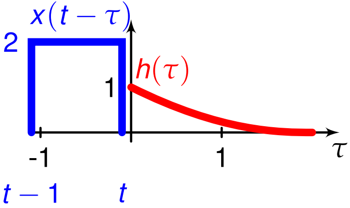




Solution 3

With the given impulse response and input signal, we get three cases when calculating the convolution $y(t) = x(t) * h(t) = \int h(\tau)x(t-\tau)\mathrm{d}\tau$:

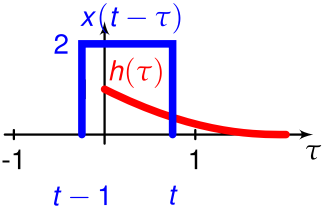
When $t < 0$:



the output becomes

$$y(t) = 0$$

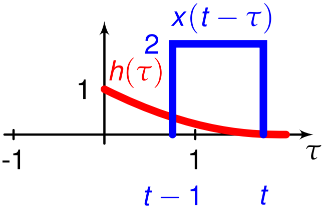
When $0 \leq t < 1$:



the output becomes

$$y(t) = \int_0^t 2e^{-\tau}\mathrm{d}\tau$$

When $t \geq 1$:



the output becomes

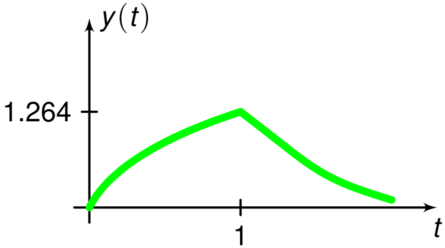
$$y(t) = \int_{t-1}^t 2e^{-\tau}\mathrm{d}\tau$$

Solution 3 (cont.)

Performing the integrations on the previous slide, we get the following output from our system:

$$y(t) = \begin{cases} 0, & t < 0 \\ 2(1 - e^{-t}), & 0 \leq t < 1 \\ 2(e-1)e^{-t}, & t \geq 1 \end{cases}$$

which looks something like this in a plot:



LUND UNIVERSITY

Causal systems

A system is said to be **causal** if the reaction always comes after the action[3], i.e., if the output $y(t_0)$ at any given time t_0 is only influenced by inputs up to this time t_0, i.e., influenced only by $x(t)$ for $t \leq t_0$.
This means that for causal systems we have

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{t} x(\tau)h(t-\tau)\mathrm{d}\tau$$

which implies that τ in the second integral never exeeds t.
For the last equality to hold, it is necessary that all causal systems have impulse responses that fulfill

$$h(t) = 0 \text{ for } t < 0.$$

Important property to remember for causal systems!

[3] All real, physical, systems have this property and what we know as speed of light, c m/s, is actually the speed of causality. If input and output are separated in space by d m, the delay is $\geq d/c$ s.

LECTURE SUMMARY

- Linear and time invariant (LTI) systems
  - Linearity
  - Superopsition and linearity condition
  - Time invariance
- Impulses and step functions
  - Dirac delta function/unit impulse and its properties
  - Step function and its properties
  - Generalized functions/distributions
  - Relation between delta function and step function
- LTI systems and impulse response
  - What is the impulse response
  - How can we use it to calculate outputs
  - Convolution
  - Causality and causal systems

LUND
UNIVERSITY