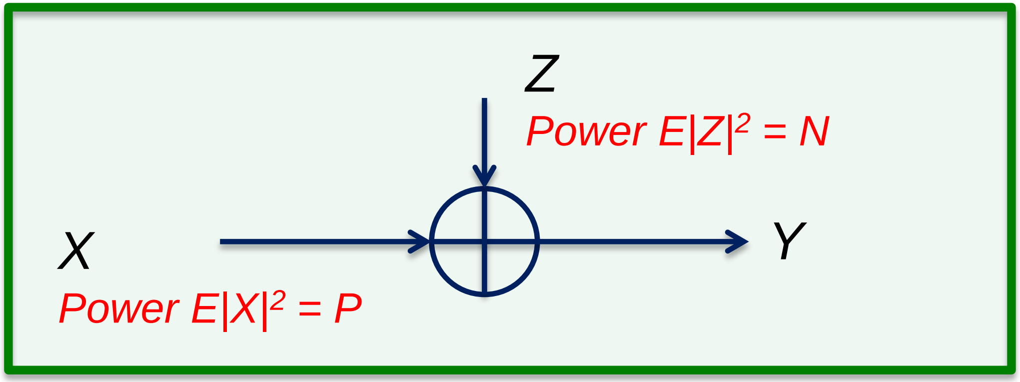
Digital Communications

Fredrik Rusek

Chapter 14,
Fading Channels II: Capacity (but almost no coding)
Proakis-Salehi

Brief review of capacity

Capacity of AWGN channel (real-valued)



$$C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

Brief review of capacity

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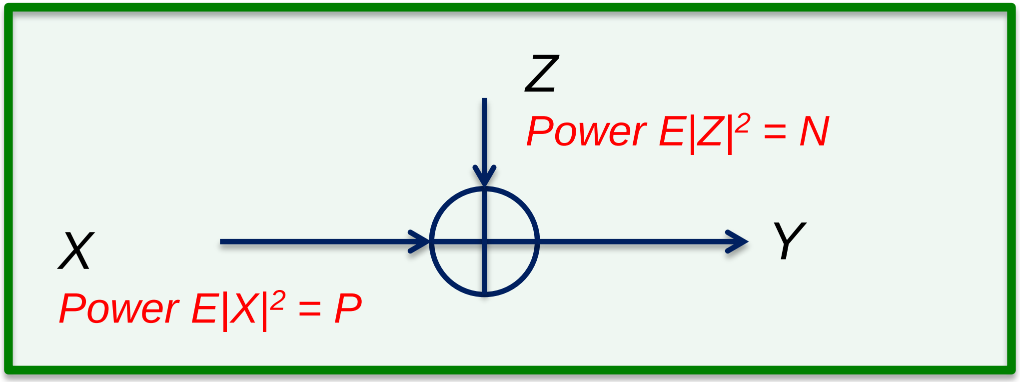


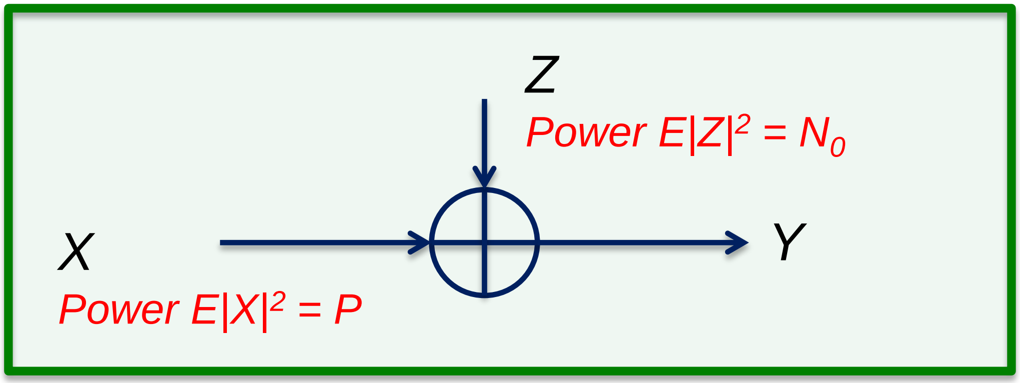
$$C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

Let $N=N_0/2$

$$C = \frac{1}{2}\log\left(1 + \frac{2P}{N_0}\right)$$

Brief review of capacity

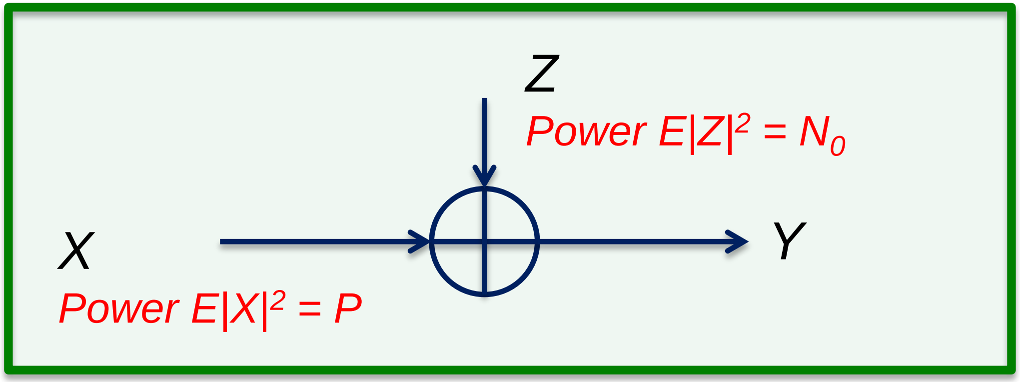
Capacity of AWGN channel **(Complex valued)**



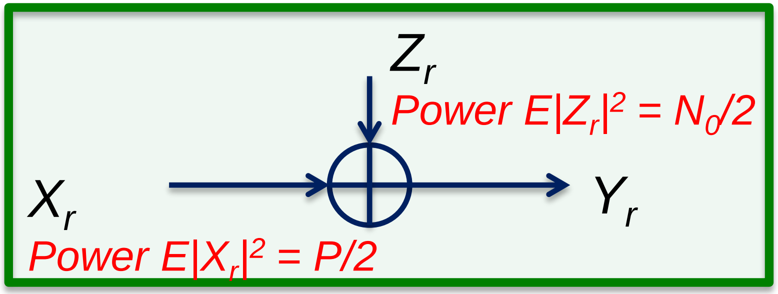
Note: N_0 and not $N_0/2$

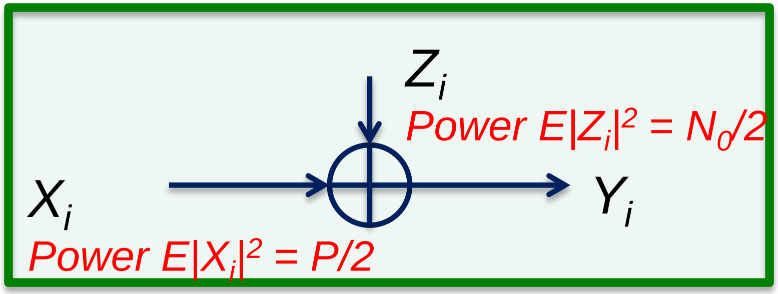
Brief review of capacity

Capacity of AWGN channel **(Complex valued)**



A complex channel is equivalent to two real valued channels.





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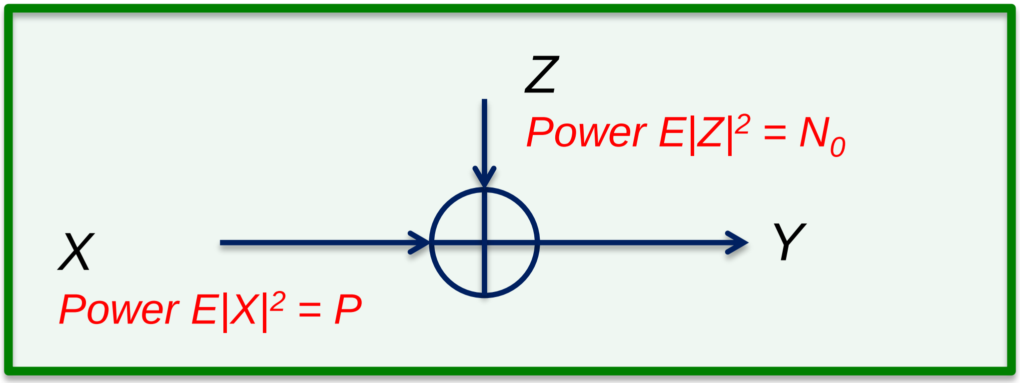

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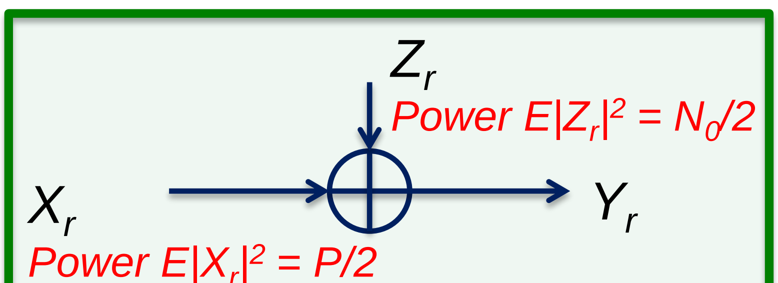

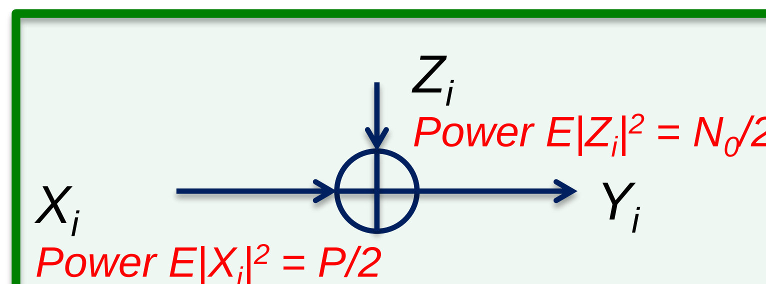

$$\boxed{C = \log\left(1 + \frac{P}{N_0}\right)}$$

Brief review of capacity

Assume a channel bandlimited to W positive Hz and input power of P W

Due to the sampling theorem, bandwidth W specifies 2W orthonormal dimensions per second. Power in each is P/2W

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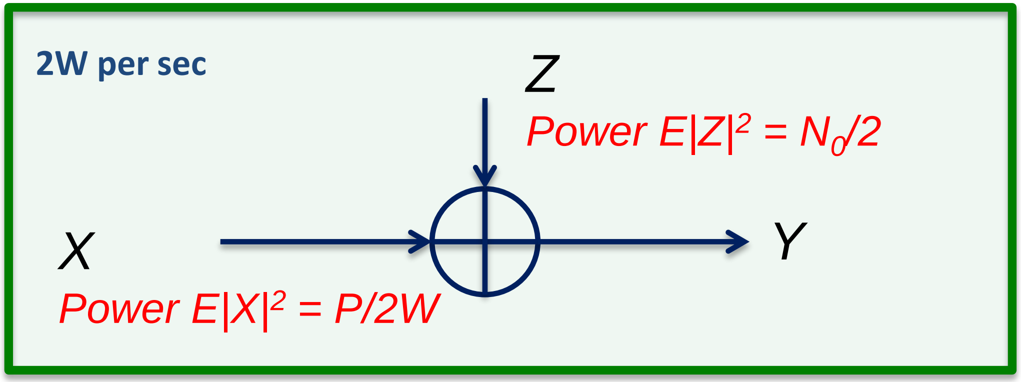

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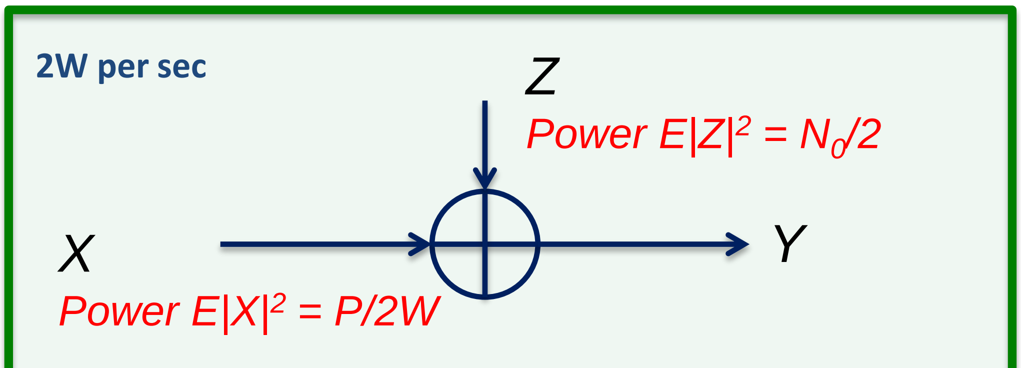

