CHAPTER 4

Transformation Properties of Plane, Spherical and Cylindrical Scalar and Vector Wave Functions*

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1. Introduction

In this chapter we give a systematic exposition of the transformation properties of three commonly used sets of stationary wave fields, namely the plane, the spherical and the cylindrical waves. The resulting formulas are given in a form intended to be directly useful for applications in scattering of stationary acoustic, electromagnetic and elastic waves. However, the results are equally relevant for more general classical and quantum-mechanical potential scattering problems.

The scalar wave functions to be considered are solutions of the reduced wave equation, i.e. the Helmholtz equation:

$$(\nabla^2 + k^2)\psi = 0. \tag{1.1}$$

For the vector wave functions we give a formulation which is appropriate for elastodynamics. Thus, the vector wave functions are solutions to the reduced wave equation of elastodynamics:

$$k_{\mathrm{p}}^{-2}\nabla(\nabla\cdot\boldsymbol{\psi}) - k_{\mathrm{s}}^{-2}\nabla\times(\nabla\times\boldsymbol{\psi}) + \boldsymbol{\psi} = \mathbf{0}, \tag{1.2}$$

where k_{p} and k_{s} are the longitudinal and transverse wavenumbers, respectively. By putting $k_{\mathrm{p}} = k_{\mathrm{s}} = k$ (which is unphysical in elastodynamics) we obtain the vector Helmholtz equation and, by further deleting the longitudinal part, we arrive at the reduced wave equation of electromagnetics:

$$(\nabla^2 + k^2)\boldsymbol{\psi} = \mathbf{0}. \tag{1.3}$$

First of all, it is obvious that the plane, spherical and cylindrical waves are very useful when the scattering surface is exactly a plane, a sphere, or a circular cylinder, respectively. Their usefulness is, however, by no means limited to such cases. They form complete sets over more general surfaces, and they can, therefore, be used as sets of basis functions for the expansion of wave fields in more general situations. For instance, the scalar spherical waves are complete and orthogonal over a sphere but they are also complete, although nonorthogonal, for a much larger class of closed surfaces (see, e.g. Millar 1973, 1983). Analogous properties hold for the plane and cylindrical waves. Furthermore, for nonspherical but finite scatterers one usually expands the scattered field in terms of outgoing spherical waves in regions not too close to the scatterer, and similarly one uses outgoing cylindrical waves to describe the scattering from a straight but noncircular cylinder.

The need to have a fairly complete knowledge of the transformation properties which will be given here, arises also in large classes of scattering problems involving two or more scattering surfaces. Consider, for instance, the case of two finite scatterers. The field which is scattered off one of the scatterers is then naturally described in terms of outgoing (irregular) spherical waves emanating from that scatterer. When these waves hit the second scatterer, they are regular everywhere in the vicinity of this scatterer, i.e. what is needed in this case is an expansion of an outgoing spherical wave in terms of regular spherical waves referred to a different origin. On the other hand, at some distance away from the two scatterers, one may sometimes wish to refer the scattered field to an origin within one of the scatterers and sometimes to an origin somewhere outside the two scatterers. In this case the relevant transformation is one where outgoing spherical waves are transformed into outgoing spherical waves referred to a new origin. If, instead, one of the two scatterers is a cylinder, corresponding expansions between irregular and regular, spherical and cylindrical waves are needed.

Thus, it is evident that in applications of this kind, the transformation properties of both the regular and irregular waves are equally important and they will both be treated in the same detail. On the other hand, we shall limit ourselves to the case of lossless media, i.e. to real wave numbers k. However, it is expected that most of the results can be extended to some region in the complex k plane, but considerable detailed analysis remains to be carried out concerning this problem.

Although we give a fairly complete review of existing results, we must, in order to keep the length of this chapter within reasonable bounds, omit many proofs. The reader can find most of the basic facts concerning the different sets of wave functions which we shall consider in well-known textbooks such as Stratton (1941), Morse and Feshbach (1953) and Felsen and Marcuwitz (1973). Concerning particular aspects of the transformation properties we call attention to the very useful expository articles by Devaney and Wolf (1974), and Danos and Maximon (1965). They both consider the expansion of spherical waves in terms of plane waves, while the article by Danos and Maximon (1965) also treats the translation of spherical waves. Both articles contain very readable accounts of the history of their subjects, including numerous references. The transformations between the spherical and cylindrical waves are treated by Stratton (1941), Pogorzelski and Lun (1976), Boström (1980b) and Boström and Olsson (1981), and those between the cylindrical and plane waves are treated by Stratton (1941). Translation of plane waves is, of course, trivial and translation of regular cylindrical waves is essentially given by the well-known Graf's addition theorem for Bessel functions. Rotation of the spherical waves reduces to a rotation of the spherical harmonics, a subject which is thoroughly covered in textbooks like those of Rose (1957), Edmonds (1957) (in the language of the orbital angular momentum formalism) and Gelfland et al. (1963). Rotation of the plane waves is also essentially trivial. It is included here for completeness and because it can be used as an auxiliary tool, particularly when one considers rotation of the cylindrical waves.

