Information Transmission, Chapter 2, Sinusoidal functions & the Fourier transform

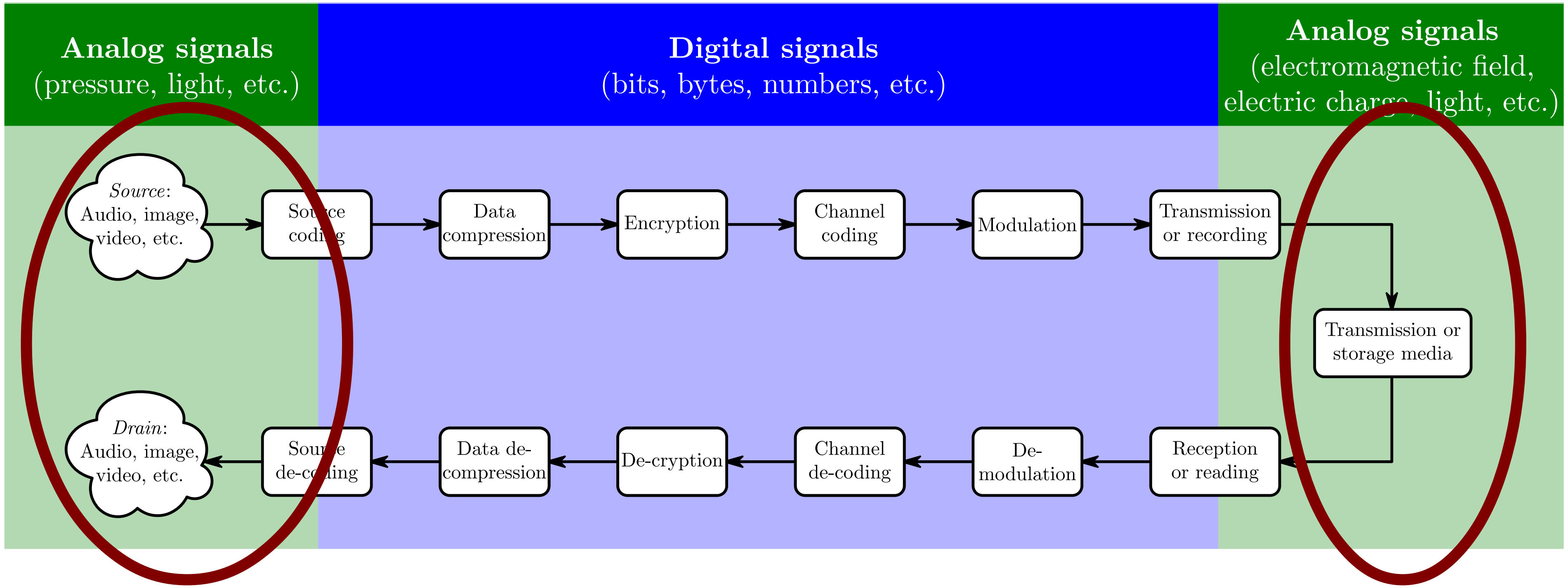
OVE EFORS

ELECTRICAL AND INFORMATION TECHNOLOGY

Learning outcomes

- The student should
  - understand how sinusoidal inputs to LTI systems generate sinusoidal outputs (sinusoidals being eigenfunctions of LTI systems),
  - be able to calculate the frequency function/transfer function of an LTI system,
  - understand and be able to calculate the Fourier transform of a time signal, using an integral,
  - understand how the Fourier transform relates to the frequency content (spectrum) of a signal,
  - be able to use Fourier transform properties and Fourier transform pairs listed in the formula collection to quickly find Fourier transforms,
  - understand the relationship between convolution (in time) and multiplication of Fourier transforms (in frequency), and how it can be used to simplify analysis of LTI systems.

Where are we in the BIG PICTURE?



