Chapter 2: Linear systems & sinusoids

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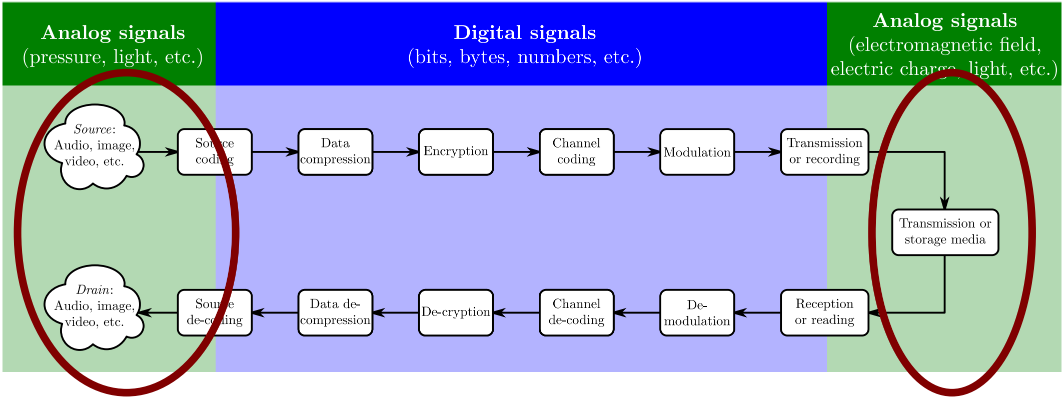
Learning outcomes

After this lecture, the student should

- understand what a linear system is, including linearity conditions, their consequences, and the superposition principle,
- understand what time-invariance is, and its connection to linear systems – linear and time-invariant (LTI) systems,
- understand what impulses and step functions are and how they are defined,
- have a basic understanding of what generalized functions are,
- understand what the impulse response of an LTI system is and how it can be used to calculate output signals,
- be able to calculate the output signal of an LTI system, when given an input signal and the system's impulse response,
- understand causality and why it is an important property of real, physical, systems, and
- be able to find out if a certain given LTI system is causal or not.

Where are we in the BIG PICTURE?



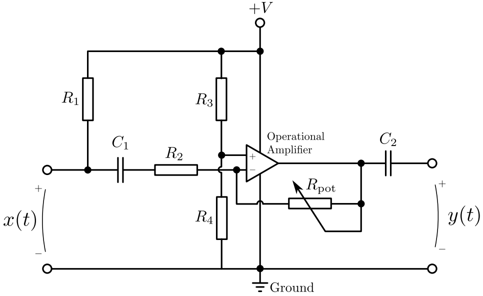
Electronics for analog input and output, including sampling and reconstruction.

Lecture relates to pages 30–43 in textbook.

Models of transmission and storage media.

EXAMPLES: Input/output electronics

Microphone amplifier



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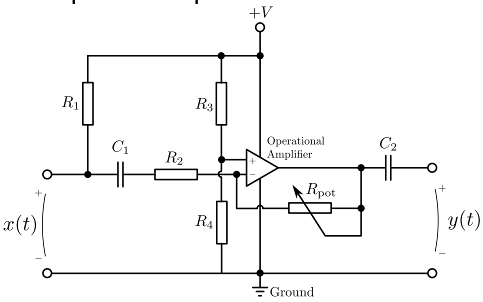
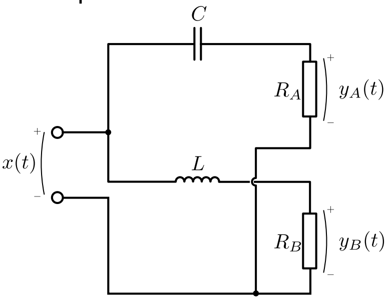


Loudspeaker crossover network



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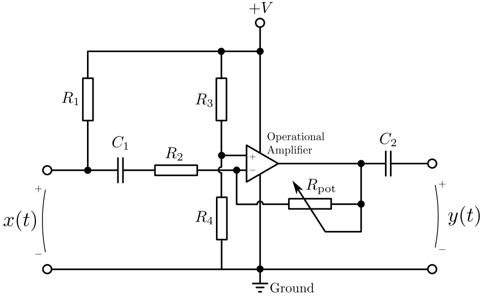

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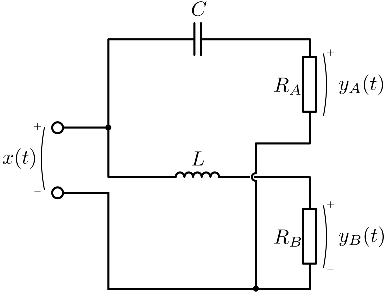
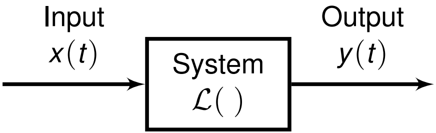


Both these are examples of linear systems.

