Chapter 2

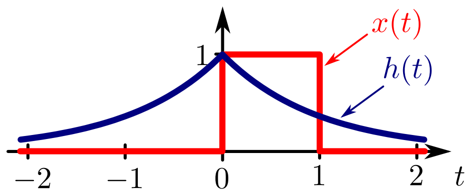
Problem 2.2

Given impulse response

$$h(t) = e^{-|t|}$$

and input signal

$$x(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$



(a) The output signal $y(t)$ is given by the convolution between $h(t)$ and $x(t)$, i.e.,

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\mathrm{d}\tau \quad (1)$$

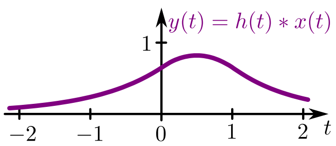
$$= x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau \quad (2)$$

We can use both (1) and (2) for our calculation. Using either these two, the integration can be divided into three different cases.

After performing these three integrals, we can collect them into a single expression gives the output

$$\boxed{y(t) = \begin{cases} e^{t}(1-e^{-1}), & \text{for } t < 0 \\ 2 - e^{t-1} - e^{-t}, & \text{for } 0 \leq t < 1 \\ -e^{-t}(1-e^{1}), & \text{for } t \geq 1 \end{cases}}$$

A plot of $y(t)$ looks like this:



(b) For a system to be causal, the output signal has to be zero as long as the input signal is zero. In this example, the output $y(t)$ is non-zero for all times t, despite the input $x(t)$ being zero for $t < 0$. Hence, **the system cannot be causal**.

Another way of seeing this is that the impulse response $h(t)$ is non-zero for $t < 0$. Any non-zero values on the impulse response at negative times will result in an output before the input "arrives", i.e., the system is non-causal.

(c) The frequency function $H(f)$ of an LTI system is given by the Fourier transform of its impulse response $h(t)$, i.e.,

$$H(f) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\mathrm{d}t\bigg|_{\omega=2\pi f}.$$

Given this, there are basically two ways to approach this problem: i) calculate the Fourier transform of $h(t)$ using the transform integral above, or ii) try to use the table of Fourier transform pairs, where we identify $h(t)$ with one of the given pairs. Like in most cases, the second way is the fastest.

Using either of them, we should end up at

$$H(f) = \boxed{\frac{2}{1+\omega^2}}$$

(d) With a complex sinusoidal input $x(t) = e^{j\omega_0 t}$, whose angular frequency is $\omega = \omega_0$, the corresponding output of the system is "the same complex sinusoidal multiplied by the system's frequency function at that frequency", i.e., (for $\omega_0 = 2\pi f_0$)

$$y(t) = H(f_0)e^{j\omega_0 t} = \boxed{\frac{2}{1+\omega_0^2}e^{j\omega_0 t}}$$

Problem 2.3

Given impulse response

$$h(t) = e^{-t}u(t). \tag{3}$$

(a) The frequency function is given by the Fourier transform of $h(t)$ and, again, there are basically two ways to approach this problem: i) calculate the Fourier transform of $h(t)$ using the transform integral, or ii) try to use the table of Fourier transform pairs, where we identify $h(t)$ with one of the given pairs. Again, the second way is the fastest.

Either way, we should end up at

$$H(f) = \boxed{\frac{1}{1+j\omega}.}$$

(b) With a complex sinusoidal input

$$x(t) = e^{j\omega_0 t}$$

the corresponding output becomes (for $w_0 = 2\pi f_0$)

$$y(t) = H(f_0)e^{j\omega_0 t} = \boxed{\frac{1}{1+j\omega_0}e^{j\omega_0 t}}$$

or, if we take the simplification one step further – which will help us in (c)

$$y(t) = \boxed{\frac{1}{\sqrt{1+\omega_0^2}}e^{j(\omega_0 t - \arctan(\omega_0))}.}$$

An important observation here is that the *magnitude* of the frequency function $|H(\omega_0)| = 1/\sqrt{1+\omega_0^2}$ amplifies the amplitude of the input signal (one in this case) and the *argument* $\angle H(\omega_0) = -\arctan(\omega_0)$ results in a phase shift of the input signal. This is a general result that applies to all LIT systems with (complex) sinusoidal input.

(c) With a real input

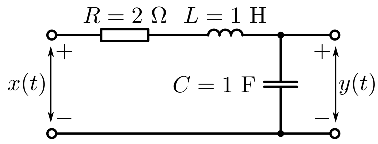
$$x(t) = \cos(\omega_0 t)$$

we have the same amplitude and phase change as in the complex sinusoidal input in (b). Therefore, with input $x(t) = \cos \omega_0 t$ we get output

$$y(t) = \boxed{\frac{1}{\sqrt{1+\omega_0^2}} \cos(\omega_0 t - \arctan \omega_0).}$$

Problem 2.5

Given the circuit



(a) To calculate the frequency function of our circuit, we replace signals and components with their frequency dependent equivalents, i.e., $X(f)$, $Y(f)$, $j\omega L$ and $1/(j\omega C)$, where $\omega = 2\pi f$, as usual. After that we do voltage division to calculate $Y(f)$ as a function of $X(f)$. Rearranging this expression to $H(f) = Y(f)/X(f)$ and performing some simplification gives the frequency function

$$H(f) = \boxed{\frac{1}{(1+j\omega)^2}} \tag{4}$$

Note: The values used on L and C (and R) in the problem are not terribly realistic in most practical situations when dealing with electronics. A 1 H inductor is very large compared to typical inductors with values in the nH–mH range. The same goes for the 1 F capacitor. Typical capacitors used in electronics are in the range pF–mF. Resistors are also rarely in the Ω range, but rather in the range kΩ–MΩ.

