Information Transmission
Chapter 2, repetition

OVE EDFORS
ELECTRICAL AND INFORMATION TECHNOLOGY

LINEAR AND TIME-INVARIANT SYSTEMS

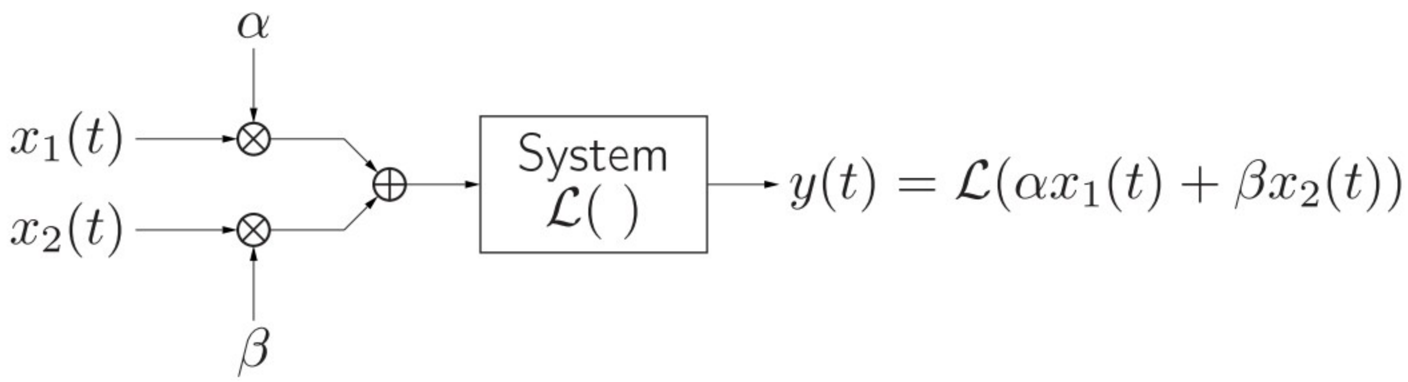
Linear, time-invariant (LTI) systems

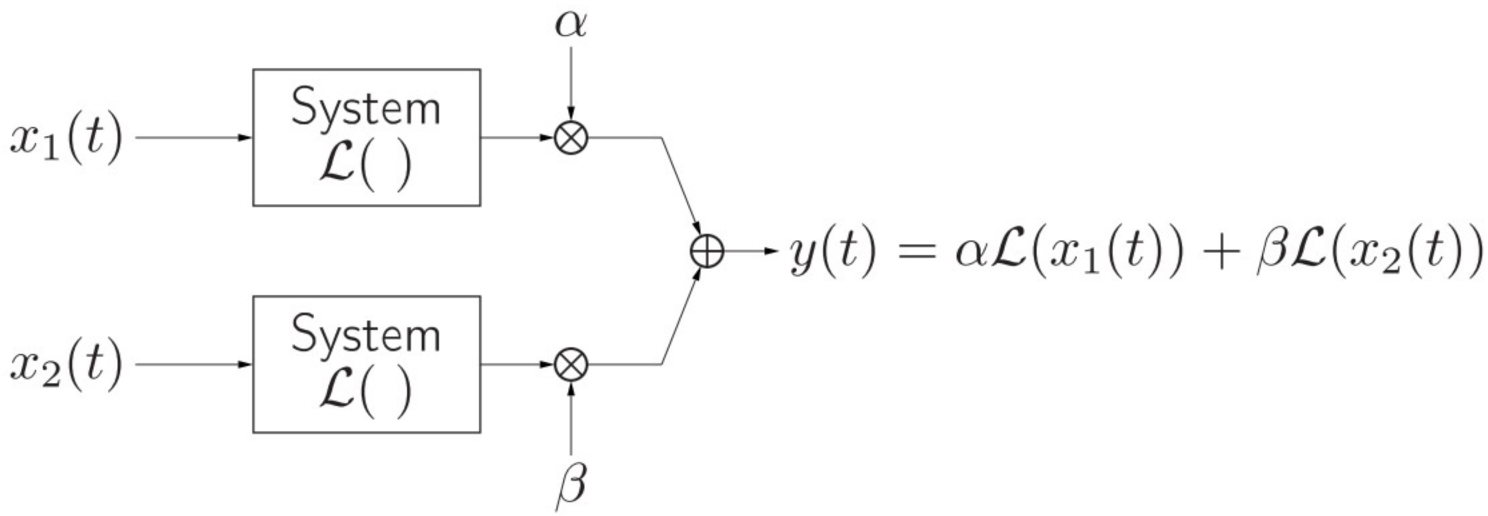
A system $\mathcal{L}$ is said to be *linear* if,
whenever an input $x_1(t)$ yields an output $\mathcal{L}(x_1(t))$
and an input $x_2(t)$ yields an output $\mathcal{L}(x_2(t))$,

We also have $L(\alpha x_1(t) + \beta x_2(t)) = \alpha L(x_1(t)) + \beta \mathcal{L}(x_2(t))$

where α, β are arbitrary real or complex constants.