Linear systems, what's that?

Linear systems – Definition



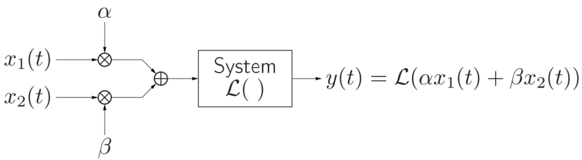
Definition (Linear system)

A system $\mathcal{L}$ is said to be **linear** if, whenever an input $x(t) = x_1(t)$ yields an output $y(t) = \mathcal{L}(x_1(t))$ and an input $x(t) = x_2(t)$ yields an output $y(t) = \mathcal{L}(x_2(t))$ this also leads to that we have $\mathcal{L}(\alpha x_1(t) + \beta x_2(t)) = \alpha \mathcal{L}(x_1(t)) + \beta \mathcal{L}(x_2(t))$, where α and β are real or complex valued constants.

Short version for memory: Weighted sum of inputs leads to same weighted sum of outputs.

Linear systems – Property illustration

This system with two superpositioned input signals



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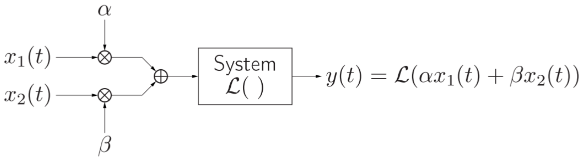
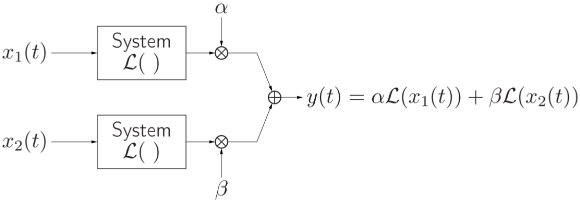


is **equivalent** to these two parallel systems, each with its own input, but outputs superpositioned the same way (same α and β)



The linearity condition – infinite sums

In order to extend the linearity condition

$$\mathcal{L}\left(\alpha x_1(t) + \beta x_2(t)\right) = \alpha \mathcal{L}\left(x_1(t)\right) + \beta \mathcal{L}\left(x_2(t)\right)$$

to infinite sums (integrals – think Riemann sum) we must assume that the system $\mathcal{L}$ is “sufficiently smooth or continuous”.

In other words: “Small” changes in input must give “small” changes in output.

The linearity condition – equivalent formulation

The linearity condition

$$\mathcal{L}(\alpha x_1(t) + \beta x_2(t)) = \alpha\mathcal{L}(x_1(t)) + \beta\mathcal{L}(x_2(t))$$

can be replaced by two simpler conditions

$$\begin{aligned} \mathcal{L}(x_1(t) + x_2(t)) &= \mathcal{L}(x_1(t)) + \mathcal{L}(x_2(t)) && \text{[Additivity]} \\ \mathcal{L}(\alpha x(t)) &= \alpha\mathcal{L}(x(t)) && \text{[Homogeneity]} \end{aligned}$$

Important note: Both simplified conditions must be true for the first (linearity) condition to be true!

Linear system properties

A consequence of the linearity condition

Assuming that both input $x_1(t)$ and $x_2(t)$ are zero, i.e.,

$$x_1(t) = x_2(t) = 0$$

we obtain [additivity]

$$\mathcal{L}(0 + 0) = \mathcal{L}(0) + \mathcal{L}(0),$$

which can **only** be true if $\mathcal{L}(0) = 0$.

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Zero input results in **zero output** for ALL linear systems!

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An important consequence

Zero input results in **zero output** for ALL linear systems!

Note: This property is not true for nonlinear systems, which may e.g. “oscillate” despite a zero-input.