(b) Using $\omega_0 = 1$ rad/sec we calculate the frequency function $H(f_0)$ for this value, $f = 2\pi\omega_0$. The amplitude and angle of the complex number $H(f_0)$ tells us how the filter scales the amplitude and delays the phase of an input signal with frequency f_0. The amplitude is scaled by $1/2$ and the phase delayed is $-\pi/2$, which results in (where we have used that $\cos(t - \pi/2) = \sin(t)$)

$$y(t) = \boxed{\frac{1}{2}\sin t.}$$

(c) Since the input $x(t) = u(t)$ is not a simple sinusoid, we need to change our strategy here. There are several different ways of doing this. One is to first find the impulse response, using the (inverse) Fourier transform ... of $H(f)$ from (a). The table of Fourier transform pairs can be used together with a simple convolution (if we see $H(f)$ as the transform of "something" convolved with itself). This gives, after some calculation,

$$h(t) = te^{-t}u(t)$$

which we can use to calculate the ourput, using a convolution. We get

$$y(t) = \boxed{(1 - (t+1)\,e^{-t})u(t).}$$

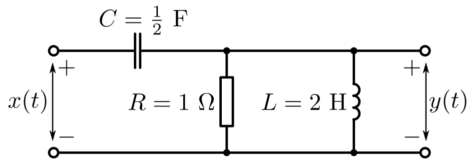
Another way to arrive at this solution is to find the impulse response by, e.g., identifying that

$$H(f) = \frac{1}{(1+j\omega)^2} = -\frac{1}{2\pi f}\frac{d}{df}\left(\frac{1}{1+j\omega}\right)$$

and use transform pair (l), in combination with transform Property 7, in Table 2.3, which relates to differentiation in the frequency domain.

Problem 2.13

Given the circuit



(a) To find the impulse response we first introduce the frequency dependent equivalents of the components and use the fact that the input signal $X(f)$ is split across two impedances, which gives a frequency function

$$H(f) = (j\omega)^2 \frac{1}{(1+j\omega)^2}.$$

Identifying the first factor $(j\omega)^2$ as a double time-differentiation of the corresponding time function. From Problem 2.5 (c) we know that $te^{-t}u(t)$ is the time function corresponding to $1/(1+j\omega)^2$, and we get

$$h(t) = \frac{d^2}{dt^2}\left(te^{-t}u(t)\right)$$

which we can differentiate using the chain rule and finally obtain

$$h(t) = \boxed{\delta(t) + e^{-t}(t-2)u(t)}$$

Problem 2.16

Given the LTI system impulse response

$$h(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

and input signal

$$x(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

the output can be calculated using the convolution integral, using four different cases. Combining all four cases into a single output function we get

$$y(t) = \boxed{\begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 \leq t < 1 \\ \frac{1}{2}(2-t)t, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases}}$$

Problem 2.17

With the basis signal

$$\psi(t) = \mathrm{rect}(t/T)$$

(a) The Fourier bandwidth determines how much bandwidth $\psi(t)$ takes up in the frequency domain (as described by its Fourier transform). Since $\psi(t)$ is time-limited, $\Psi(f)$ will extend to infinity. Since infinite bandwidth is not meaningful, we define the bandwidth as the largest frequency of the main lobe of $\Psi(f)$, that is

$$W = \boxed{\frac{1}{T}.}$$

(b) The Nyquist shift is given by the smallest time T_N which makes $\phi(t)$ orthogonal to all time-shifted copies $\phi(t - kT_N)$, for non-zero integers k. For the given $\psi(t)$ we get

$$T_N = \boxed{T.}$$

(c) The Shannon bandwidth describes how much bandwidth is "needed" (in terms of number of orthogonal dimensions per second) and for the given $\phi(t)$ it is

$$B_{\text{Shannon}} = \boxed{\frac{1}{2T}.}$$

Problem 2.18

With the basis signal

$$\psi(t) = \begin{cases} 1 - \frac{2}{T}|t|, & |t| < T/2 \\ 0, & \text{otherwise} \end{cases}$$

solve the same sub-problems as in Problem 2.17.

(a) The Fourier bandwidth becomes (again, largest frequency of main lobe of $\Psi(f)$)

$$W = \boxed{\frac{2}{T}.}$$

(b) The Nyquist shift for the given $\psi(t)$ is

$$T_N = \boxed{T.}$$

(c) The Shannon bandwidth for the given $\phi(t)$ is

$$B_{\text{Shannon}} = \boxed{\frac{1}{2T}}.$$