What does this mean?





$$L(\alpha x_1(t) + \beta x_2(t)) = \alpha L(x_1(t)) + \beta\mathcal{L}(x_2(t))$$

What does this mean?

- Input zero results in output zero for all linear systems!

Superposition:

- the output resulting from an input that is a weighted sum of signals

is the same as

- the weighted sum of the outputs obtained when the input signals are acting separately.

The delta function

When the duration a pulse with “unit area” approaches 0, the pulse approaches the *delta function* $\delta(t)$

(also called *Dirac's delta function* or, the *unit impulse*)

$$\boxed{\begin{gathered}\delta(t) = 0, \quad t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)dt = 1\end{gathered}}$$

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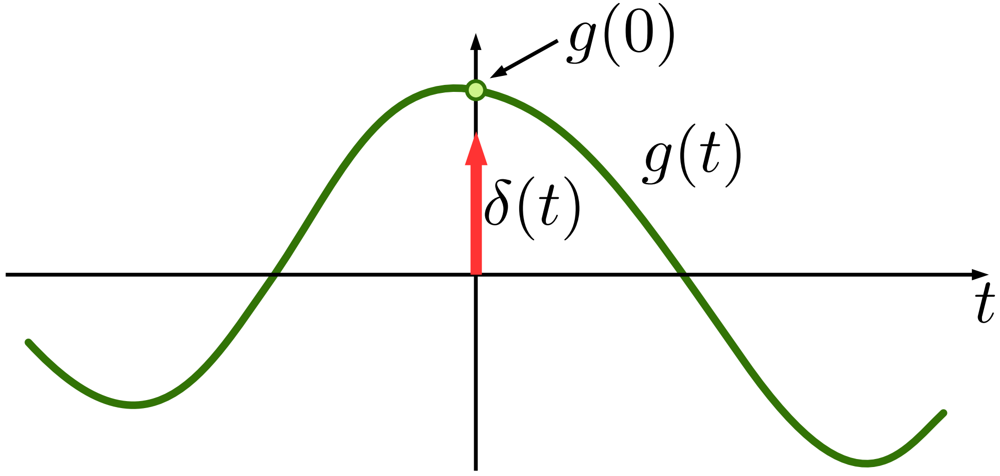
Properties of the delta function

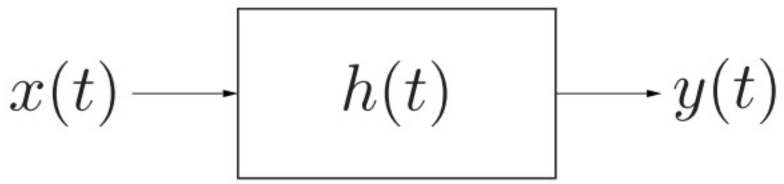
The delta function $\delta(t)$ is defined by the property

$$\int_{-\infty}^{\infty} \delta(t) g(t) dt = g(0)$$

where $g(t)$ is an arbitrary function, continuous at the origin.



The output of LTI systems



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

The integral is called convolution and is denoted

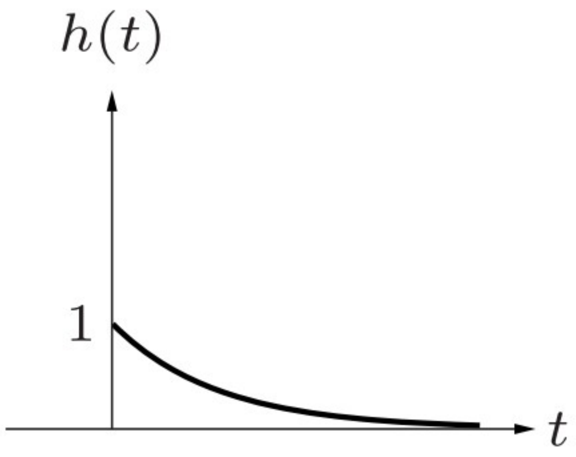
$$y(t) = x(t) * h(t) = h(t) * x(t)$$

The output $y(t)$ of a linear, time-invariant system is the convolution of its input $x(t)$ and impulse response $h(t)$.

Example 3

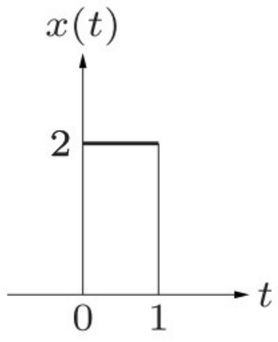
Consider an LTI system with impulse response