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Superposition principle

A system is said to be linear if *the output resulting from an input that is a weighted sum of signals* is the same as *the weighted sum of the outputs obtained when the input signals are acting separately*. This means

$$\mathcal{L}\big(\underbrace{\alpha x_1(t) + \beta x_2(t)}_{\substack{\text{Weighted sum of} \\ \text{input signals}}}\big) = \underbrace{\alpha \mathcal{L}(x_1(t)) + \beta \mathcal{L}(x_2(t))}_{\substack{\text{Corresponding weighted} \\ \text{sum of output signals}}}$$

In other words: Per definition, linear systems obey the superposition principle. (See definition of *linear system*.)

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Linear and time invariant systems

Time invariance

A system $\mathcal{I}$ is said to be *time-invariant* if a delay of the input by any amount of time only causes the same delay of the output. More precisely:

Definition (Time invariance)

A system $\mathcal{I}$ with output

$$y(t) = \mathcal{I}(x(t))$$

is time-invariant if

$$y(t-\tau) = \mathcal{I}(x(t-\tau))$$

holds for all delays (shifts) τ.

What it means: The system as such does not change its behavior with time. Its components do not "grow old".

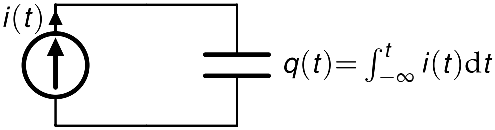
Linear and time-invariant (LTI) systems

A system that is both **linear** and **time-invariant** is called an **LTI system**.

LTI systems are an important class of systems which have many applications in real life!

A circuit example: Impulses and step functions

Consider the output when charging a capacitor with a current pulse:



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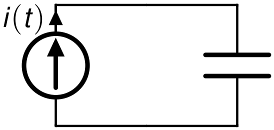
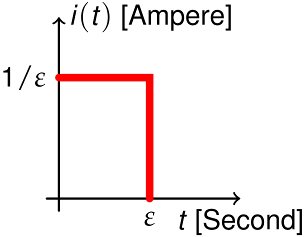


$q(t) = \int_{-\infty}^{t} i(t)\,\mathrm{d}t$

Input signal



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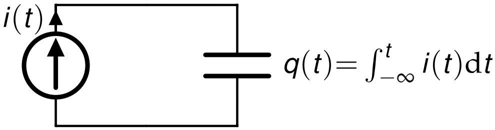

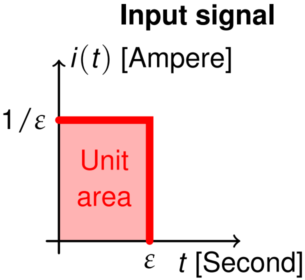

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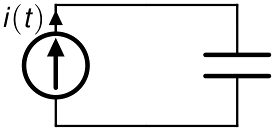

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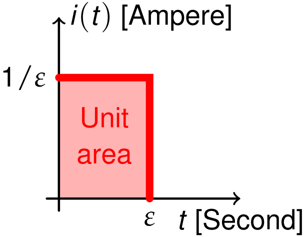
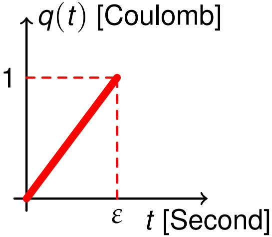


Out signal



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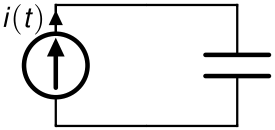

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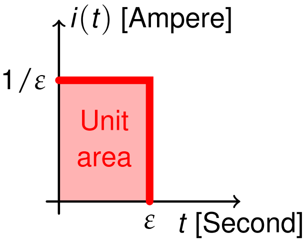

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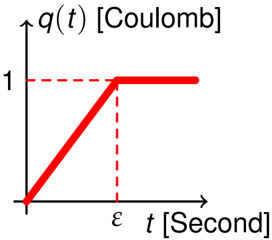

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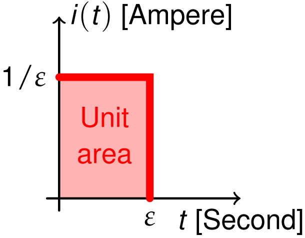

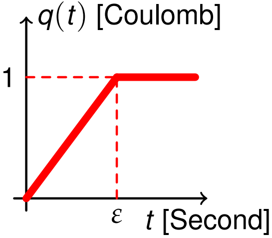

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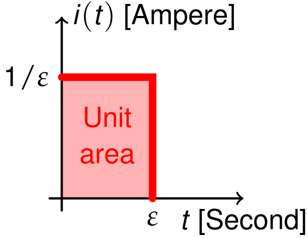

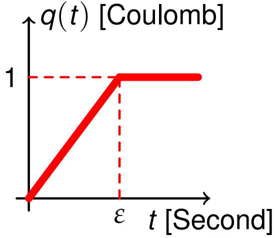

Now, let the current pulse become shorter and shorter, approaching zero-length in time.