Capacity per channel use $C = \frac{1}{2} \log\left(1 + \frac{P}{N_0 W}\right)$

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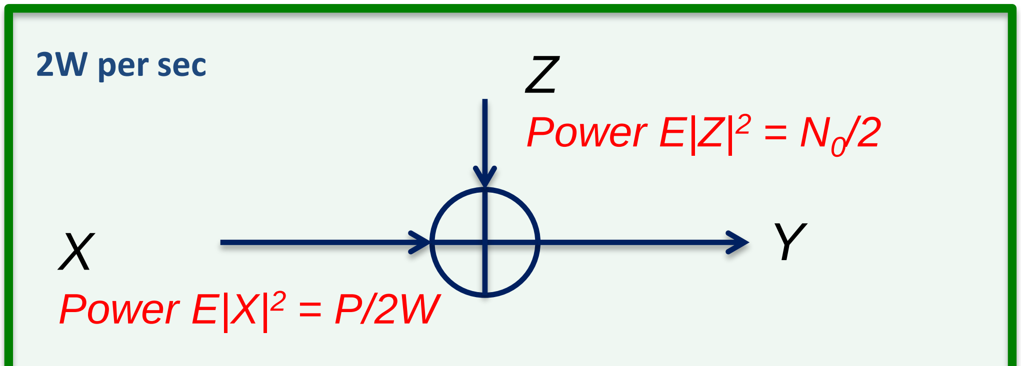

Capacity per channel use $$C = \frac{1}{2}\log\left(1 + \frac{P}{N_0 W}\right)$$

Capacity per second $$C = W\log\left(1 + \frac{P}{N_0 W}\right)$$

Brief review of water filling

Now assume a frequency selective channel $y(t) = x(t) \star c(t) + n(t)$

From chapter 11 (easy)

$$C = \frac{1}{2}\int_{-\infty}^{\infty} \log\left(1 + \frac{P(f)|C(f)|^2}{S_n(f)}\right) df$$

Waterfilling: **Graphical solution to the optimization problem**

$$\max_{P(f)} \quad C = \frac{1}{2}\int_{-\infty}^{\infty} \log\left(1 + \frac{P(f)|C(f)|^2}{S_n(f)}\right) df$$

$$\text{S.t.} \quad \int_{-\infty}^{\infty} P(f)\, df = 1$$

Brief review of water filling

Solution.

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Add a Lagrange multiplier to handle the constraint

$$\int_W \left\{ \log_2\left[1 + \frac{P(f)|C(f)|^2}{S_{nn}(f)}\right] + \lambda P(f) \right\} df$$

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Euler - Lagrange equation

With $S(q) = \int_a^b L(t, q(t), q'(t))\, \mathrm{d}t$

condition for a stationary point

$$L_q(t, q(t), q'(t)) - \frac{\mathrm{d}}{\mathrm{d}t} L_{q'}(t, q(t), q'(t)) = 0.$$

S **is functional (function from the space of functions to the reals)**

q **is function of** ***t***

L **is function of three variables**

Brief review of water filling

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In our case:

- *P is q, f is t.*
- *Derivative of L with respect to q' is 0*

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$$\text{S.t.} \quad \int_{-\infty}^{\infty} P(f)\, df = 1$$

This gives

$$\frac{1}{P(f) + S_{nn}(f)/|C(f)|^2} + \lambda = 0$$

$$P(f) + S_{nn}(f)|C(f)|^2$$ is a constant adjusted to satisfy the power constraint

Waterfilling

$$\boxed{P(f) = \left(K - \frac{S_n(f)}{|C(f)|^2}\right)^+}$$

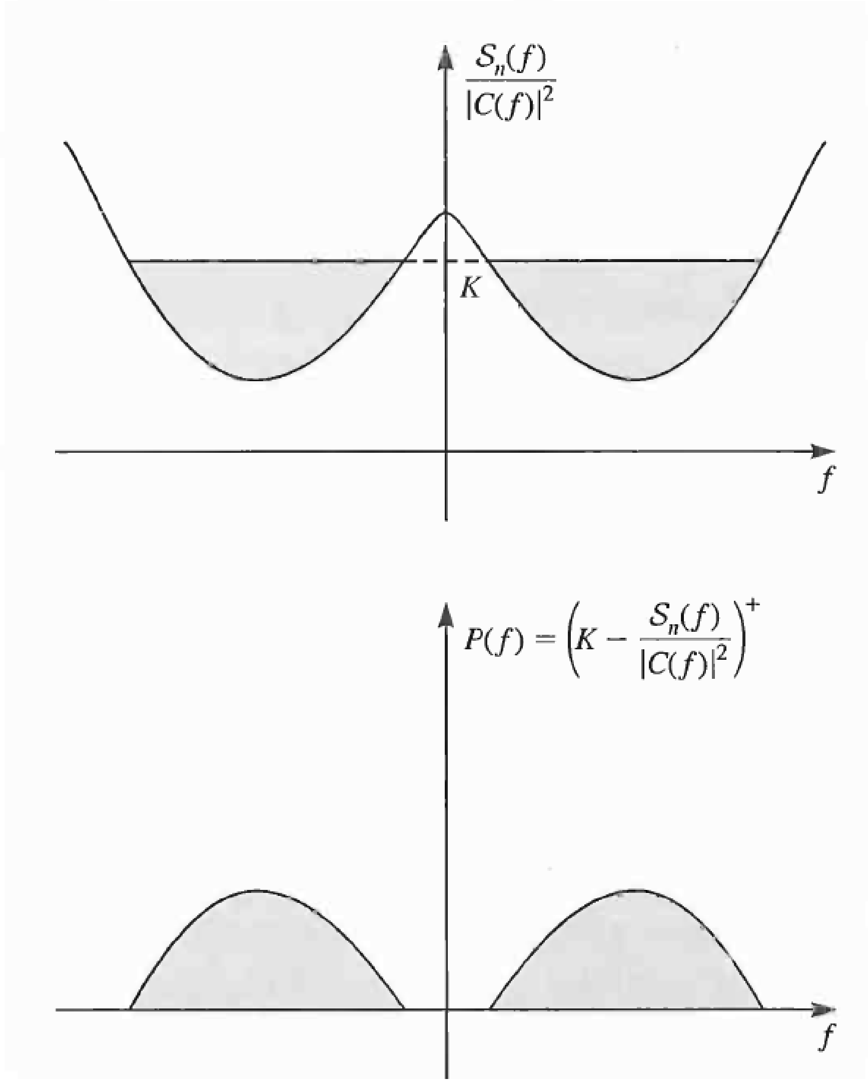
$$x^+ = \max\{0, x\}$$

Brief review of water filling

This gives

$$\frac{1}{P(f) + \mathcal{S}_{nn}(f)/|C(f)|^2} + \lambda = 0$$

$$P(f) + \mathcal{S}_{nn}(f)|C(f)|^2$$



Waterfilling

$$\boxed{P(f) = \left(K - \frac{\mathcal{S}_n(f)}{|C(f)|^2}\right)^+}$$

$$x^+ = \max\{0, x\}$$

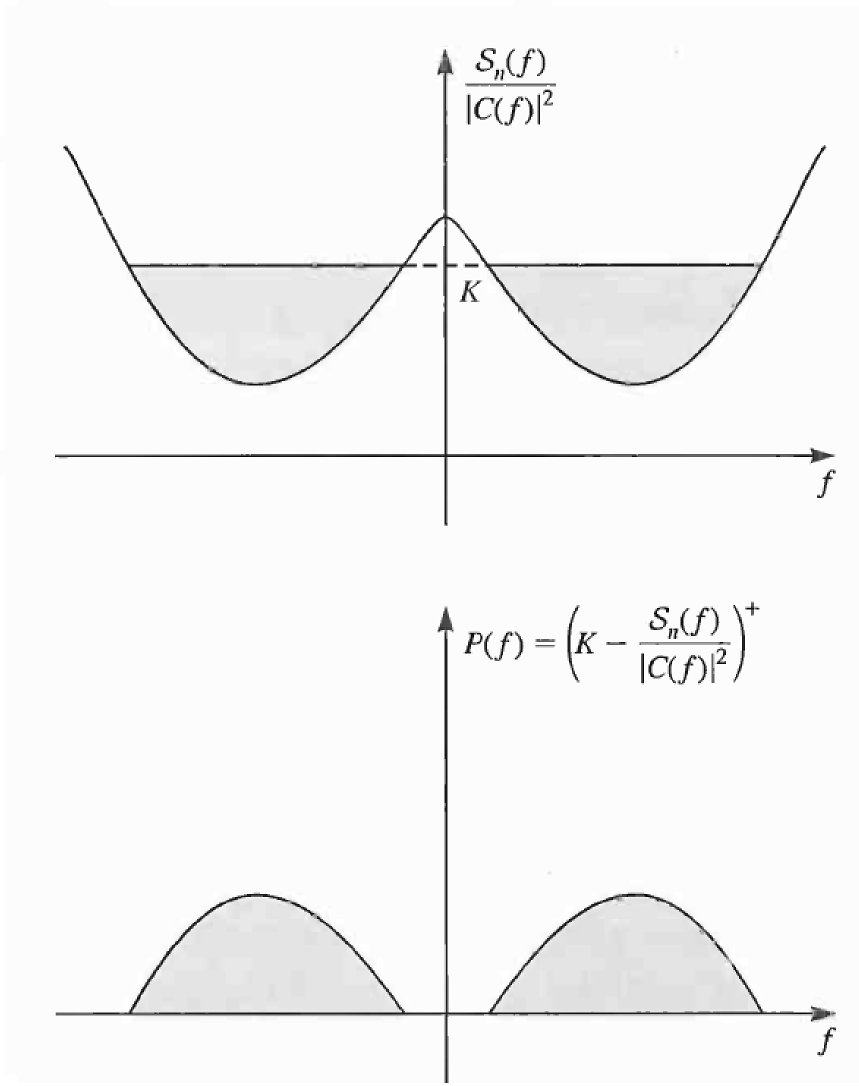
Brief review of water filling

Works also for N discrete parallel channels

$$C = \frac{1}{2}\sum_{i=1}^{N}\log\left(1+\frac{P_i}{N_i}\right)$$

Waterfilling

$$\boxed{\begin{gathered} P_i = (K - N_i)^+ \\ \sum_{i=1}^{N} P_i = P \end{gathered}}$$



14.1-1 finite state machines

Warning: hard to read. (p.904)

$$C = \max_{p(t)} I(T; Y|V) \qquad (14.1\text{–}19)$$

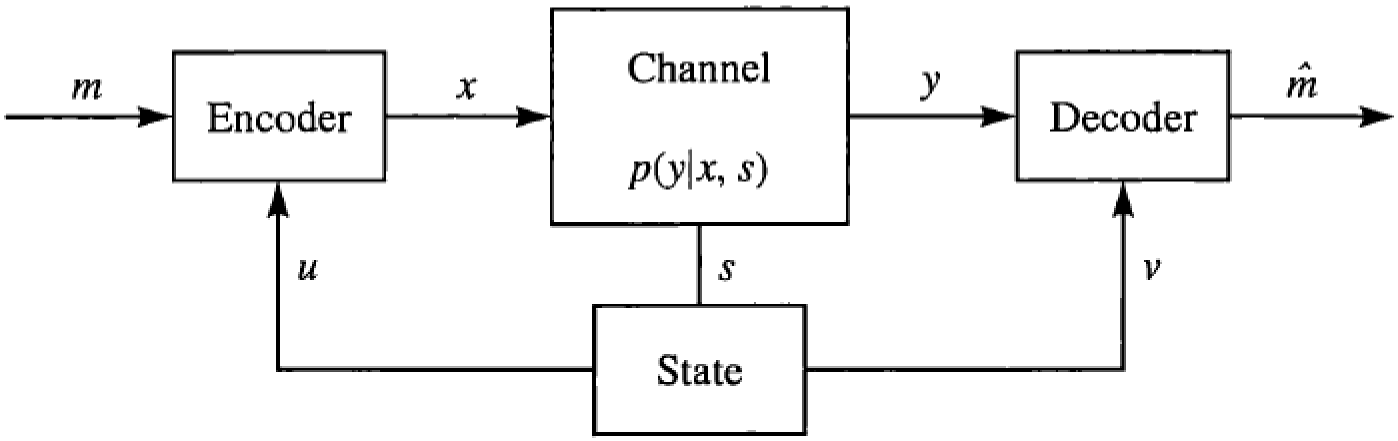
T is (at least to me) completely undefined. I don't know what it is.