$$h(t) = e^{-t}u(t)$$



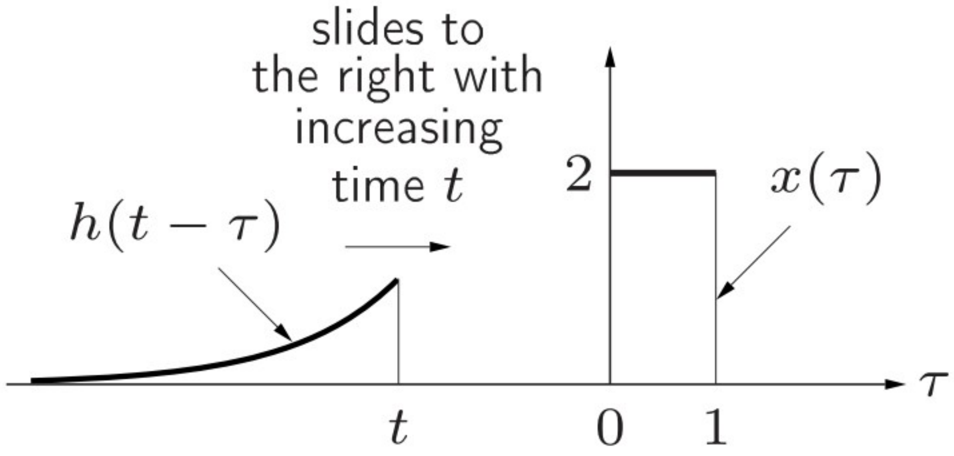
and input

$$x(t) = \begin{cases} 2, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

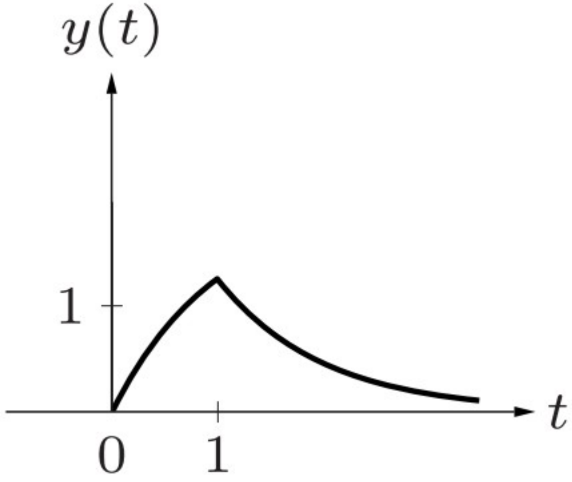


What is the output?

Example 3 (cont.)



$$y(t) = \begin{cases} 0, & t < 0 \\ 2(1-e^{-t}), & 0 \le t < 1 \\ 2(e-1)e^{-t}, & t \ge 1 \end{cases}$$



Euler's formula

In school we all learned about complex numbers and in particular about Euler's remarkable formula for the complex exponential

$$e^{j\phi} = \cos\phi + j\sin\phi$$

Where $j = \sqrt{-1}$

$\cos\phi$ is the real part $\Re\{e^{j\phi}\}$, of $e^{j\phi}$

$\sin\phi$ is the imaginary part $\Im\{e^{j\phi}\}$, of $e^{j\phi}$

THE TRANSFER FUNCTION

The transfer function

$$H(f_0) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau} d\tau$$

is called the *frequency function* or the *transfer function* for the LTI system with impulse response $h(t)$.

Remember the two notations for frequency:

$$f_0 = \frac{\omega_0}{2\pi}$$

Phase and amplitude functions

The frequency function is in general a complex function of the frequency:

$$H(f) = A(f)e^{j\phi(f)}$$

where

$$A(f) = |H(f)|$$

is called the *amplitude function* and

$$\phi(f) = \arctan\frac{\Im\{H(f)\}}{\Re\{H(f)\}}$$

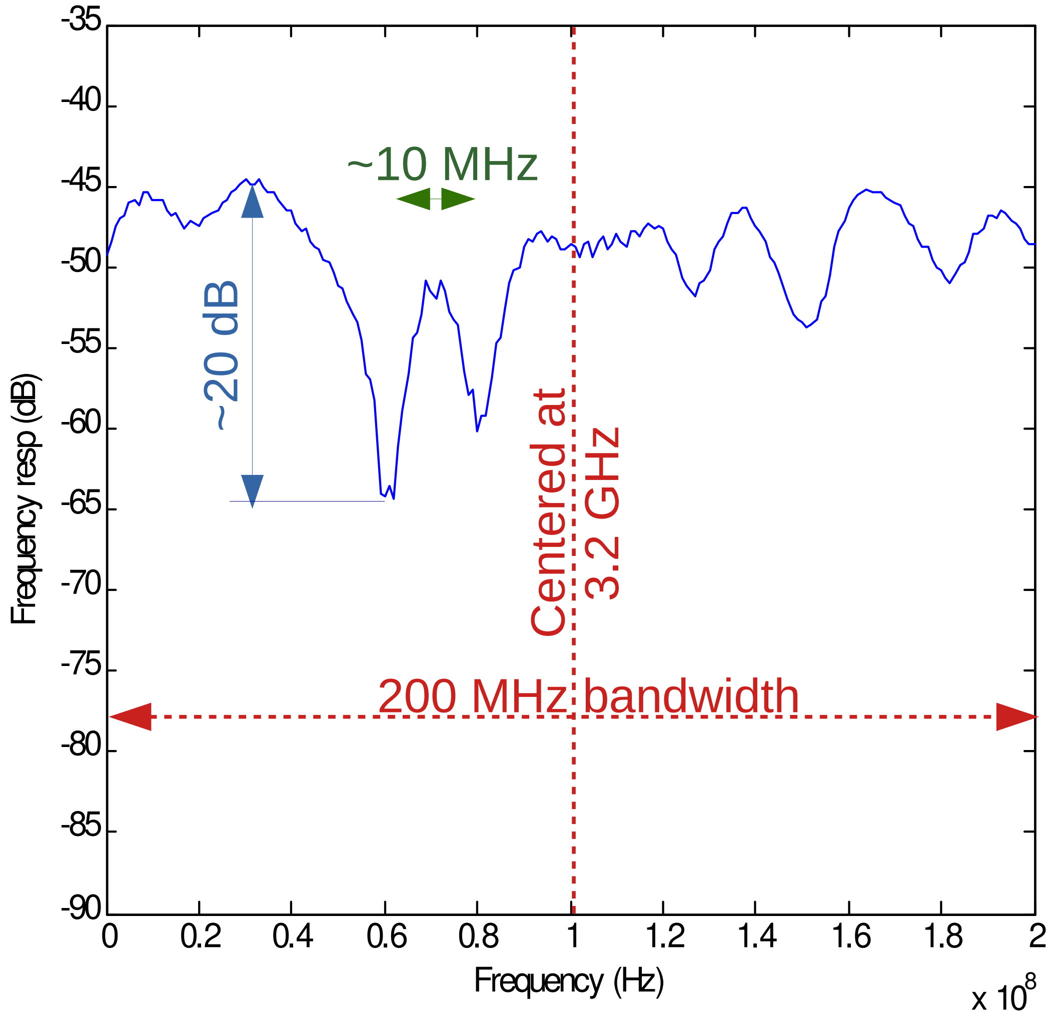
is called the *phase function*.

An example with a real measured radio channel

Measurement example, the radio channel

- Measurement in the lab
- Center frequency 3.2 GHz
- Measurement bandwidth 200 MHz at 201 equidistant frequency points (1 MHz separation)
- 60 measurement positions over 60 cm, spaced 1 cm apart
- Measured with a vector network analyzer