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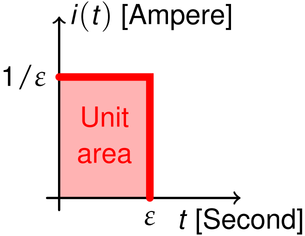

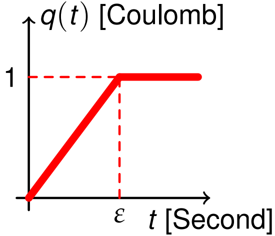

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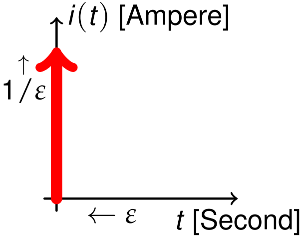

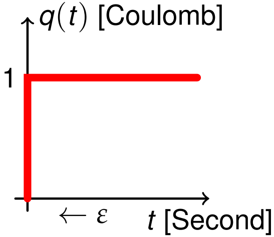

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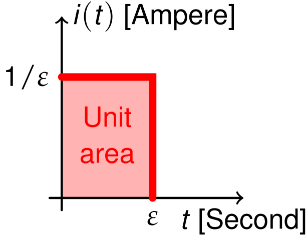

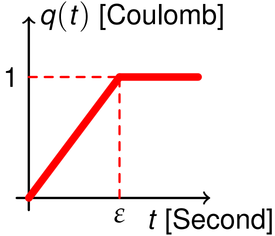

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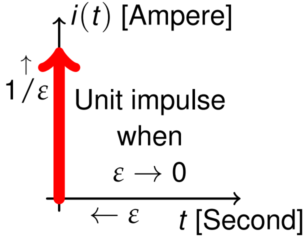

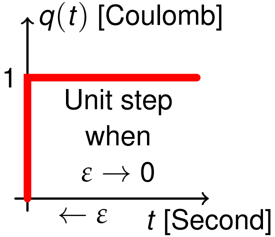

Impulses, steps and impulse responses

The delta/unit-impulse function

When the length of our pulse (with unit area) in the circuit example approaches zero, the pulse approaches the *delta-function* $\delta(t)$ – also called the *unit impulse* or *Dirac's delta-function*[1].

[1] Named after Paul Dirac (1902-1984), one of the founders of quantum mechanics.

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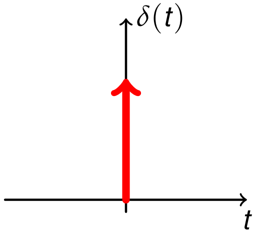
Definition (Delta function)

The delta function $\delta(t)$ (unit impulse) is a function that has the following two properties:

$$\delta(t) = 0 \text{ for all } t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) \mathrm{d}t = 1$$



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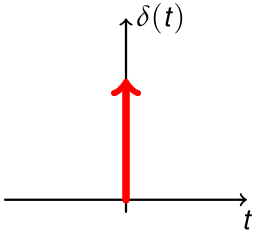

This is a very important function. We will use it throughout the course!

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The unit-step function

If a delta/unit-impulse function acts as input in our circuit example, the charge in the capacitor will rise immediately from 0 to 1 Coulomb, at $t = 0$, and stay there forever. We say that when the pulse length on the input approaches zero, the output (charge) approaches the *unit-step function* $u(t)$ – also called Heaviside's step-function[2].