Switch these two

ular interest. The special case where $\underline{U} = S$ and $\underline{V}$ is a degenerate random variable corresponds to the case when complete *channel state information* (CSI) is available at the receiver and no channel state information is available at the transmitter. In this case

14.1-1 finite state machines

System model

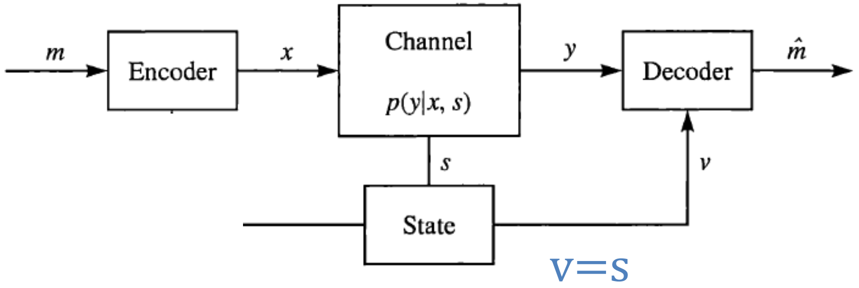


- Channel behavior changes over time. Channel is described by variable *s*
- A finite number of different *s*
- Transmitter has some knowledge about the state, represented by u
- Receiver has some knowledge about the state, represented by v
- u = s = v is perfect CSI both at Rx and Tx
- u = 0. v=s is perfect CSI at Rx, no CSI at Tx

What is the capacity?

14.1-1 finite state machines

Let us start with the case of perfect CSI at Rx, no CSI at Tx



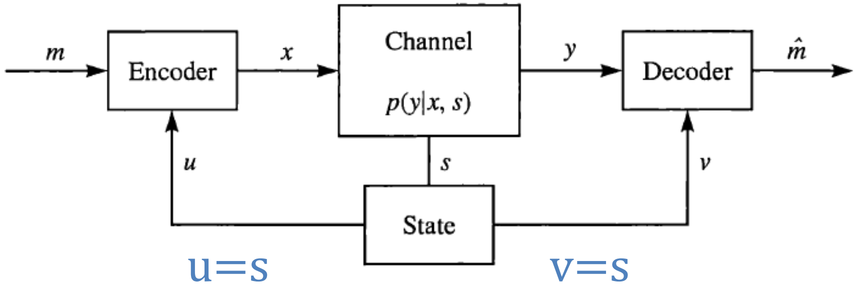
Capacity is [Schlomo Shamai, Guiseppe Caire '1999]

$$C = \max_{p(x)} I(X;Y|S) = \max_{p(x)} \sum_{s} p(s) I(X;Y|S=s)$$

The task of the Tx is to choose the distribution on x, that maximizes the averaged mutual information over the channel states.

14.1-1 finite state machines

Perfect CSI at Rx and Tx



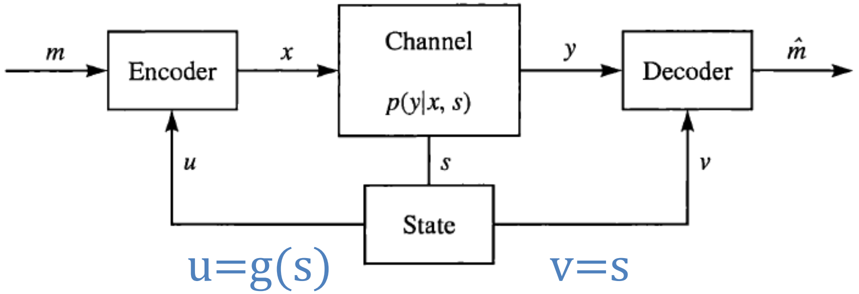
Capacity is

$$C = \max_{p(x|s)} I(X;Y|S) = \sum_{s} p(s) \max_{p(x|s)} I(X;Y|S=s)$$

The task of the Tx is to choose the distribution on x, that maximizes the instantaneous mutual information.

This is the expectation of the channel capacity

14.1-1 finite state machines



Limited feedback

g(s) can be quantization

Capacity is

$$C = \sum_{u} p(u) \max_{p(x|u)} I(X;Y|S,U=u)$$

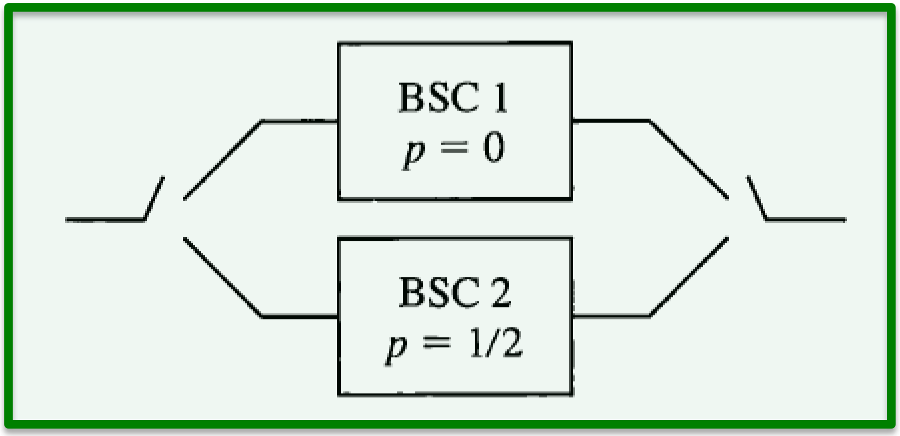
The task of the Tx is to choose the distribution on x, that maximizes the average of mutual informations over the channels that can be expected to occur around a certain u

14.2 Ergodic and outage capacity

System model

Capacity $C_1 = 1$

$C_2 = 0$



Either a perfect channel is used (BSC 1), or a useless channel (BSC 2)

1. In channel model 1 the input and output switches choose the top channel (BSC 1) with probability δ and the bottom channel (BSC 2) with probability $1-\delta$, *independently for each transmission*. In this channel model each symbol is transmitted independently of the previous symbols, and the state of the channel is also selected independently for each symbol.
2. In channel model 2 the top and the bottom channels are selected at the beginning of the transmission with probabilities δ and $1-\delta$, respectively; but once a channel is selected, it will not change for the entire transmission period.

14.2 Ergodic and outage capacity

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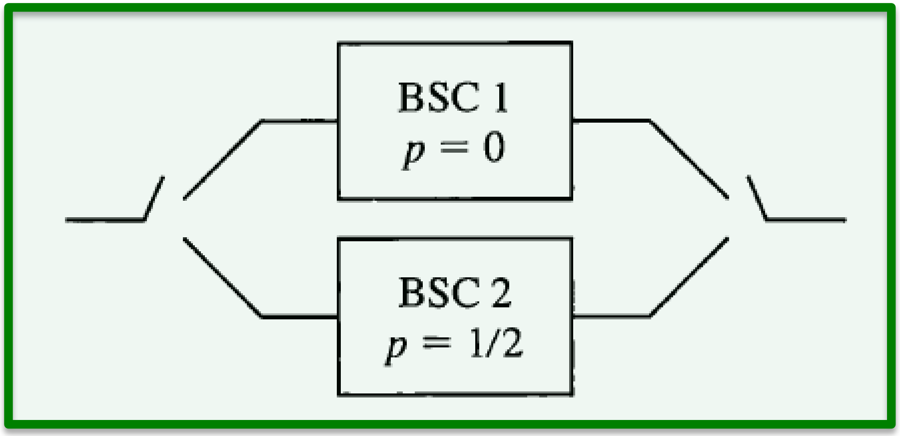

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Ergodic capacity. In a fraction δ of the time, the capacity is 1. In (1-δ), the capacity is 0. The ergodic capacity is the mean capacity that will be experienced over a very long period of time.

This capacity can be reached with a very long codeword spanning all states of the channel.

14.2 Ergodic and outage capacity

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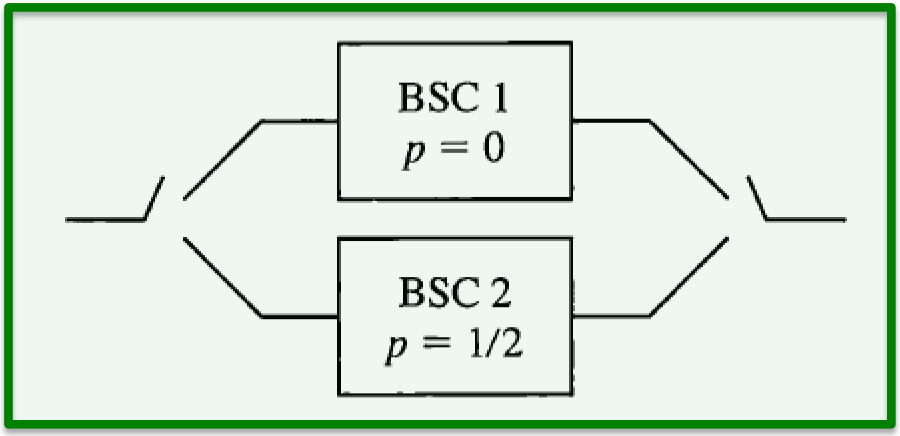

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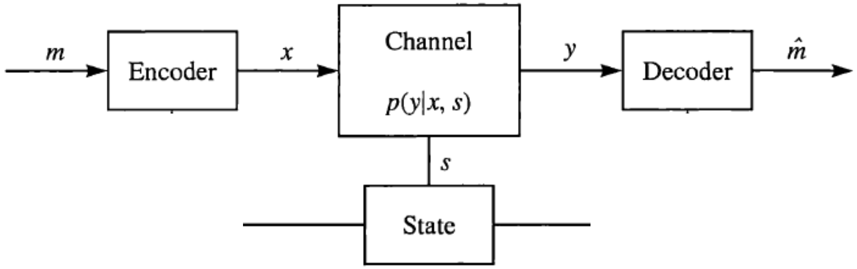
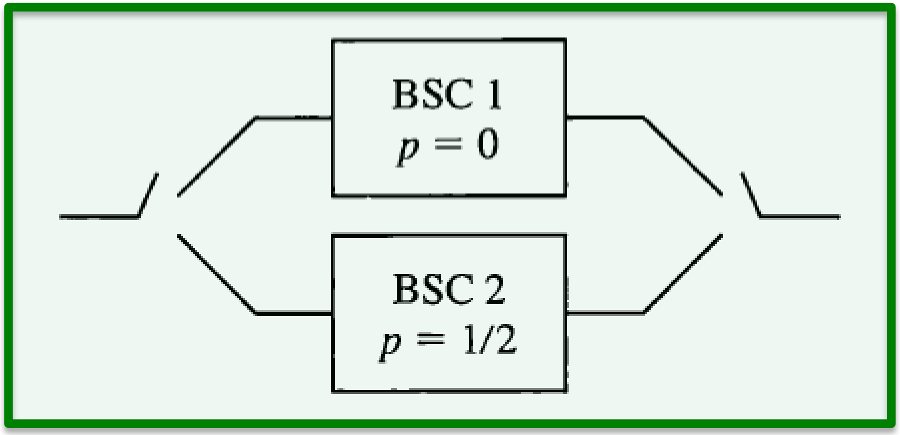
Now further break case 1 this into 3 sub-cases

14.2 Ergodic and outage capacity

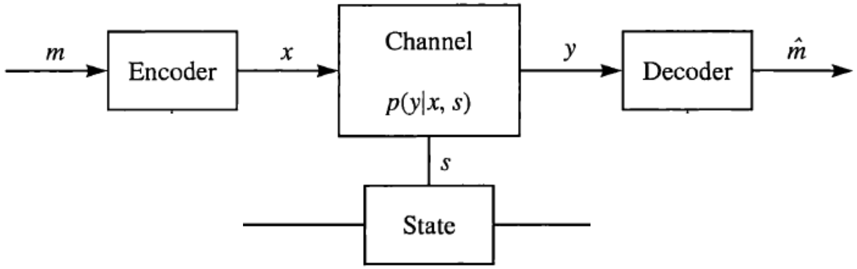


Sub case 1: No CSI at Rx or Tx



In this case an amount (1-δ)/2 of the bits are in error, the errors are randomly spread over the block, as the switching between BSC 1 and BSC 2 is random.