Transfer function, at one position

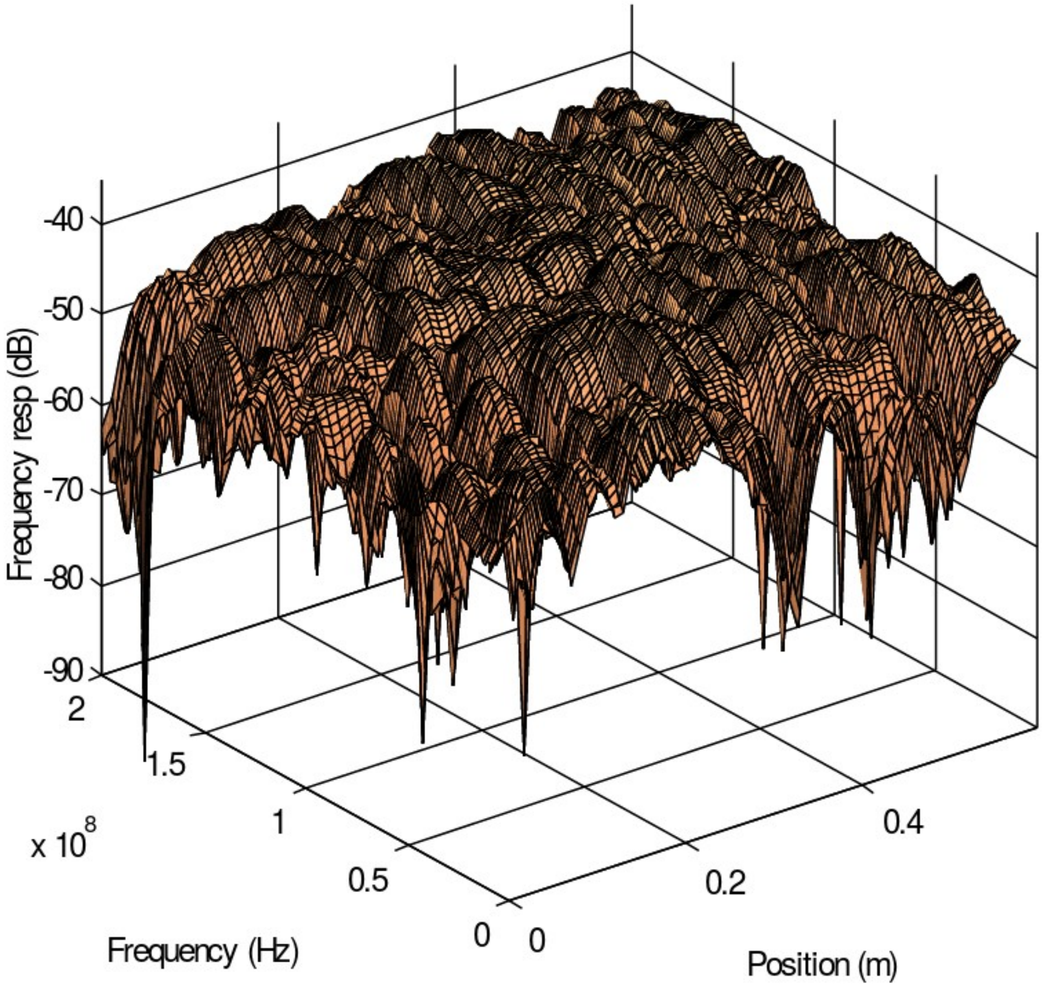


What we can say from such a measurement, e.g.:

- Some frequencies are attenuated 20 dB more than others
- The transfer function seems to be somewhat smooth and not change totally over about at least 10 MHz

The two observations above have proper technical terms and are called *fading depth* and *coherence bandwidth.* These are very important when designing wireless communication systems. (Beyond this course.)

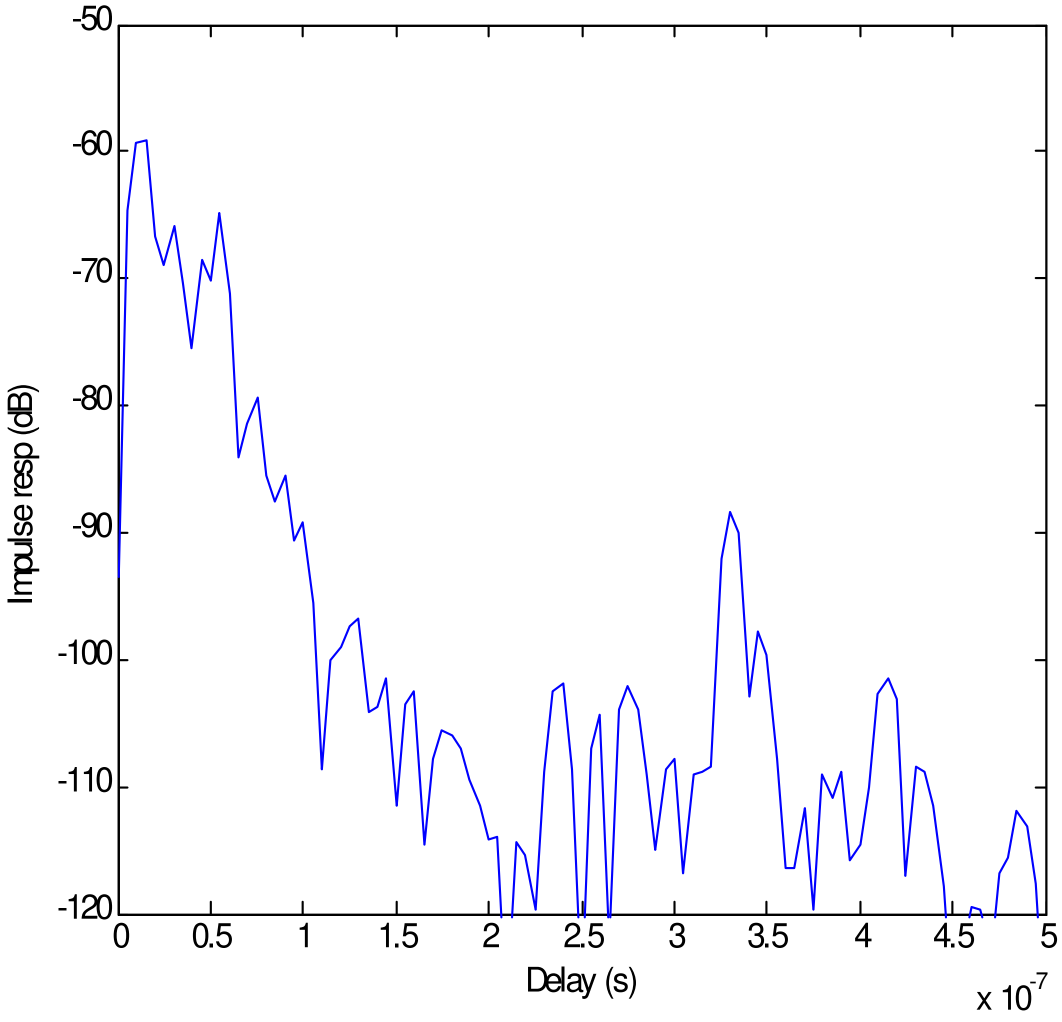
Transfer function, all 60 positions



Here we can see more of the transfer function, but it is also more difficult to interpret.

One thing we can see is that the transfer function changes quite considerably from position to position.