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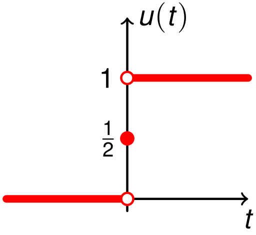
$$u(t) = \begin{cases} 0 & , \quad t < 0 \\ 1/2 & , \quad t = 0 \\ 1 & , \quad t > 0 \end{cases}$$



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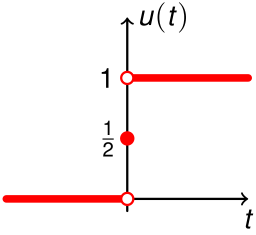

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Generalized functions and LTI systems

Generalized functions/distributions

The delta function is not a function in the ordinary sense – it a *generalized function* or a *distribution*. These need to be treated in a special way.
The derivative of a generalized function/distribution is indirectly defined:

Definition (Derivative of generalized function/distribution)

The derivative $g'(t)$ of a generalized function/distribution $g(t)$ is given by

$$\int_{-\infty}^{\infty} g'(t)\varphi(t)\mathrm{d}t \stackrel{\text{def}}{=} -\int_{-\infty}^{\infty} g(t)\varphi'(t)\mathrm{d}t$$

where $\varphi(t)$ is a test function which is *infinitely differentiable* (smooth) and has *compact support* (zero outside some bounded interval).

Important property of the delta function

Using the definition of the delta function $\delta(t)$ we can conclude that

$$\int_{-\infty}^{\infty} \delta(t) f(t) \mathrm{d}t = f(0)$$

if $f(t)$ is a function which is continuous at $t = 0$.

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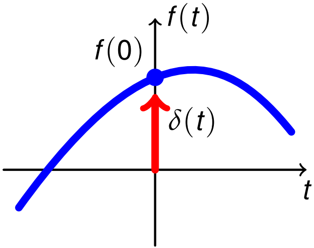
We can see this as the above integral *sampling* (picking up the value of) the function $f(t)$ at $t = 0$.



QUIZ: Sampling at a certain time

As we have seen, this expression

$$\int_{-\infty}^{\infty} \delta(t) f(t) \mathrm{d}t = f(0)$$

accomplishes a sampling of $f(t)$ at $t = 0$.
What would the corresponding expression sampling $f(t)$ at an arbitrary time $t = t_0$?

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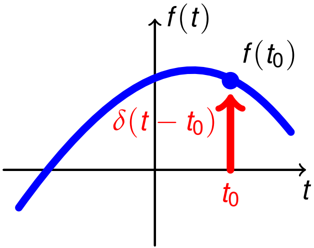
What would the corresponding expression sampling $f(t)$ at an arbitrary time $t = t_0$?

Solution: By moving (delaying) the $\delta(t)$ function by t_0 units along the t-axis, we get

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t) \mathrm{d}t = f(t_0)$$

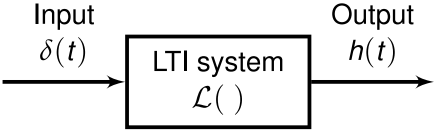


Derivatives and integrals

Between delta and step functions, $\delta(t)$ and $u(t)$, we have the following relationships (derivative defined for generalized functions):

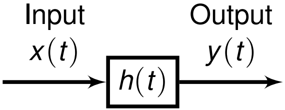
$$u'(t) = \delta(t)$$
$$\int_{-\infty}^{t} \delta(t)\mathrm{d}t = u(t)$$

LTI system impulse response

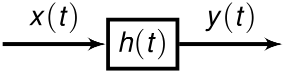


The **impulse response** $h(t)$ of an LTI system is defined as the output when an (Dirac/unit) impulse at $t = 0$ acts as input.

We often draw the block diagram of such as system:



LTI system input-output relation

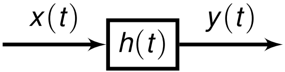


The input-output relation for an LTI system with impulse response $h(t)$ is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\mathrm{d}\tau$$

where $x(t)$ is the input and $y(t)$ the corresponding output.

LTI system input-output relation



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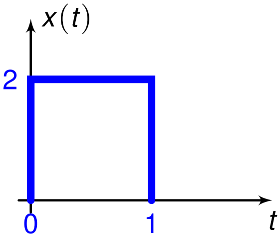
where $x(t)$ is the input and $y(t)$ the corresponding output.

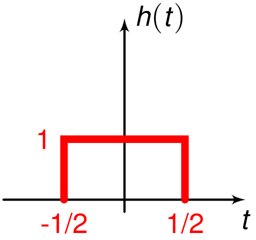
This particular integral is called a **convolution** and is *one of the most important mathematical concepts in the course*. Convolution is commutative and denoted by $*$. We write

$$\boxed{y(t) = x(t) * h(t) = h(t) * x(t).}$$

Example 1 (Example 2.2 in textbook)

Assume we have an LTI system with input $x(t)$ and impulse response $h(t)$, according to the figures below.





What will the output $y(t)$ be?