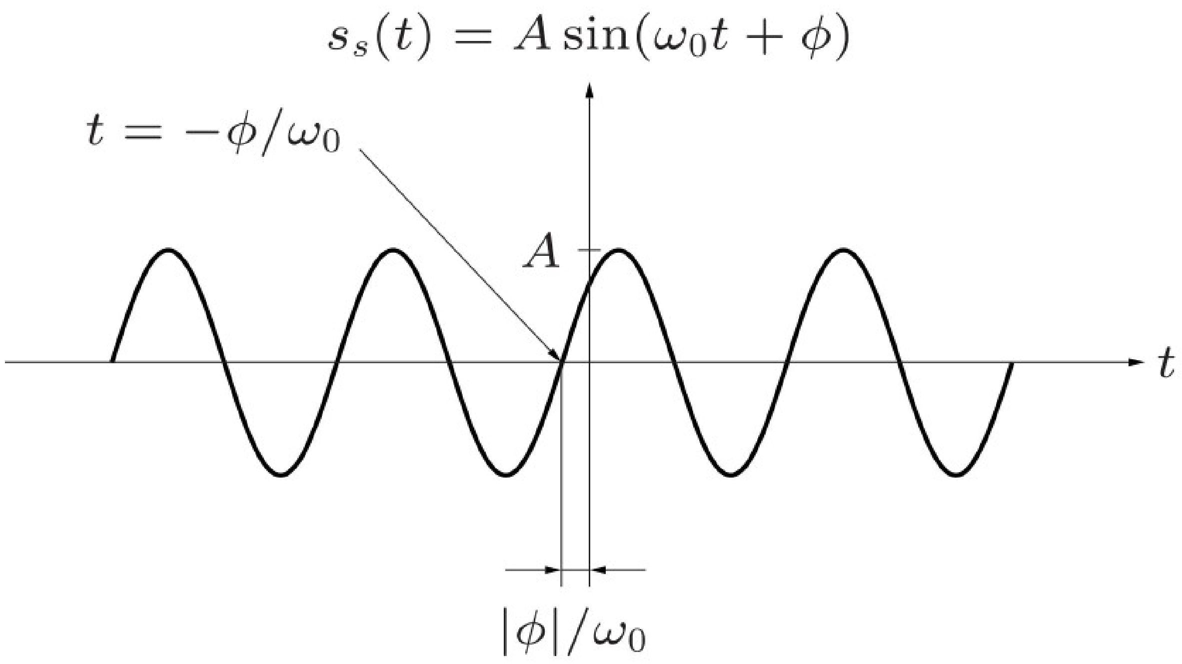
Electronics for analog input and output, including sampling and reconstruction.

Lecture relates to pages 43–58 in textbook.

Models of transmission and storage media.

On the importance of being sinusoidal

Frequency notation



Two notations for frequency:

f Hertz

ω radians per second

$$2\pi f = \omega$$

Some trigonometric identities

1a	$\sin\alpha = -\sin(-\alpha) = \sin(\pi-\alpha)$ $= \cos(\pi/2-\alpha) = \mp\cos(\alpha\pm\pi/2)$
1b	$\cos\alpha = -\cos(-\alpha) = -\cos(\pi-\alpha)$ $= \sin(\pi/2-\alpha) = \pm\sin(\alpha\pm\pi/2)$
2	$\sin^2\alpha + \cos^2\alpha = 1$
3a	$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
3b	$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
4a	$\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2}$
4b	$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
4c	$\cos\alpha - \cos\beta = -2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$

Some trigonometric identities

5a	$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$
5b	$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta))$
5c	$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))$
6a	$\sin 2\alpha = 2\sin\alpha\cos\alpha$
6b	$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
7a	$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$
7b	$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$
8a	$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$
8b	$\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$

Typo in the book
on page 45, Table 2.2.

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Euler's formula

In school we all learned about complex numbers and in particular about Euler's remarkable formula for the complex exponential

$$e^{j\phi} = \cos\phi + j\sin\phi$$

where

$$j = \sqrt{-1}$$

$\cos\phi$ is the real part $\Re\{e^{j\phi}\}$, of $e^{j\phi}$

$\sin\phi$ is the imaginary part $\Im\{e^{j\phi}\}$, of $e^{j\phi}$

A complex input signal split into its real and imaginary part

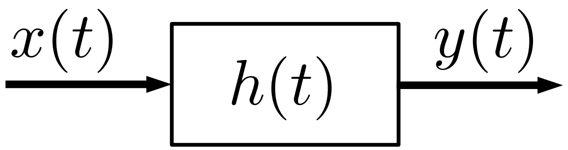
$$x(t) = e^{j\omega_0 t} \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