14.2 Ergodic and outage capacity



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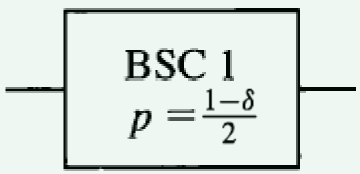
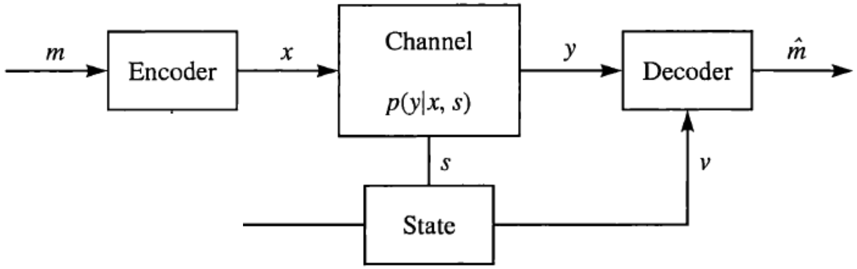


$$\overline{C} = 1 - H_b\left(\frac{1-\delta}{2}\right)$$

In this case an amount (1-δ)/2 of the bits are in error, the errors are randomly spread over the block, as the switching between BSC 1 and BSC 2 is random.

Scenario is indistinguishable from a single BSC

14.2 Ergodic and outage capacity

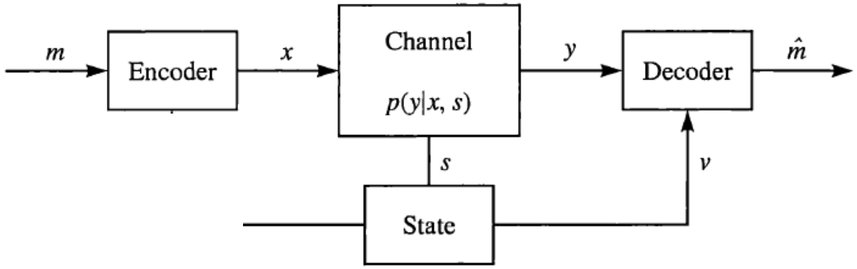


Sub case 2: CSI at Rx

$$C = \max_{p(x)} I(X;Y|S) = \max_{p(x)} \sum_{s} p(s) I(X;Y|S=s)$$

Find the input distribution that maximizes the average of the capacities.

14.2 Ergodic and outage capacity



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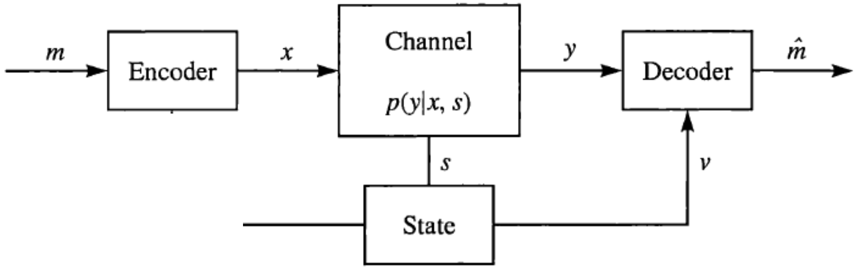
However, the capacity of a BSC channel is maximized by a uniform input distribution. That is, p(x=1)=p(x=0)=0.5

14.2 Ergodic and outage capacity



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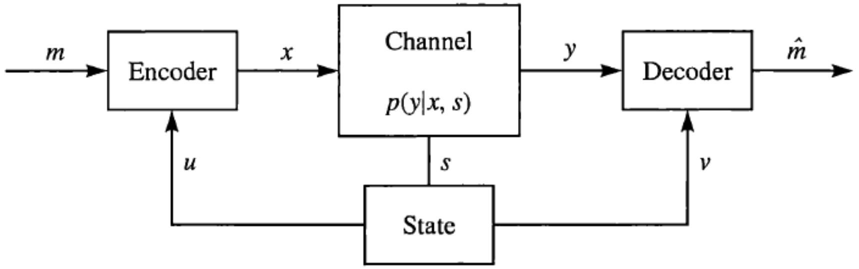
$$\overline{C} = \delta C_1 + (1-\delta) C_2 = \delta$$

14.2 Ergodic and outage capacity

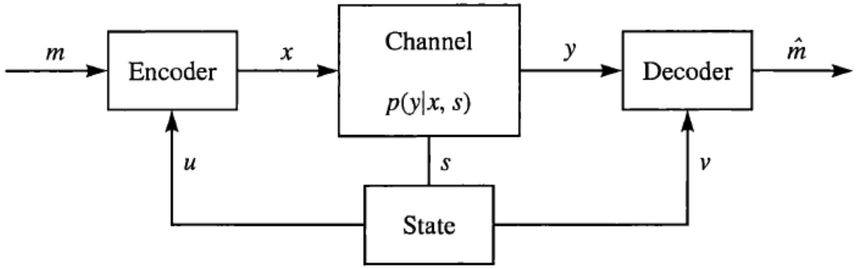


Sub case 3: CSI at Rx and Tx

$$C = \max_{p(x|s)} I(X;Y|S) = \sum_{s} p(s) \max_{p(x|s)} I(X;Y|S=s)$$

Find the input distributions that maximize the capacities of the two channels individually.

14.2 Ergodic and outage capacity



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Find the input distributions that maximize the capacities of the two channels individually.

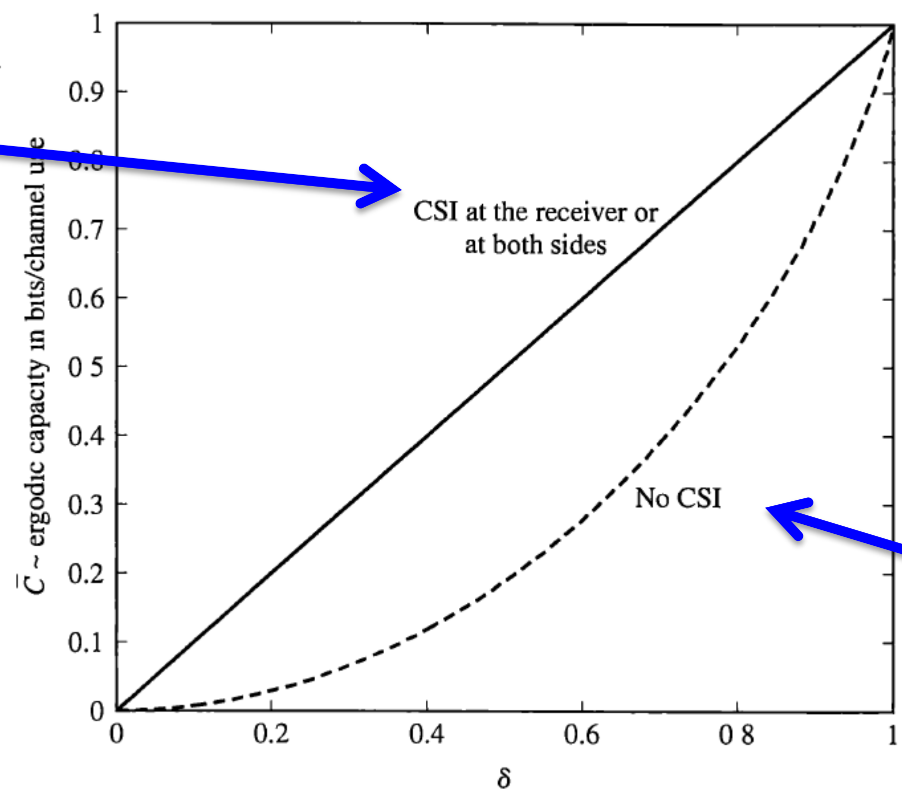
In both cases, this is the uniform distribution, so Tx feedback does not help in this case

$$\overline{C} = \delta C_1 + (1-\delta) C_2 = \delta$$

14.2 Ergodic and outage capacity

Comparison of sub case 1 and 2/3

$$\overline{C} = \delta C_1 + (1-\delta)C_2 = \delta$$



$$\overline{C} = 1 - H_b\left(\frac{1-\delta}{2}\right)$$

14.2 Ergodic and outage capacity

Case 2

2. In channel model 2 the top and the bottom channels are selected at the beginning of the transmission with probabilities δ and $1-\delta$, respectively; but once a channel is selected, it will not change for the entire transmission period.

Suppose that we select a transmission rate R>0

14.2 Ergodic and outage capacity

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Now suppose that BSC 2 is selected

Capacity $C_2 = 0 < R$

14.2 Ergodic and outage capacity

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Suppose that we select a transmission rate R>0

Now suppose that BSC 2 is selected (happens with non-zero probability)

Capacity $C_2 = 0 < R$

Hence, reliable communication at any nonzero rate is not possible

Shannon capacity of the channel is 0

14.2 Ergodic and outage capacity

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2. In channel model 2 the top and the bottom channels are selected at the beginning of the transmission with probabilities δ and $1 - \delta$, respectively; but once a channel is selected, it will not change for the entire transmission period.

In case 2, the channel capacity is a random function (0 or 1).

The notion of ergodic capacity is not applicable

This calls for the concept of outage capacity

14.2 Ergodic and outage capacity

Case 2

If we transmit at a rate R>0, there is a probability that the channel capacity C is less than R, and then the channel is in *outage.*

Probability of this is called outage probability

$$\boxed{\mathrm{P}_{\mathrm{out}}(R) = \mathrm{P}\left[C < R\right] = F_C(R^-)}$$

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CDF of C

Limit of approaching R from the left

14.2 Ergodic and outage capacity

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ε-outage-capacity: Highest rate that keeps the outage probability below ε

$$\boxed{C_\epsilon = \max\left\{R : \mathrm{P}_{\mathrm{out}}(R) \leq \epsilon\right\}}$$

14.2 Ergodic and outage capacity

In the previous 2-BSC example, we get the following outage capacity

$$\boxed{C_\epsilon = \begin{cases} 0 & \text{for } 0 \le \epsilon < 1-\delta \\ 1 & \text{for } 1-\delta \le \epsilon < 1 \end{cases}}$$

$$\boxed{C_\epsilon = \max\{R : \mathrm{P}_{\mathrm{out}}(R) \le \epsilon\}}$$

14.2-1 Ergodic capacity of Rayleigh fading

$$y_i = R_i x_i + n_i$$

$$\mathcal{CN}(0, N_0)$$

All variables are complex.
R_i has Rayleigh distributed amplitude, and uniform phase

$$p(r) = \begin{cases} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} & r > 0 \\ 0 & r \le 0 \end{cases}$$

$$E[R^2] = 2\sigma^2 \qquad \rho = |R_i|^2$$

$$p(\rho) = \begin{cases} \frac{1}{2\sigma^2} e^{-\frac{\rho}{2\sigma^2}} & \rho > 0 \\ 0 & \rho \le 0 \end{cases}$$

14.2-1 Ergodic capacity of Rayleigh fading

$$y_i = R_i x_i + n_i$$

$\mathcal{CN}(0, N_0)$

All variables are complex.
R_i has Rayleigh distributed amplitude, and uniform phase

$$p(r) = \begin{cases} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} & r > 0 \\ 0 & r \leq 0 \end{cases}$$

Means that R_i is A_i+iB_i, where A_i and B_i are independent normal variables with variance σ^2

$$E[R^2] = 2\sigma^2 \qquad \rho = |R_i|^2$$

$$p(\rho) = \begin{cases} \frac{1}{2\sigma^2} e^{-\frac{\rho}{2\sigma^2}} & \rho > 0 \\ 0 & \rho \leq 0 \end{cases}$$

Means that ρ is chi-2 with 2 degrees of freedom

14.2-1 Ergodic capacity of Rayleigh fading

$$y_i = R_i x_i + n_i$$

$\mathcal{CN}(0, N_0)$

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R_i has Rayleigh distributed amplitude, and uniform phase

$$p(r) = \begin{cases} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} & r > 0 \\ 0 & r \le 0 \end{cases}$$

$$E[R^2] = 2\sigma^2 \qquad \rho = |R_i|^2$$

$$p(\rho) = \begin{cases} \frac{1}{2\sigma^2} e^{-\frac{\rho}{2\sigma^2}} & \rho > 0 \\ 0 & \rho \le 0 \end{cases}$$

Received power

$$P_r = 2\sigma^2 P_t$$

Assume

$$2\sigma^2 = 1$$

Thus

$$P_t = P_r = P$$

14.2-1 Ergodic capacity of Rayleigh fading

No CSI at Rx or Tx

In this case, we can drop the sub-index ”i” on the signals, as the channel is totally random identically distributed from one time instant to the other.

$$y = Rx + n$$

$\mathcal{CN}(0, 2\sigma^2)$

$\mathcal{CN}(0, N_0)$

14.2-1 Ergodic capacity of Rayleigh fading

No CSI at Rx or Tx

In this case, we can drop the sub-index ”i” on the signals, as the channel is totally random identically distributed from one time instant to the other.

$$y = Rx + n$$

Phase of R is unknown and uniform -> phase of x cannot carry information.