Solution 1

The output of an LTI system with input $x(t)$ and impulse response $h(t)$ is determined by the convolution

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau.$$

Let's start with performing the convolution by using the integral.
First, what does the factors of the integrand, $x(\tau)$ and $h(t-\tau)$, look like on the τ-axis?

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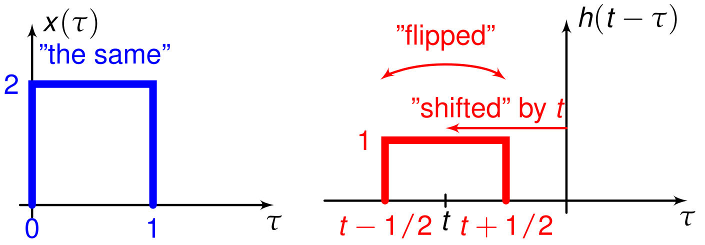
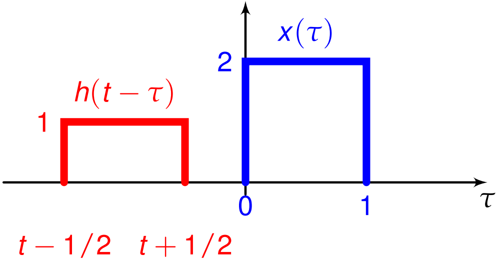


Solution 1 (cont.)

Result of

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau}$$

Let's perform the convolution:



$t < -1/2$:

$$y(t) = 0$$

Solution 1 (cont.)

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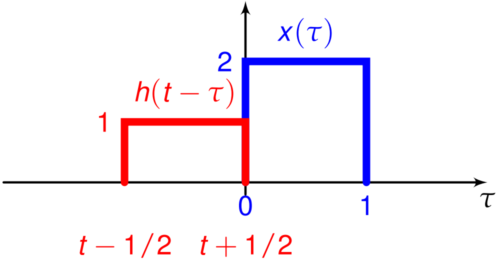

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Solution 1 (cont.)

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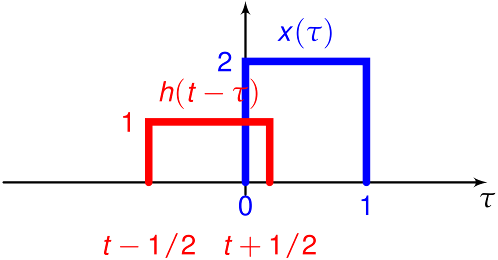


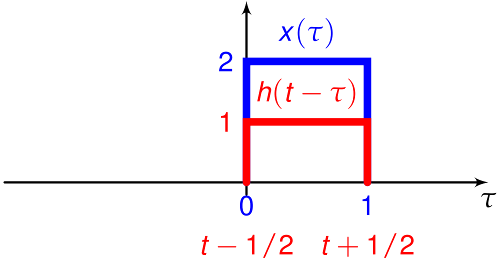
Result of

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau}$$

$$t < -1/2:$$
$$y(t) = 0$$
$$-1/2 \leq t < 1/2:$$
$$y(t) = 2(t+1/2)$$

Solution 1 (cont.)

Let's perform the convolution:



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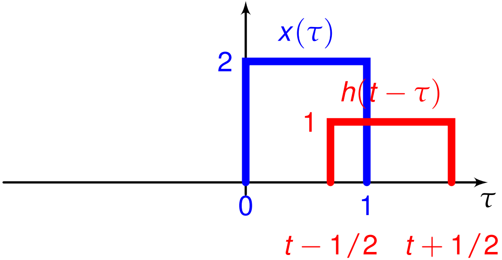

Result of

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau}$$

$$
\begin{aligned}
t < -1/2 &: \\
& y(t) = 0 \\
-1/2 \le t < 1/2 &: \\
& y(t) = 2(t+1/2) \\
1/2 \le t < 3/2 &: \\
& y(t) = 2(3/2-t)
\end{aligned}
$$

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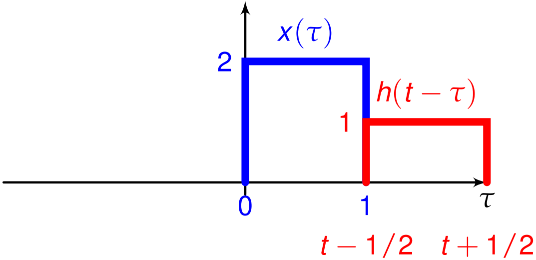
Solution 1 (cont.)