Impulse response, one position

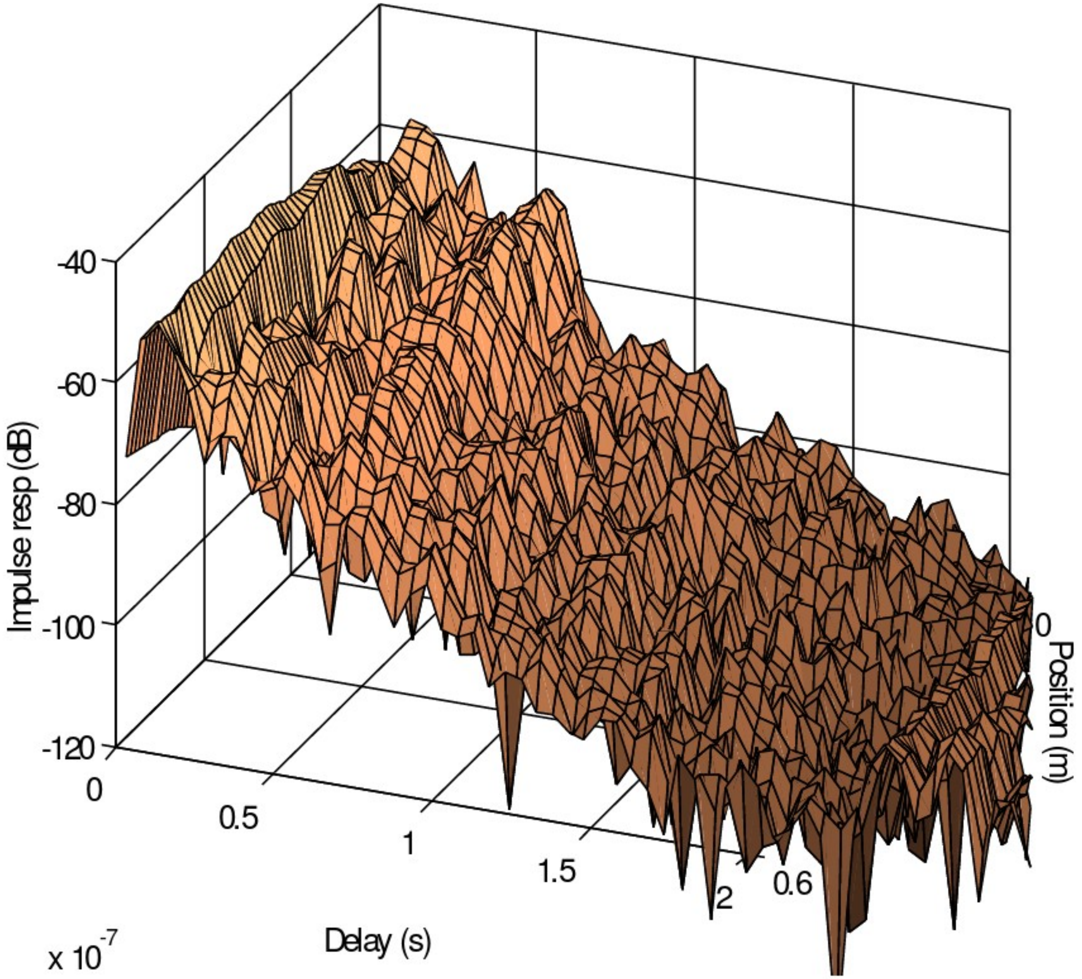


What are the delays?

How is the signal affected for different delays?

How does it change with time?

Impulse response, all 60 positions



We can see more of the impulse responses, but it is again more difficult to interpret.

One thing we can see is that, like the transfer function, the impulse response changes quite considerably from position to position.

THE FOURIER TRANSFORM

The Fourier transform

The Fourier transform of the signal *x(t)* is given by the formula

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

This function is in general complex and can be written in polar form

$$X(f) = A(f) e^{j\phi(f)}$$

where $A(f) = |X(f)|$ is called the (amplitude) spectrum of *x(t)*

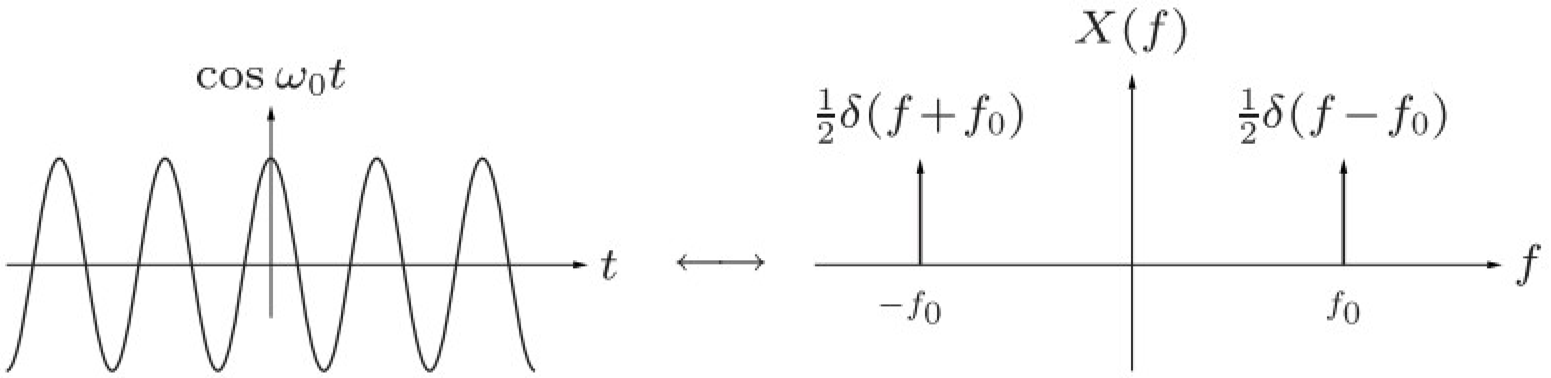
and $\phi(f)$ its phase angle.