$\Updownarrow$ IF IMPULSE RESPONSE IS REAL

$$\Re\{x(t)\} = \cos w_0 t \longrightarrow \boxed{\text{System}} \longrightarrow \Re\{y(t)\}$$

$$\Im\{x(t)\} = \sin \omega_0 t \longrightarrow \boxed{\text{System}} \longrightarrow \Im\{y(t)\}$$

Complex sinusoidal input to an LTI system



In the previous lecture we learned that the output from an LTI system with impulse response h(t) is calculated as the convolution

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Using $x(t) = e^{j\omega_0 t}$ as input, we get

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{j\omega_0(t-\tau)}d\tau = e^{j\omega_0 t}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-j\omega_0\tau}d\tau}$$

The same sinusoidal as output … but multiplied by a complex number, which typically depends on the frequency of the sinusoidal.

The transfer function

$$H(f_0) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau} d\tau$$

is called the *frequency function* or the *transfer function* for the LTI system with impulse response *h(t)*.

Remember the two notations for frequency:

$$f_0 = \frac{\omega_0}{2\pi}$$

Phase and amplitude functions

The frequency function is in general a complex function of the frequency:

$$H(f) = A(f)e^{j\phi(f)}$$

where

$$A(f) = |H(f)|$$

is called the *amplitude function* and

$$\phi(f) = \arctan \frac{\Im\{H(f)\}}{\Re\{H(f)\}}$$

is called the *phase function*.

Finally…

For a linear, time-invariant system with a (bi-infinite) sinusoidal input, we always obtain a (bi-infinite) sinusoidal output!

The Fourier transform

The transfer function

$$H(f_0) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau} d\tau$$

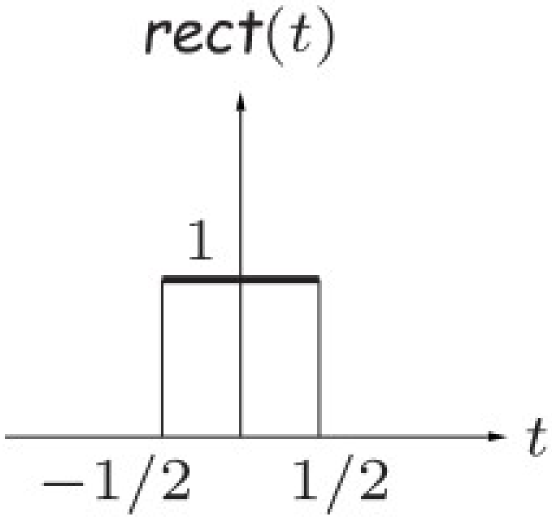
is the frequency function (transfer function) for the LTI system with impulse response *h(t)*.

The frequency function $H(f_0)$ specifies how the amplitude and phase of the sinusoidal input of frequency f_0 are changed by the LTI system.

Frequency content of a pulse?

Which frequencies does a pulse contain?

$$\text{rect}(t) = \begin{cases} 1, & -1/2 \leq t \leq 1/2 \\ 0, & \text{otherwise} \end{cases}$$



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The Fourier transform

There is a mathematical way of solving this problem, namely using the Fourier transform of the signal *x(t)* given by the formula

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

This function is in general complex:

$$X(f) = A(f) e^{j\phi(f)}$$

where $A(f) = |X(f)|$ is called the (amplitude) spectrum of *x(t)*

and $\phi(f)$ its phase angle.

Spectrum of a consine

Consider now the sinusoidal signal $\cos\omega_0 t$

where $\omega_0 = 2\pi f_0$.

Which frequencies does it contain?