Let's perform the convolution:



Result of

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau}$$

$t < -1/2$:

$$y(t) = 0$$

$-1/2 \leq t < 1/2$:

$$y(t) = 2(t+1/2)$$

$1/2 \leq t < 3/2$:

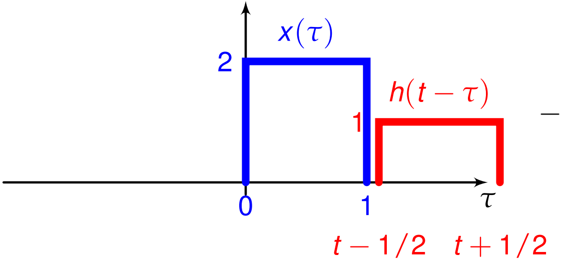
$$y(t) = 2(3/2-t)$$

Solution 1 (cont.)

Result of

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau}$$

Let's perform the convolution:



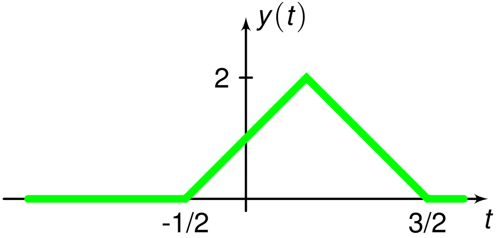
$$\begin{aligned} t < -1/2: & \\ & y(t) = 0 \\ -1/2 \le t < 1/2: & \\ & y(t) = 2(t+1/2) \\ 1/2 \le t < 3/2: & \\ & y(t) = 2(3/2-t) \\ t \ge 3/2: & \\ & y(t) = 0 \end{aligned}$$

Solution 1 (cont.)

Plotting the graph of the output signal

$$y(t) = \begin{cases} 0, & t < -1/2 \\ 2(t + 1/2), & -1/2 \leq t < 1/2 \\ 2(3/2 - t), & 1/2 \leq t < 3/2 \\ 0, & t \geq 3/2 \end{cases}$$

we get



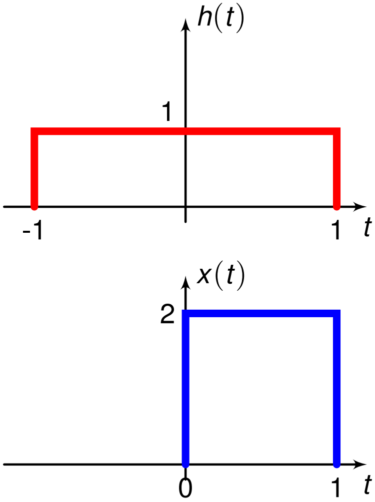
Example 2 (Example 2.3 in textbook)

Consider an LTI system with impulse response

$$h(t) = \begin{cases} 1, & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

and input signal

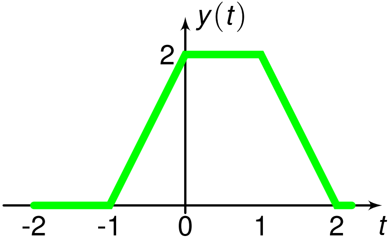
$$x(t) = \begin{cases} 2, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$



Solution 2

Following the same type of reasoning as in Example 1, we get a piece-wise linear output (since $x(t)$ and $h(t)$ are piece-wise constant), but now with four corners rather than three (since $x(t)$ and $h(t)$ are of different length):

$$y(t) = \begin{cases} 0, & t < -1 \\ 2t + 2, & -1 \leq t < 0 \\ 2, & 0 \leq t < 1 \\ -2t + 4, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases}$$



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Example 3 (Example 2.4 in textbook)

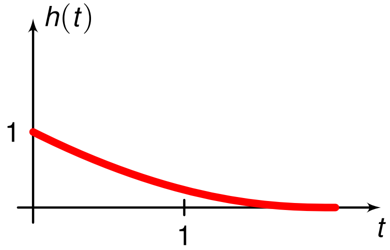
Consider an LTI system with impulse response

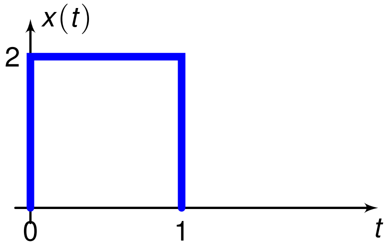
$$h(t) = e^{-t}u(t)$$

and the same input signal

$$x(t) = \begin{cases} 2, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

as in Example 2.