(Given the phase of x, the phase of y is still uniformly distributed over [0,2π])

To compute capacity, we need p(y|x)

$$p(y|x) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} p(y|x, r, \theta) p(r)\, dr\, d\theta$$

14.2-1 Ergodic capacity of Rayleigh fading

No CSI at Rx or Tx

In this case, we can drop the sub-index ”i” on the signals, as the channel is totally random identically distributed from one time instant to the other.

$$y = Rx + n$$

Phase of R is unknown and uniform -> phase of x cannot carry information.

(Given the phase of x, the phase of y is still uniformly distributed over [0,2π])

To compute capacity, we need p(y|x)

$$\boxed{p(y|x) = \frac{1}{\pi\left(N_0 + |x|^2\right)} e^{-\frac{|y|^2}{N_0+|x|^2}}}$$

Clearly seen that all phase information in x has been lost

14.2-1 Ergodic capacity of Rayleigh fading

No CSI at Rx or Tx

To find capacity entails the expectations

Entropy of y: $H(y) = - E_y \log_2 [p(y)]$

Entropy of y|x: $H(y|x) = - E_y \log_2 [p(y|x)]$

Then, one should optimize over the distribution of x

This optimization does not allow for a closed-form solution

14.2-1 Ergodic capacity of Rayleigh fading

No CSI at Rx or Tx

To find capacity entails the expectations

Entropy of y: $H(y) = - E_y \log_2 [p(y)]$

Entropy of y|x: $H(y|x) = - E_y \log_2 [p(y|x)]$

Then, one should optimize over the distribution of x

This optimization does not allow for a closed-form solution

Facts about it

- **Gaussian inputs are not optimal (they are with Rx CSI)**
- **Optimal distribution is discrete**
- **At low SNR, on-off keying is optimal**
- **At low SNR, ergodic capacity is**

$$\boxed{\overline{C} = \frac{1}{\ln 2} \frac{P}{N_0} \approx 1.44 \frac{P}{N_0}}$$

14.2-1 Ergodic capacity of Rayleigh fading

No CSI at Rx or Tx

With full CSI at the Rx, we get

$$C = \log\left(1 + \frac{P}{N_0}\right)$$

With small P, we get

$$C = \frac{1}{\ln 2}\frac{P}{N_0} \approx 1.44\,\frac{P}{N_0}$$

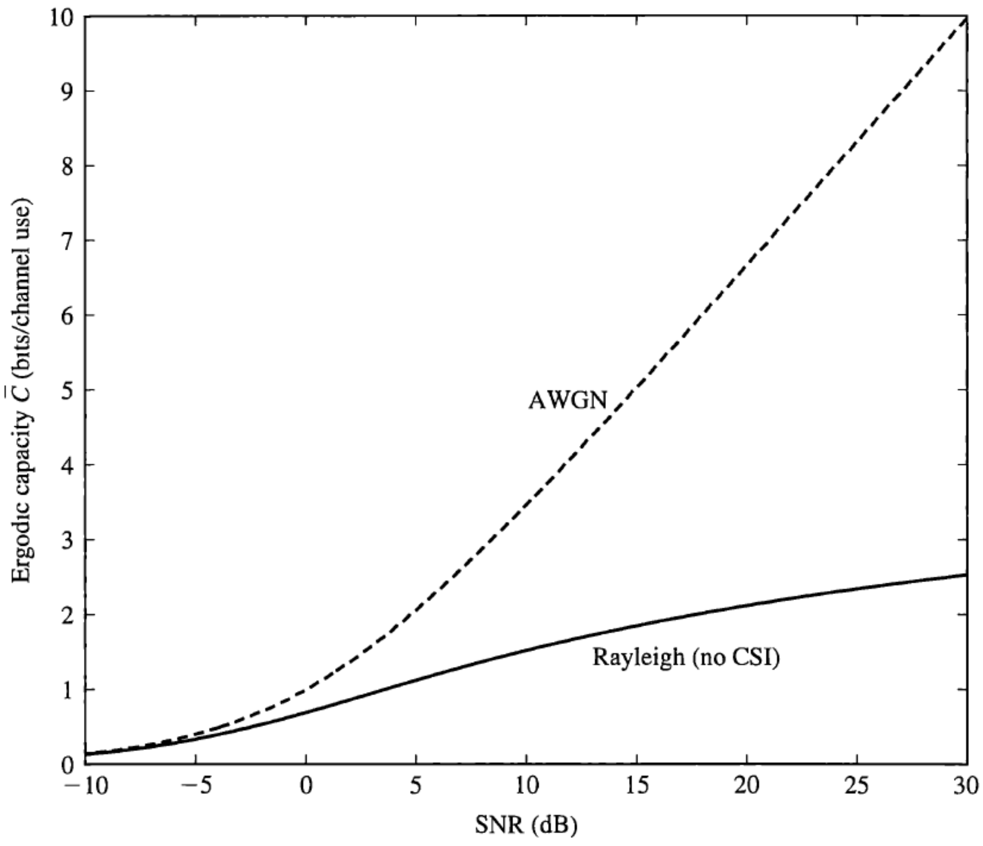
Hence, at low SNR, the loss of having no CSI at Rx vanishes

- At low SNR, ergodic capacity is

$$\boxed{\overline{C} = \frac{1}{\ln 2}\frac{P}{N_0} \approx 1.44\,\frac{P}{N_0}}$$

14.2-1 Ergodic capacity of Rayleigh fading

No CSI at Rx or Tx



≈ log (SNR)

≈ log log (SNR)

14.2-1 Ergodic capacity of Rayleigh fading

CSI at Rx

With Rx CSI, the receiver can compensate for the phase of the channel

We can wlog assume the channel R to be real

The effect is a scaling of the SNR by an amount ρ

Ergodic capacity is the expectation of

$$\boxed{C = \log\left(1 + \rho\frac{P}{N_0}\right)}$$

14.2-1 Ergodic capacity of Rayleigh fading

CSI at Rx

Log is a concave function -> Jensen's inequality applies:

$$\begin{aligned}\overline{C} &= \mathrm{E}\left[\log\left(1+\rho\frac{P}{N_0}\right)\right] \\ &\leq \log\left(1+\mathrm{E}\left[\rho\right]\frac{P}{N_0}\right) \\ &= \log\left(1+\frac{P}{N_0}\right)\end{aligned}$$

Ergodic capacity of time-varying channel is never better than the capacity of a static channel with the same SNR

14.2-1 Ergodic capacity of Rayleigh fading

CSI at Rx

Exact computation of ergodic capacity

$$p(\rho) = \begin{cases} \frac{1}{2\sigma^2}\, e^{-\frac{\rho}{2\sigma^2}} & \rho > 0 \\ 0 & \rho \le 0 \end{cases}$$

$$2\sigma^2 = 1$$

$$p(\rho) = \begin{cases} e^{-\rho} & \rho > 0 \\ 0 & \rho \le 0 \end{cases}$$

14.2-1 Ergodic capacity of Rayleigh fading

CSI at Rx

Exact computation of ergodic capacity

$$
\begin{aligned}
\overline{C} &= \int_0^{\infty} \log\left(1 + \rho \frac{P}{N_0}\right) e^{-\rho}\, d\rho \\
&= \frac{1}{\ln 2}\, e^{\frac{N_0}{P}}\, \Gamma\left(0, \frac{N_0}{P}\right) \\
&= \frac{1}{\ln 2}\, e^{\frac{1}{\mathrm{SNR}}}\, \Gamma\left(0, \frac{1}{\mathrm{SNR}}\right)
\end{aligned}
$$

$$\Gamma(a, z) = \int_z^{\infty} t^{a-1} e^{-t}\, dt$$

complementary gamma function

14.2-1 Ergodic capacity of Rayleigh fading

CSI at Rx

Exact computation of ergodic capacity

$$\overline{C} = \int_0^\infty \log\left(1 + \rho\frac{P}{N_0}\right) e^{-\rho}\, d\rho$$
$$= \frac{1}{\ln 2} e^{\frac{N_0}{P}}\, \Gamma\left(0, \frac{N_0}{P}\right)$$
$$= \frac{1}{\ln 2} e^{\frac{1}{\mathrm{SNR}}}\, \Gamma\left(0, \frac{1}{\mathrm{SNR}}\right)$$

$$\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t}\, dt$$

complementary gamma function

Low SNR

Cannot be worse than no Rx CSI
Cannot be better than AWGN
due to Jensen's

$$\overline{C} = 1.44\,\mathrm{SNR}$$

14.2-1 Ergodic capacity of Rayleigh fading

CSI at Rx

Exact computation of ergodic capacity

$$\overline{C} = \int_0^\infty \log\left(1 + \rho \frac{P}{N_0}\right) e^{-\rho}\, d\rho$$

$$= \frac{1}{\ln 2} e^{\frac{N_0}{P}}\, \Gamma\left(0, \frac{N_0}{P}\right)$$

$$= \frac{1}{\ln 2} e^{\frac{1}{\mathrm{SNR}}}\, \Gamma\left(0, \frac{1}{\mathrm{SNR}}\right)$$

$$\boxed{\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t}\, dt}$$

complementary gamma function

Low SNR

Cannot be worse than no Rx CSI
Cannot be better than AWGN
due to Jensen's

$$\overline{C} = 1.44\,\mathrm{SNR}$$

High SNR

$$\log\left(1 + \rho \frac{P}{N_0}\right) \approx \log\left(\rho \frac{P}{N_0}\right)$$

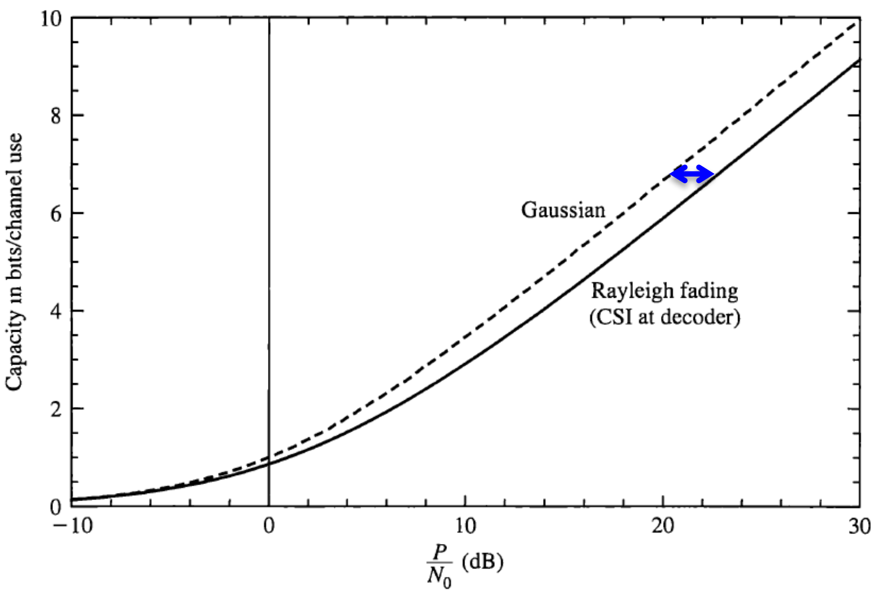
$$\overline{C} \approx \frac{1}{\ln 2} \int_0^\infty \log\left(\rho \frac{P}{N_0}\right) e^{-\rho}\, d\rho$$

$$= \log \mathrm{SNR} + \frac{1}{\ln 2} \int_0^\infty (\ln \rho) e^{-\rho}\, d\rho$$

$$= \log \mathrm{SNR} - 0.8327$$

14.2-1 Ergodic capacity of Rayleigh fading

CSI at Rx



At high SNR, there is about 2.5 dB SNR loss from having a Rayleigh fading channel compared with an AWGN channel