We note that, for example, in electrostatics, magnetostatics and elastostatics involving several bodies, one encounters analogous transformation problems for sets of solutions to the Laplace equation. Many of the results required for this case can be obtained by studying the limit $k \to 0$ in the corresponding transformation formulas for solutions to the Helmholtz equation. We shall not pursue this matter here, and we refer to Sack (1964a, b, c, 1967) for a discussion of transformation properties of solutions to the Laplace equation.

Besides the sets of wave functions which are treated in this chapter there are, of course, several other sets which are used in scattering theory, such as spheroidal waves, conical waves, etc. However, the three sets treated here are distinguished in a definite sense that is instructive to dwell upon briefly. We are interested in wave functions which are solutions to the (scalar or vector) Helmholtz equation. This equation is closely related to the representation theory of the three-dimensional Euclidean group E(3) of rotations and translations, inasmuch as the Helmholtz equation corresponds to the eigenvalue equation for the Casimir operator in an irreducible representation of E(3) (see, e.g. Miller 1964, 1968). In view of this, the theory of Lie-group representations provides a very useful and, from a more fundamental point of view, a very natural framework for the subject matter of the present chapter. Therefore, we have included a few remarks on this connection, but the main text does not rely on any knowledge of the group-theoretical background.

For real values of k the representations of E(3) are unitary and considerable simplifications occur in this case. One can choose different sets of basis functions which can, furthermore, be chosen as simultaneous eigenfunctions of various sets of commuting operators [in the enveloping algebra of E(3)]. These operators are, in turn, related to specific transformations in the three-dimensional space. A separate index, the eigenvalue, corresponds to each of the operators. The collection of these eigenvalues will then characterize the basis functions and they will also appear in the functions which describe the transformation properties of this basis. Simple transformation properties are obtained when the indices characterizing a basis are ordered in a definite hierarchy. Hierarchies of this kind are naturally obtained by choosing the sets of commuting operators as the subgroup invariants in a decomposition of E(3) into a sequence of subgroups. The bases with the simplest transformation properties are then those for which the chain of subgroups is as long as possible, i.e. starts with as large a subgroup as possible. All the Lie-subgroup decompositions of E(3) are known and one finds that there are only three such decompositions which start with a three- or four-parameter subgroup (see, e.g. Kalnins et al. 1973). The corresponding three basis systems are closely related to the plane, spherical and cylindrical waves.

Thus, a systematic treatment of the relevant properties of the *regular* plane, spherical, and cylindrical waves can be based on the E(3) representation theory which gives a systematic and unified approach to a large body of results. The wave functions treated in this chapter are the most useful ones in the sense that any other set will have, on the whole, more complicated transformation properties. When working with other wave functions it is consequently often advantageous to go over to one of the three distinguished sets, perform the transformation in that set, and then go back to the original set. This is illustrated in the literature, e.g. by the fact that when one wishes to translate the spheroidal functions, this is usually done by means of a transformation to the spherical waves, where the translation is performed (King and van Buren 1973).

As was indicated above, the irregular waves are equally important in scattering theory and in order to treat these one employs analytic techniques. Some of the results concerning the irregular waves, such as some of the transformations between the spherical and cylindrical waves, in both the scalar and vector cases (Boström 1980, Boström and Olsson 1981), were not available in the literature until fairly recently. Taking these into account, we now have a fairly complete picture of the transformation properties of all the relevant functions (regular and irregular, scalar and vector), and, therefore, a review of these properties, using systematic and condensed notations and normalizations, seems appropriate at this time, as no effort has been made to present the results in such a way that they can be used directly in acoustic, electromagnetic, and elastic wave scattering. The notations for the vector wave functions are chosen so as to be suitable for applications in elastodynamics. However, the formulas relevant for the electromagnetic case are obtained by means of simple specializations as is explained above and in Section 2. The normalizations are chosen so that the expansion formulas for the Green's function take a simple form.