Solution 3

With the given impulse response and input signal, we get three cases when calculating the convolution $y(t) = x(t) * h(t) = \int h(\tau)x(t-\tau)\mathrm{d}\tau$:

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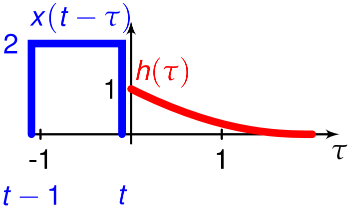
When $t < 0$:



the output becomes

$$y(t) = 0$$

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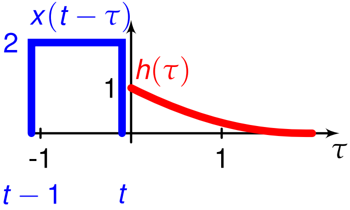

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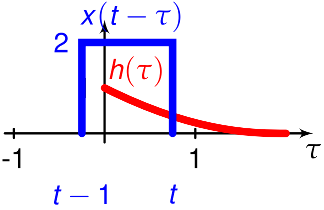
When $0 \leq t < 1$:



the output becomes

$$y(t) = \int_0^t 2e^{-\tau}\mathrm{d}\tau$$

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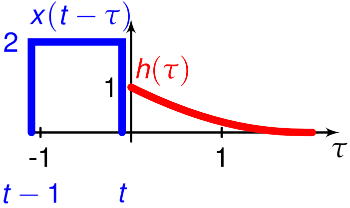

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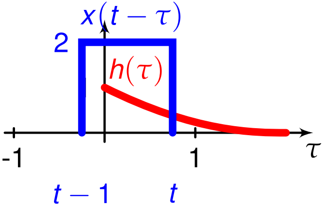
When $0 \leq t < 1$:



the output becomes

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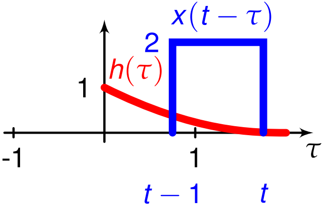
When $t \geq 1$:



the output becomes

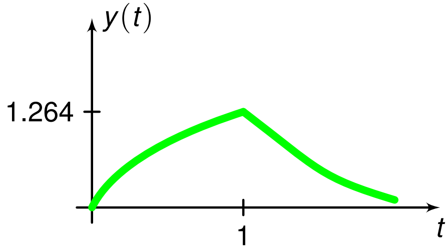
$$y(t) = \int_{t-1}^t 2e^{-\tau}\mathrm{d}\tau$$

Solution 3 (cont.)

Performing the integrations on the previous slide, we get the following output from our system:

$$y(t) = \begin{cases} 0, & t < 0 \\ 2(1 - e^{-t}), & 0 \leq t < 1 \\ 2(e - 1)e^{-t}, & t \geq 1 \end{cases}$$

which looks something like this in a plot:



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Causal systems

A system is said to be **causal** if the reaction always comes after the action[3], i.e., if the output $y(t_0)$ at any given time t_0 is only influenced by inputs up to this time t_0, i.e., influenced only by $x(t)$ for $t \leq t_0$.

[3] All real, physical, systems have this property and what we know as speed of light, c m/s, is actually the speed of causality. If input and output are separated in space by d m, the delay is $\geq d/c$ s.

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This means that for causal systems we have

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{t} x(\tau)h(t-\tau)\mathrm{d}\tau$$

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which implies that τ in the second integral never exeeds t.

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This means that for causal systems we have

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which implies that τ in the second integral never exeeds t.
For the last equality to hold, it is necessary that all causal systems have impulse responses that fulfill

$$h(t) = 0 \text{ for } t < 0.$$

Important property to remember for causal systems!

[3] All real, physical, systems have this property and what we know as speed of light, c m/s, is actually the speed of causality. If input and output are separated in space by d m, the delay is $\geq d/c$ s.

LECTURE SUMMARY

- Linear and time invariant (LTI) systems
  - Linearity
  - Superopsition and linearity condition
  - Time invariance
- Impulses and step functions
  - Dirac delta function/unit impulse and its properties
  - Step function and its properties
  - Generalized functions/distributions
  - Relation between delta function and step function
- LTI systems and impulse response
  - What is the impulse response
  - How can we use it to calculate outputs
  - Convolution
  - Causality and causal systems

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