Spectrum of a cosine

$$X(f) = \int_{-\infty}^{\infty} \cos \omega_0 t \; e^{-j\omega_0 t} dt = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

Hence we have a Fourier transform pair

$$\cos \omega_0 t \leftrightarrow \frac{1}{2}\delta(f - f_0) + \frac{1}{2}(f + f_0)$$



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Properties of the Fourier transform

1. Linearity

$$ax_1(t) + bx_2(t) \leftrightarrow aX_1(f) + bX_2(f)$$

2. Inverse

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j\omega t} df$$

3. Translation (time shifting)

$$x(t - t_0) \leftrightarrow X(f)\, e^{-j\omega t_0}$$

4. Modulation (frequency shifting)

$$x(t)\, e^{j\omega_0 t} \leftrightarrow X(f - f_0)$$

Properties of the Fourier transform

5. Time scaling

$$x(at) \leftrightarrow \frac{1}{|a|}X(f/a)$$

6. Differentiation in the time domain

$$\frac{d}{dt}x(t) \leftrightarrow j\omega X(f)$$

7. Integration in the time domain

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega}X(f)$$

8. Duality

$$X(t) \leftrightarrow x(-f)$$

Transformation of differential equations (to easier-to-solve algebraic equations)

Properties of the Fourier transform

9. Conjugate functions

$$x^*(t) \leftrightarrow X^*(-f)$$

10. Convolution in the time domain

$$x_1(t) * x_2(t) \leftrightarrow X_1(f)X_2(f)$$

11. Multiplication in the time domain

$$x_1(t)x_2(t) \leftrightarrow X_1(f) * X_2(f)$$

Simplify convolution operation

12. Parseval's formulas

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = \int_{-\infty}^{\infty} X_1(f)X_2^*(f)df$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