In order to answer this fundamental question we use Euler's formula as

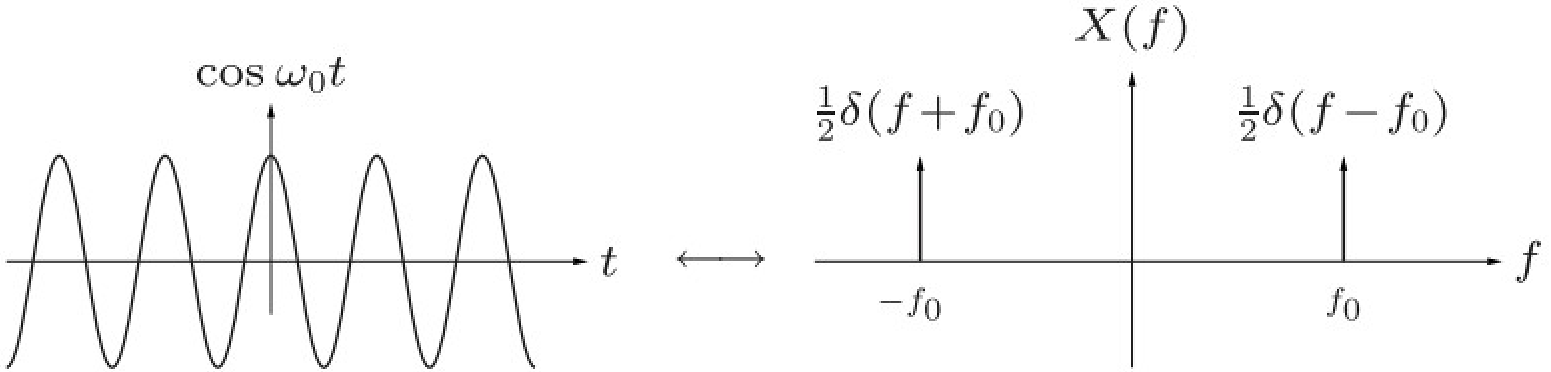
$$\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

Spectrum of a consine

$$X(f) = \int_{-\infty}^{\infty} \cos \omega_0 t \; e^{-j\omega_0 t} dt = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

Hence we have a Fourier transform pair

$$\cos \omega_0 t \leftrightarrow \frac{1}{2}\delta(f - f_0) + \frac{1}{2}(f + f_0)$$



Frequency content of a pulse?

Which frequencies does a pulse contain?

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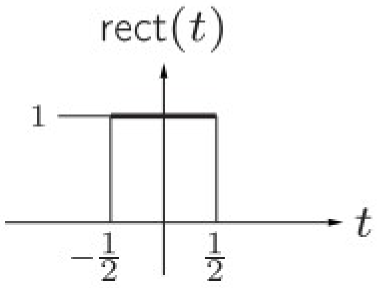
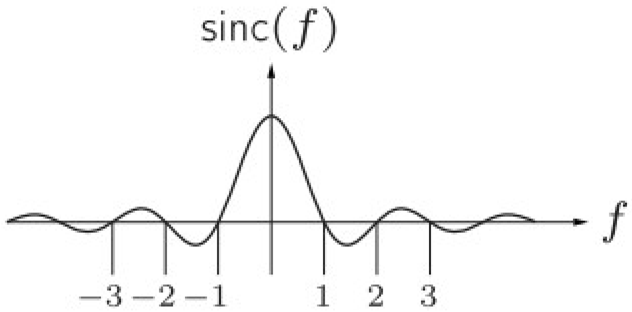


$\longleftrightarrow$



$$rect(t) \leftrightarrow sinc(f)$$

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Properties of the Fourier transform

1. Linearity

$$a x_1(t) + b x_2(t) \leftrightarrow a X_1(f) + b X_2(f)$$

2. Inverse

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j\omega t} df$$

3. Translation (time shifting)

$$x(t - t_0) \leftrightarrow X(f)\, e^{-j\omega t_0}$$

4. Modulation (frequency shifting)

$$x(t)\, e^{j\omega_0 t} \leftrightarrow X(f - f_0)$$

Properties of the Fourier transform

5. Time scaling

$$x(at) \leftrightarrow \frac{1}{|a|} X(f/a)$$

6. Differentiation in the time domain

$$\frac{d}{dt} x(t) \leftrightarrow j\omega X(f)$$

7. Integration in the time domain

$$\int_{-\infty}^{t} x(\tau) d\tau \leftrightarrow \frac{1}{j\omega} X(f)$$

8. Duality

$$X(t) \leftrightarrow x(-f)$$

Properties of the Fourier transform

9. Conjugate functions

$$x^*(t) \leftrightarrow X^*(-f)$$

10. Convolution in the time domain

$$x_1(t) * x_2(t) \leftrightarrow X_1(f)X_2(f)$$

11. Multiplication in the time domain

$$x_1(t)x_2(t) \leftrightarrow X_1(f) * X_2(f)$$

12. Parseval's formulas

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = \int_{-\infty}^{\infty} X_1(f)X_2^*(f)df$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Fourier transform of a convolution