Low SNR

Cannot be worse than no Rx CSI
Cannot be better than AWGN
due to Jensen's

$$\overline{C} = 1.44\,\mathrm{SNR}$$

High SNR

$$\log\left(1+\rho\frac{P}{N_0}\right) \approx \log\left(\rho\frac{P}{N_0}\right)$$

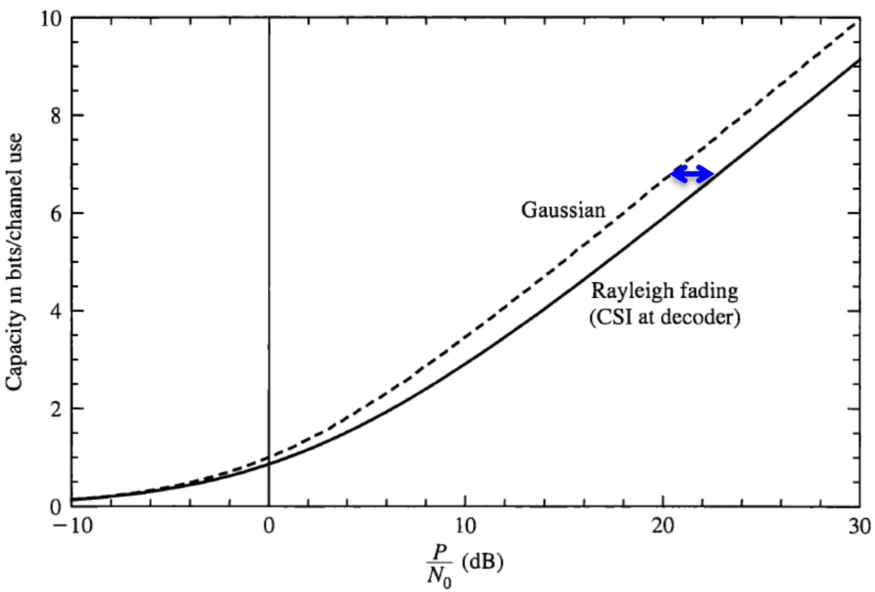
$$\overline{C} \approx \frac{1}{\ln 2}\int_0^\infty \log\left(\rho\frac{P}{N_0}\right) e^{-\rho}\,d\rho$$

$$= \log \mathrm{SNR} + \frac{1}{\ln 2}\int_0^\infty (\ln\rho)e^{-\rho}\,d\rho$$

$$= \log \mathrm{SNR} - 0.8327$$

14.2-1 Ergodic capacity of Rayleigh fading

CSI at Rx

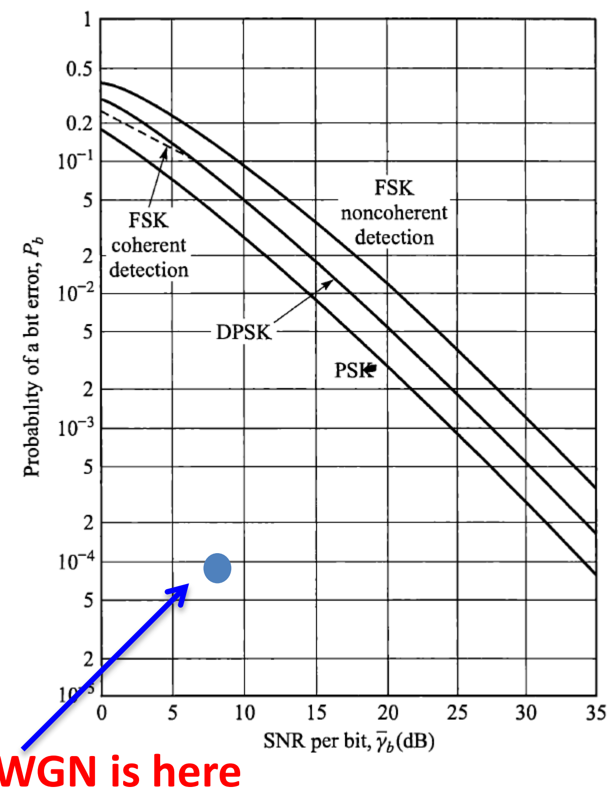


At high SNR, there is about 2.5 dB SNR loss from having a Rayleigh fading channel compared with an AWGN channel

Comparison with BER

See 13.3-13 for eqs

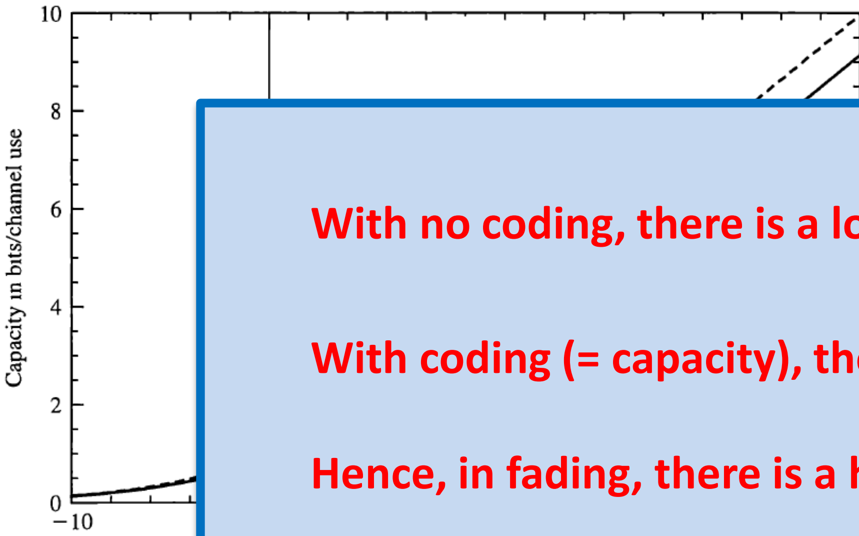
$P_b \approx 1/4\overline{\gamma}_b$ for coherent PSK



30 dB difference in BER

14.2-1 Ergodic capacity of Rayleigh fading

CSI at Rx



At high SNR, there is about 2.5 dB SNR loss from having a Rayleigh fading channel compared with an AWGN channel

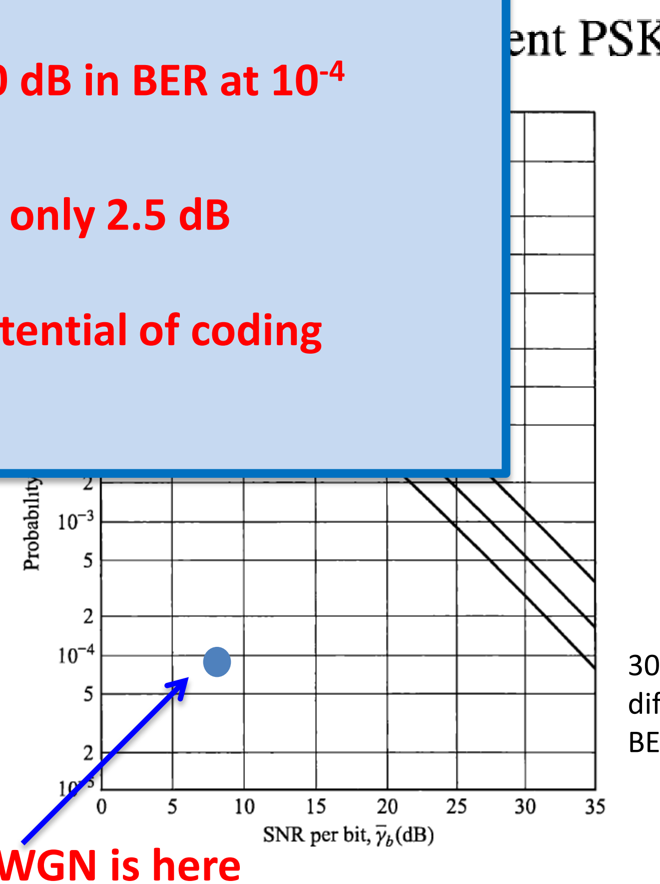
With no coding, there is a loss of 30 dB in BER at 10^{-4}

With coding (= capacity), the loss is only 2.5 dB

Hence, in fading, there is a huge potential of coding

Comparison with BER

See 13.3-13 for eqs



30 dB difference in BER

14.2-1 Ergodic capacity of Rayleigh fading

CSI at Rx and Tx

Transmit power is a function of the channel gain

$$\overline{C} = \int_0^\infty \log\left(1 + \rho\,\frac{P(\rho)}{N_0}\right) e^{-\rho}\, d\rho$$

14.2-1 Ergodic capacity of Rayleigh fading

CSI at Rx and Tx

Transmit power is a function of the channel gain

$$\overline{C} = \int_0^\infty \log\left(1 + \rho\,\frac{P(\rho)}{N_0}\right) e^{-\rho}\, d\rho$$

We can now apply the waterfilling principle in the time domain

$$\boxed{\begin{aligned} &\frac{P(\rho)}{N_0} = \left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)^+ \\ &\int_0^\infty \left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)^+ e^{-\rho}\, d\rho = \frac{P}{N_0} \end{aligned}}$$

Waterfilling

Satisfy average power constraint

14.2-1 Ergodic capacity of Rayleigh fading

CSI at Rx and Tx

Transmit power is a function of the channel gain

$$\overline{C} = \int_0^\infty \log\left(1 + \rho\,\frac{P(\rho)}{N_0}\right) e^{-\rho}\, d\rho$$

Parametric solution exists

$$\frac{P(\rho)}{N_0} = \left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)^+$$

$$\int_0^\infty \left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)^+ e^{-\rho}\, d\rho = \frac{P}{N_0}$$

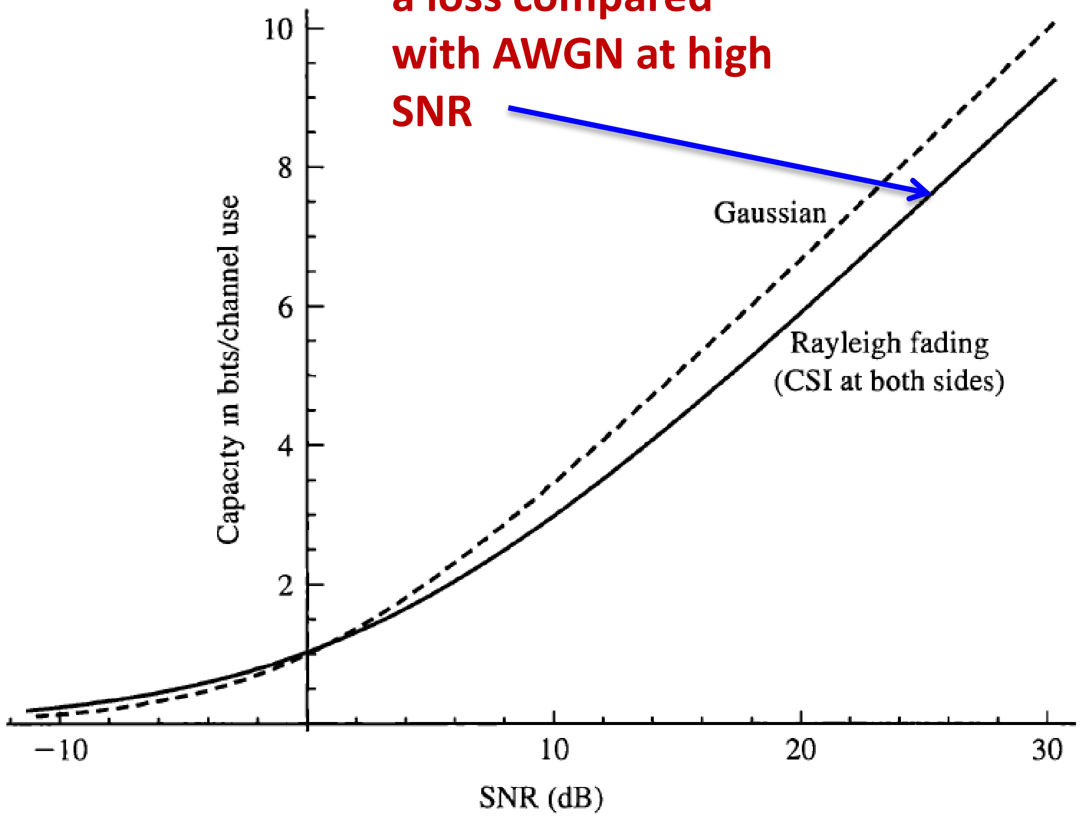
$$\frac{e^{-\rho_0}}{\rho_0} - \Gamma(0, \rho_0) = \frac{P}{N_0}$$