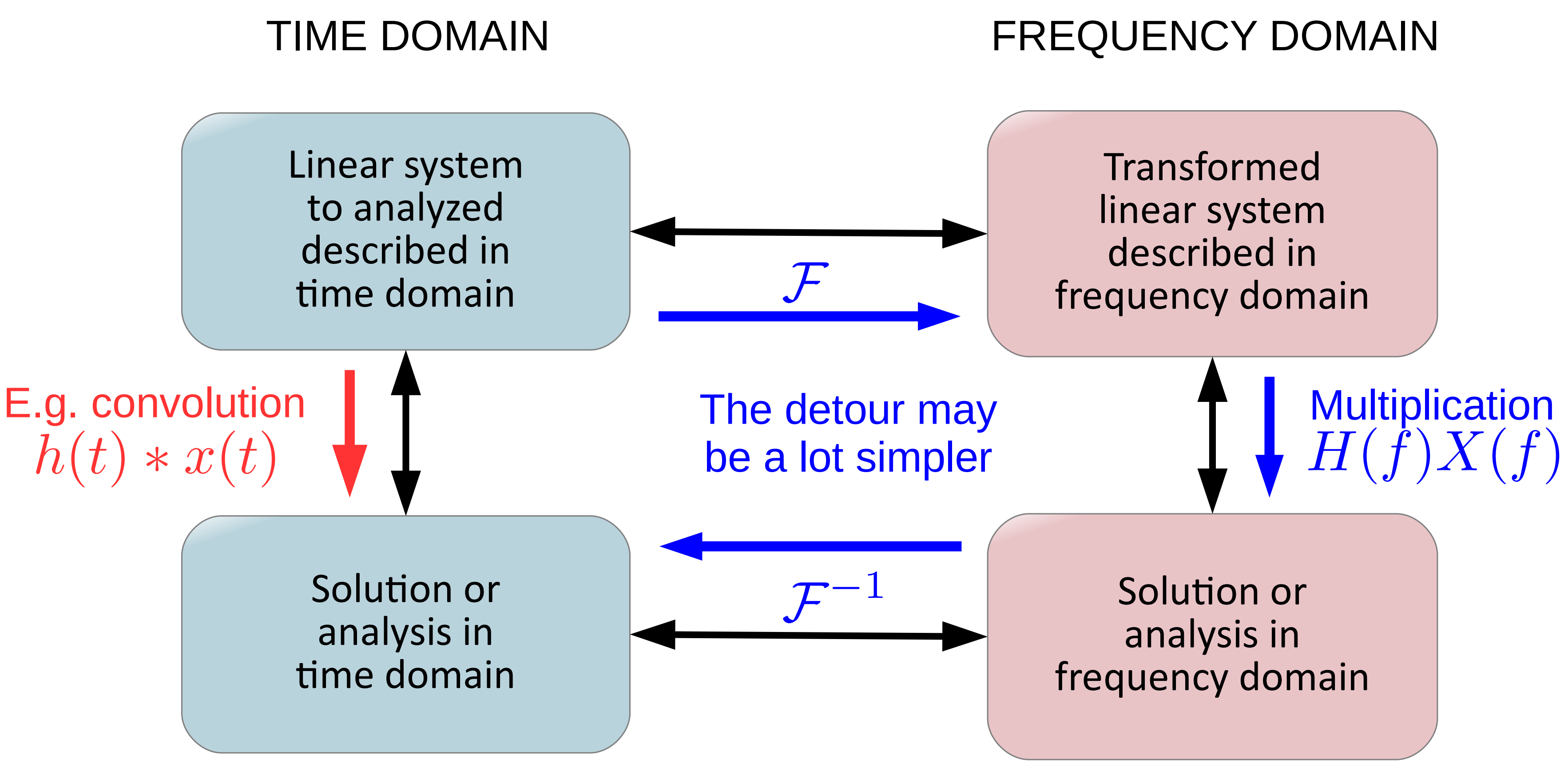
Calculation of orthogonality and energy

Fourier transform of a convolution

Since the output $y(t)$ of an LTI system is the convolution of its input $x(t)$ and impulse response $h(t)$ it follows from Property 10 (Convolution in the time domain) that the Fourier transform of its output $Y(f)$ is simply the product of the Fourier transform of its input $X(f)$ and its frequency function $H(f)$, that is,

$$Y(f) = X(f)H(f) = H(f)X(f)$$

Why use Fourier transforms?



Some useful Fourier transform pairs

(a) *Impulse in the time domain*

$$\delta(t) \leftrightarrow 1$$

(b) *Impulse in the frequency domain*

$$1 \leftrightarrow \delta(f)$$

(c) *Sign function*

$$sgn(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases} \quad \leftrightarrow \quad \frac{2}{j\omega}$$

(d) *Unit step function*

$$u(t) = \begin{cases} 1, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0 \end{cases} \quad \leftrightarrow \quad \frac{1}{j\omega} + \frac{1}{2}\delta(f)$$

An impulse in time contains ”all” frequencies

A constant (DC) signal in time has a single impulse in frequency (at *f*=0)

The sign and step functions are related:

$$u(t) = \frac{1}{2}(1 + \mathrm{sgn}(t))$$

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Some useful Fourier transform pairs

(e) *Complex exponential*

$$e^{j\omega_0 t} \leftrightarrow \delta(f - f_0)$$

(f) *Cosine function*

$$\cos\omega_0 t \leftrightarrow \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

(g) *Sine function*

$$\sin\omega_0 t \leftrightarrow \frac{1}{2j}\delta(f - f_0) + \frac{1}{2j}\delta(f + f_0)$$

(h) *Rectangular pulse*

$$rect(t) \leftrightarrow sinc(f) = \frac{\sin\pi f}{\pi f}$$

A complex sinusoids in time is a single delta function in frequency

The cos and sin functions each contain two complex sinusoids (see Euler's formula), and therefore two delta functions each in frequency

Rectangular function in time, a very important signal shape, often used in digital communications

Some useful Fourier transform pairs

(i) *Sinc pulse*

$$\text{sinc}(t) \leftrightarrow \text{rect}(f) = \begin{cases} 1, & |f| < 1/2 \\ 0, & |f| > 1/2 \end{cases}$$

(j) *Triangular pulse*

$$\begin{cases} 1 - |t|, & |t| < 1 \\ 0, & |t| > 0 \end{cases} \leftrightarrow \text{sinc}^2(f)$$

(k) *Gaussian pulse*

$$e^{-\pi t^2} \leftrightarrow e^{-\pi f^2}$$

Sinc in time often used for reconstruction of sampled signals. Defined as

$$\mathrm{sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

Some useful Fourier transform pairs

(l) *One-sided exponential function* $(\alpha > 0)$

$$e^{-\alpha t}u(t) \leftrightarrow \frac{1}{\alpha + j\omega}$$

(m) *Double-sided exponential function* $(\alpha > 0)$

$$e^{-\alpha|t|} \leftrightarrow \frac{2\alpha}{\alpha^2+\omega^2}$$

(n) *Impulses spaced* T *sec. apart*

$$\sum_{i=-\infty}^{\infty} \delta(t - iT) \leftrightarrow \frac{1}{T} \sum_{j=-\infty}^{\infty} \delta\left(f - \frac{j}{T}\right)$$

Shows up often in analysis of circuits

Relates to sampling of signals in time (impulse sampling)

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