Since the output *y(t)* of an LTI system is the convolution of its input *x*(*t*) and impulse response *h*(*t*) it follows from Property 10 (Convolution in the time domain) that the Fourier transform of its output *Y*(*f*) is simply the product of the Fourier transform of its input *X*(*f*) and its frequency function *H*(*f*), that is,

$$Y(f) = X(f)H(f) = H(f)X(f)$$

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Some useful Fourier transform pairs

(a) Impulse in the time domain

$$\delta(t) \leftrightarrow 1$$

(b) Impulse in the frequency domain

$$1 \leftrightarrow \delta(f)$$

(c) Sign function

$$sgn(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases} \quad \leftrightarrow \quad \frac{2}{j\omega}$$

(d) Unit step function

$$u(t) = \begin{cases} 1, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0 \end{cases} \quad \leftrightarrow \quad \frac{1}{j\omega} + \frac{1}{2}\delta(f)$$

Some useful Fourier transform pairs

(e) *Complex exponential*

$$e^{j\omega_0 t} \leftrightarrow \delta(f - f_0)$$

(f) *Cosine function*

$$\cos \omega_0 t \leftrightarrow \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

(g) *Sine function*

$$\sin \omega_0 t \leftrightarrow \frac{1}{2j}\delta(f - f_0) + \frac{1}{2j}\delta(f + f_0)$$

(h) *Rectangular pulse*

$$rect(t) \leftrightarrow sinc(f) = \frac{\sin \pi f}{\pi f}$$

Some useful Fourier transform pairs

(i) *Sinc pulse*

$$sinc(t) \leftrightarrow rect(f) = \begin{cases} 1, & |f| < 1/2 \\ 0, & |f| > 1/2 \end{cases}$$

(j) *Triangular pulse*

$$\begin{cases} 1 - |t|, & |t| < 1 \\ 0, & |t| > 0 \end{cases} \leftrightarrow sinc^2(f)$$

(k) *Gaussian pulse*

$$e^{-\pi t^2} \leftrightarrow e^{-\pi f^2}$$

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$$e^{-\pi t^2} \leftrightarrow e^{-\pi f^2}$$

Some useful Fourier transform pairs

(l) One-sided exponential function $(\alpha > 0)$

$$e^{-\alpha t}u(t) \leftrightarrow \frac{1}{\alpha + j\omega}$$

(m) Double-sided exponential function $(\alpha > 0)$

$$e^{-\alpha|t|} \leftrightarrow \frac{2\alpha}{\alpha^2 + \omega^2}$$

(n) Impulses spaced T *sec. apart*

$$\sum_{i=-\infty}^{\infty} \delta(t - iT) \leftrightarrow \frac{1}{T} \sum_{j=-\infty}^{\infty} \delta\left(f - \frac{j}{T}\right)$$

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Example

What is the spectrum of a modulated rect signal?

The spectrum of the $rect(t)$ signal is $sinc(f)$.

$\boxed{\text{The spectrum is concentrated around } f = 0.}$

Multiply the $rect(t)$ signal by $\cos \omega_0 t$:

$$rect(t) \cos \omega_0 t \leftrightarrow sinc(f) * \left(\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0) \right)$$

Summary

- Sinusoids – real and complex
- Frequency and angular frequency
- Sinusoids are ”eigenfunctions” of LTI systems (with complex sinusoid on input, a sinusoid with the same frequency on the output, multiplied by a complex number)
- Transfer function of an LTI system and its phase and amplitude functions
- The Fourier transform (”derived” from the transfer function of an LTI system)
  - Frequency content of signals
  - Fourier transform properties
  - Forurier transform pairs

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