$$\overline{C} = \frac{1}{\ln 2}\Gamma(0, \rho_0)$$

14.2-1 Ergodic capacity of Rayleigh fading

CSI at Rx and Tx

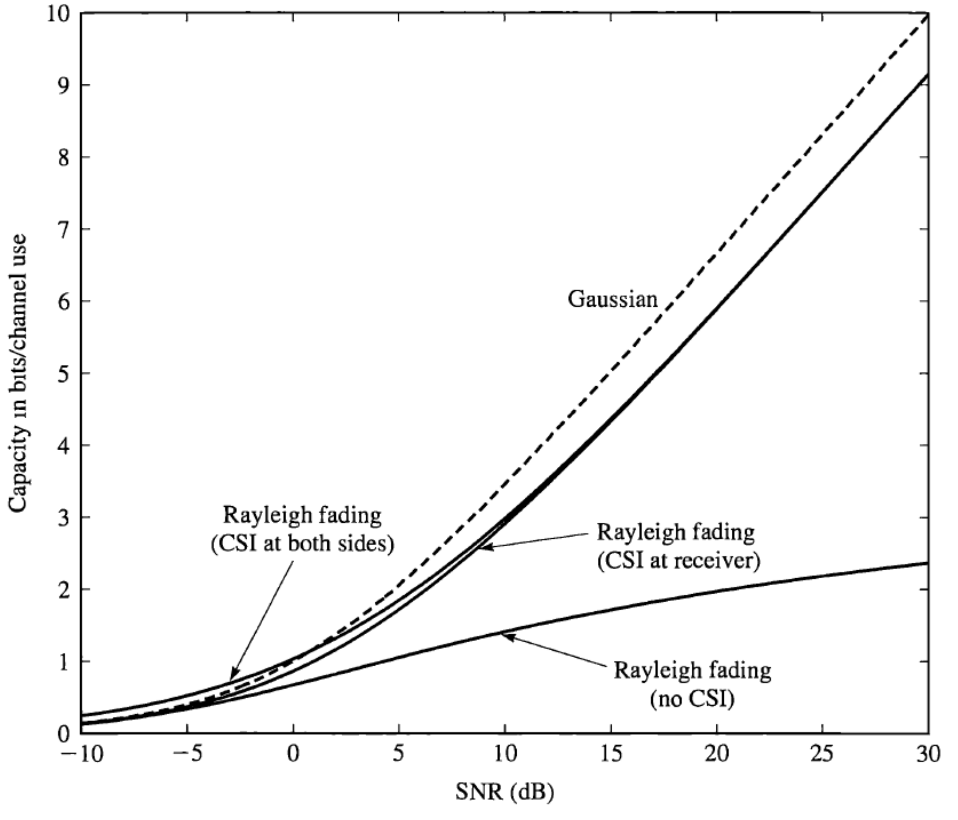
Fading channel has a loss compared with AWGN at high SNR



Fading channel can outperform AWGN at low SNR

14.2-1 Ergodic capacity of Rayleigh fading

Comparisons



Extension

Consider a channel model

$Y_i = R_i X_i + N_i$

Where neither the receiver or the transmitter has CSI.

Assume that R_i fades continuously according to a known channel transition law

$$\Pr(R_i | R_{i-1}, R_{i-2}, \ldots, R_1)$$

The capacity of this is a wide open problem, and not much is known

Excellent research topic

14.2-2 Outage capacity of Rayleigh fading

Applies when, due to latency etc, one cannot afford to interleave the data across a long codeword, so that one can average over all channel realizations.

14.2-2 Outage capacity of Rayleigh fading

$$
\begin{aligned}
C_\epsilon &= \max\{R : \mathrm{P}_{\text{out}}(R) \le \epsilon\} \\
&= \max\{R : F_C(R^-) = \epsilon\} \\
&= F_C^{-1}(\epsilon)
\end{aligned}
$$

$$C = \log(1 + \rho\, \mathrm{SNR})$$

Perfect CSI at Rx assumed
By definition, no CSI at Tx

$$\mathrm{P}_{\text{out}}(R) = \mathrm{P}[C < R]$$

Probability of outage event
for a given rate R

14.2-2 Outage capacity of Rayleigh fading

$$C_\epsilon = \max\{R : \mathrm{P}_{\text{out}}(R) \leq \epsilon\}$$
$$= \max\{R : F_C(R^-) = \epsilon\}$$
$$= F_C^{-1}(\epsilon)$$

$$C = \log(1 + \rho\, \mathrm{SNR})$$

Perfect CSI at Rx assumed
By definition, no CSI at Tx

$$\mathrm{P}_{\text{out}}(R) = \mathrm{P}\,[C < R]$$

Probability of outage event for a given rate R

$$\mathrm{P}_{\text{out}}(R) = \mathrm{P}\left[\rho < \frac{2^R - 1}{\mathrm{SNR}}\right]$$

Can be expressed as

$$= 1 - e^{-\frac{2^R - 1}{\mathrm{SNR}}}$$

Using the cdf of ρ

14.2-2 Outage capacity of Rayleigh fading

Solving for R yields

$$R = \log\left[1 - \mathrm{SNR}\ \ln\left(1 - \mathrm{P}_{\mathrm{out}}\right)\right]$$

Confusing typo in book: Between eq (14.2-36) and (14.2-37), Proakis refers to eq (14.2-36), but this should be (14.2-35)

$$\mathrm{P}_{\mathrm{out}}(R) = \mathrm{P}\left[\rho < \frac{2^R - 1}{\mathrm{SNR}}\right]$$

$$= 1 - e^{-\frac{2^R - 1}{\mathrm{SNR}}}$$

14.2-2 Outage capacity of Rayleigh fading

Solving for R yields

$$R = \log\left[1 - \mathrm{SNR}\ \ln\left(1 - \mathrm{P}_{\mathrm{out}}\right)\right]$$

Signaling at this rate gives
an outage capacity of

$$C_\epsilon = \log\left[1 - \mathrm{SNR}\ \ln\left(1 - \epsilon\right)\right]$$

14.2-2 Outage capacity of Rayleigh fading

Solving for R yields

$$R = \log\left[1 - \mathrm{SNR}\ \ln\left(1 - \mathrm{P_{out}}\right)\right]$$

Signaling at this rate gives an outage capacity of

$$C_\epsilon = \log\left[1 - \mathrm{SNR}\ \ln\left(1 - \epsilon\right)\right]$$

Low SNR

$$C_\epsilon \approx \frac{\mathrm{SNR}}{\ln 2} \ln \frac{1}{1-\epsilon}$$

log(1+x) ≈ x, for x ≈ 0

14.2-2 Outage capacity of Rayleigh fading

Solving for R yields

$$R = \log\left[1 - \mathrm{SNR}\ \ln\left(1 - \mathrm{P_{out}}\right)\right]$$

Signaling at this rate gives
an outage capacity of

$$C_\epsilon = \log\left[1 - \mathrm{SNR}\ \ln\left(1 - \epsilon\right)\right]$$

Low SNR

$$C_\epsilon \approx \frac{\mathrm{SNR}}{\ln 2} \ln \frac{1}{1-\epsilon}$$

= C_{AWGN} at low SNR

14.2-2 Outage capacity of Rayleigh fading

Solving for R yields

$$R = \log\left[1 - \mathrm{SNR}\ \ln(1 - \mathrm{P_{out}})\right]$$

Signaling at this rate gives
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$$C_\epsilon = \log\left[1 - \mathrm{SNR}\ \ln(1 - \epsilon)\right]$$

Low SNR

$$C_\epsilon \approx \frac{\mathrm{SNR}}{\ln 2} \ln \frac{1}{1 - \epsilon}$$

= C_{AWGN} at low SNR

At very small ε

$$C_\epsilon \approx \epsilon C_{\mathrm{AWGN}}$$

14.2-2 Outage capacity of Rayleigh fading

Solving for R yields

$$R = \log\left[1 - \mathrm{SNR}\ \ln(1 - \mathrm{P_{out}})\right]$$

Signaling at this rate gives
an outage capacity of

$$C_\epsilon = \log\left[1 - \mathrm{SNR}\ \ln(1 - \epsilon)\right]$$

Bring "-" inside log,
and ignore "1+"

Low SNR

$$C_\epsilon \approx \frac{\mathrm{SNR}}{\ln 2} \ln \frac{1}{1-\epsilon}$$

= C_{AWGN} at low SNR

High SNR

$$C_\epsilon \approx \log\left[\mathrm{SNR}\ \ln \frac{1}{1-\epsilon}\right]$$
$$= \log \mathrm{SNR} + \log\left(\ln \frac{1}{1-\epsilon}\right)$$

At very small ε

$$C_\epsilon \approx \epsilon C_{\mathrm{AWGN}}$$

14.2-2 Outage capacity of Rayleigh fading

Solving for R yields

$$R = \log\left[1 - \mathrm{SNR}\ \ln\left(1 - \mathrm{P_{out}}\right)\right]$$

Signaling at this rate gives
an outage capacity of

$$C_\epsilon = \log\left[1 - \mathrm{SNR}\ \ln\left(1 - \epsilon\right)\right]$$

Low SNR

$$C_\epsilon \approx \frac{\mathrm{SNR}}{\ln 2} \ln \frac{1}{1-\epsilon}$$

= C_{AWGN} at low SNR

At very small ε

$$C_\epsilon \approx \epsilon C_{\mathrm{AWGN}}$$

High SNR

$$C_\epsilon \approx \log\left[\mathrm{SNR}\ \ln \frac{1}{1-\epsilon}\right]$$

$$= \log \mathrm{SNR} + \log\left(\ln \frac{1}{1-\epsilon}\right)$$

For ε=0.1,

$$\log\left(\ln \frac{1}{1-\epsilon}\right) = -3.25$$

14.2-2 Outage capacity of Rayleigh fading

Solving for R yields

$$R = \log\left[1 - \mathrm{SNR}\ \ln\left(1 - \mathrm{P_{out}}\right)\right]$$

Signaling at this rate gives
an outage capacity of

$$C_\epsilon = \log\left[1 - \mathrm{SNR}\ \ln\left(1 - \epsilon\right)\right]$$

Low SNR

$$C_\epsilon \approx \frac{\mathrm{SNR}}{\ln 2} \ln \frac{1}{1-\epsilon}$$

= C_{AWGN} at low SNR

At very small ε

$$C_\epsilon \approx \epsilon C_{\mathrm{AWGN}}$$

High SNR

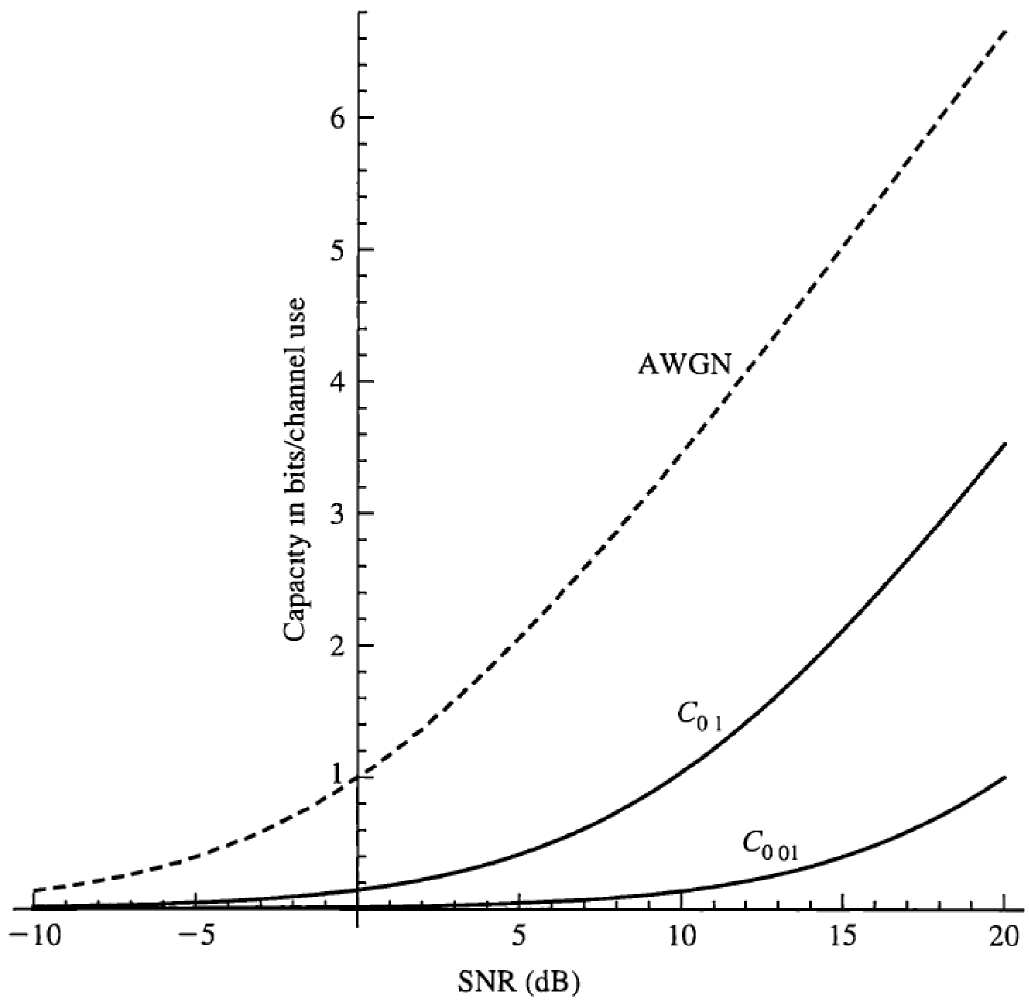
$$C_\epsilon \approx \log\left[\mathrm{SNR}\ \ln \frac{1}{1-\epsilon}\right] = \log \mathrm{SNR} + \log\left(\ln \frac{1}{1-\epsilon}\right)$$

For ε ≈ 0

$$\ln \frac{1}{1-\epsilon} \approx \epsilon$$

Loss is $\log_2 \epsilon$

14.2-2 Outage capacity of Rayleigh fading



10% outage

1% outage

14.2-2 Outage capacity of Rayleigh fading

What we did so far was for Rayleigh fading

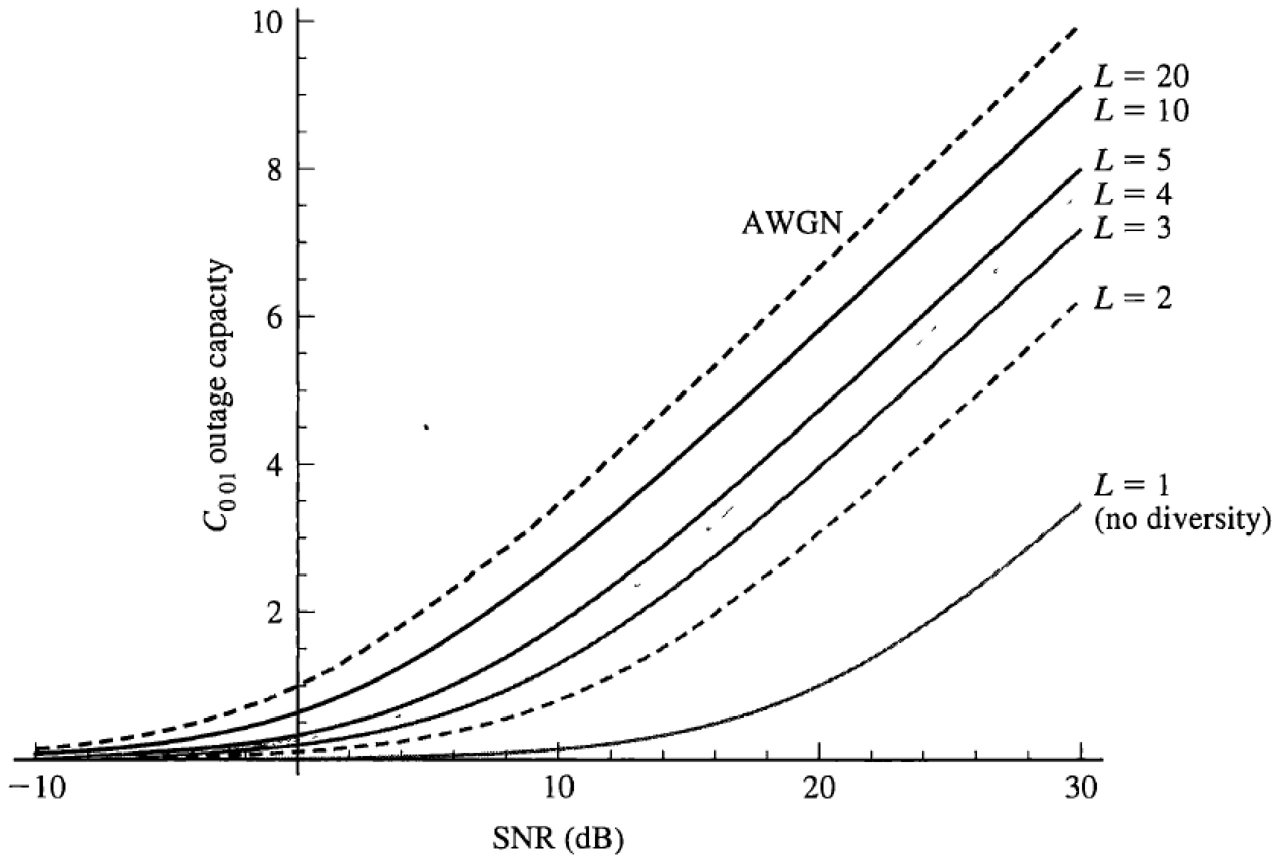
Rayleigh fading is a diversity $L = 1$ scheme

The variable ρ has a chi-2 distribution with 2 degrees of freedom

Now assume an order L diversity scheme, so that ρ is a chi-2 with $2L$ DoF

Repeat outage computations!

14.2-2 Outage capacity of Rayleigh fading



Diversity has huge impact on outage capacity!

14.3 – 14.7 Coding

Too many details…Read if you are interested
Much more time would be needed to cover this in lectures

One thing is very important

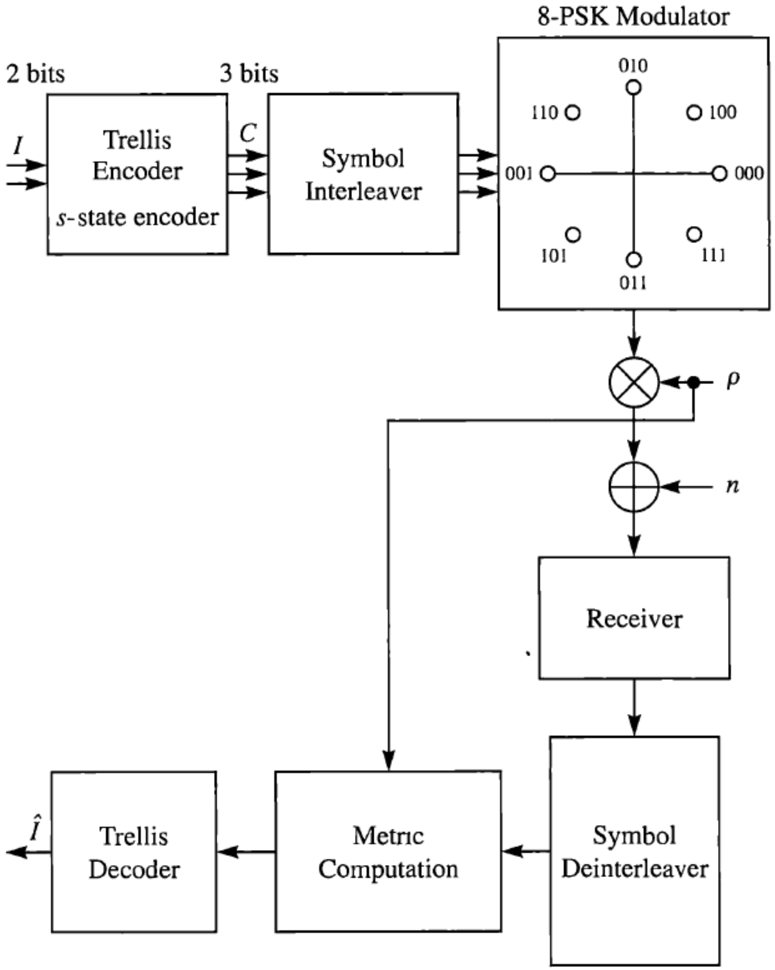
In AWGN, a good code is designed by maximizing the Euclidean distance between pairs of transmitted signals

In fading, it is the Hamming distance of the code itself that should be optimized (loosely speaking)

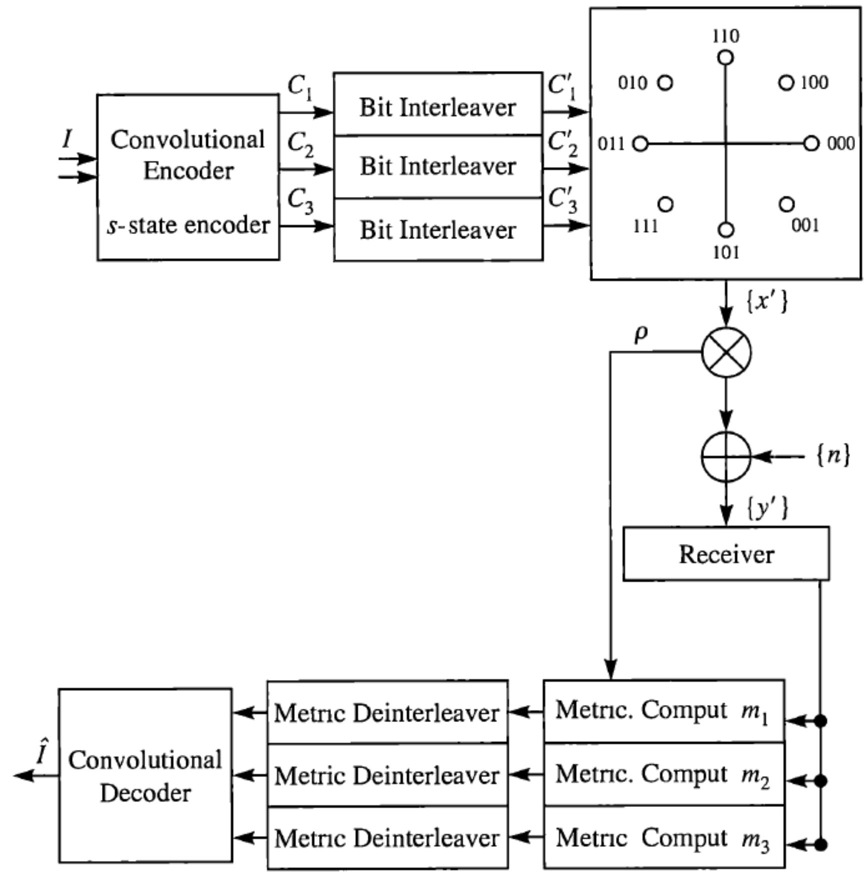
For BPSK, these Euclidean distance and Hamming distance is the same (except for scaling), but not for higher order constellations.

One way to do this is via BICM

BICM



Trellis coded modulation (TCM)



Bit-iterleaved coded modulation (BICM)

BICM

TABLE 14.6–1
Upper Bounds to Minimum Euclidean Distance and Diversity Order for TCM and BICM for 16-QAM Signaling. Average Energy is Normalized to 1 and Transmission Rate is 3 Bits per Complex Dimension.

Encoder memory	BICM d_E^2	BICM $d_{2(C)}$	TCM d_E^2	TCM $d_{M(C)}$
2	1.2	3	2	1
3	1.6	4	2.4	2
4	1.6	4	2.8	2
5	2.4	6	3.2	2
6	2.4	6	3.6	3
7	3.2	8	3.6	3
8	3.2	8	4	3

Source: From Caire et al. (1998), copyright IEEE

BICM has worse Euclidean distance, but superior Hamming distance. This is more important in fading channels

BICM is worse on AWGN channels due to reduced Euclidean distance