The chapter is organised as follows. In Sections 2 and 3 we introduce the notations and conventions for the wave functions and the (free space) Green's function in three and two dimensions, respectively. Several relevant Green's function and Green's dyadic expansions are listed. Sections 2–6 are divided into two parts for convenience. The first part treats the scalar waves and the second the vector waves. A reader interested only in the scalar case may thus read just the first part of these sections. In Section 4 we give the transformations between the three sets of wave functions. Completeness relations and relations between the transformation functions are also listed. A graphical representation of the various transformations which are available is presented in Fig. 3. The treatment of the properties of the wave functions under general transformations of the coordinate system starts in Section 5, which contains the transformations brought about by a translation of the origin. A general transformation can always be described as a combination of rotations and a translation along a specific axis, e.g. the z axis. The simplifications which appear in the translation formulas in this particular case are pointed out. The possibility of using the plane waves in an intermediate stage is also exemplified. Rotations are treated in Section 6, with due attention to the particular features of the vector case. Section 7 illustrates the application of some of the formulas to a multiple-scattering problem with two scattering surfaces, one of which is an infinite cylinder and the other a bounded surface outside the cylinder. Two appendices conclude the chapter. In Appendix A we discuss the proofs of some of the formulas for the transformation between the different wave functions. Appendix B contains some additional remarks on the connection to the E(3) representation theory, with references to the relevant literature in that field.

2. Definition of three-dimensional wave functions; expansions of the Green's function

In this section we define spherical, cylindrical, and plane scalar and vector wave functions in three dimensions. When one wants to establish a precise definition of these wave functions, one is faced with several choices concerning notations, normalizations, and phase conventions. The choices made will be reflected in the explicit form of the transformation formulas for the waves. Sometimes these choices are arbitrary, but, on the other hand, in several instances physical interpretation and/or mathematical or computational convenience favour specific choices.

Our conventions, and the motivation for them, are as follows. The first choice concerns the time factor, which we take to be $\exp\{-\mathrm{i}\omega t\}$, in view of its predominance in the current classical wave scattering literature. As usual, it will be omitted from all subsequent formulas. The spherical and cylindrical waves considered will be of the regular and outgoing kind since we have in mind applications to typical scattering situations. Thus, they contain Bessel functions and Hankel functions of the first kind.

The three coordinate systems to be considered are the spherical coordinates (r, θ, ϕ), the cylindrical coordinates (ρ, ϕ, z), and the rectangular coordinates (x, y, z). We employ a caret to denote unit vectors, e.g. $\hat{r}$ for the unit vector in the radial direction.

The dependence of the azimuthal angle ϕ is often described by complex functions $\exp\{im\phi\}$. However, we have chosen to depart from this common practice and, instead, we use real combinations $\{\cos m\phi, \sin m\phi\}$. Some relevant facts supporting this choice are the following. The decomposition of $\exp\{im\phi\}$ into $\{\cos m\phi, \sin m\phi\}$ gives, at the same time, a decomposition into the real and imaginary parts and a decomposition into even and odd functions of ϕ. This is a convenient feature since both decompositions are often useful in the implementation of one and the same problem. Two-dimensional models are frequently considered and for such models the even and odd combinations $\{\cos m\phi, \sin m\phi\}$ are useful, particularly when the scatterer has some mirror symmetry. Furthermore, in three-dimensional axially symmetric problems the decomposition into the even and odd parts of the azimuthal dependence is often a simplification. In this context the great dominance of axially symmetric models in actual computations, as opposed to formal developments, should be noted.

In general, it is advantageous to use notations which are suitable in as many situations as possible. The use of complex functions in classical scattering theory is tied to the use of a complex time factor $\exp\{\pm i\omega t\}$ for time-harmonic problems. This then entails the use of complex radial functions, but not necessarily complex functions of ϕ. Thus, when using $\exp\{im\phi\}$ together with Hankel functions of the first or second kind, these complex functions have different origins: the complex Hankel functions are a consequence of the time factor $\exp\{\pm i\omega t\}$, whereas the use of $\exp\{im\phi\}$ represents an independent arbitrary choice. As a consequence, one needs separate notations for switching between ingoing and outgoing waves (i.e. complex conjugation of the Hankel functions only) and for total complex conjugation. With the conventions used in the present chapter, this is avoided. In transient scattering problems treated by methods other than integration in the frequency domain, there is no similar widespread use of complex functions.

Finally, we have chosen all wave functions to be dimensionless and normalized so that the expansions of the Green's function and dyadic become simple. A consequence of our choice is that each of the outgoing spherical waves and each of the outgoing two-dimensional cylindrical waves carry an equal amount of energy through a sphere and a cylinder, respectively. Furthermore, the scattering matrix referred to these wave functions will be unitary and the scattering and transition matrices will be symmetric.

2.1. *Scalar wave functions*

It is convenient to consider the spherical system first. The spherical wave function is given as a product of a spherical Bessel or Hankel function and a spherical harmonic.

We define the outgoing function $\psi_{\sigma ml}(\boldsymbol{r})$ by

$$\psi_{\sigma ml}(\boldsymbol{r}) = h_l^{(1)}(kr)\,Y_{\sigma ml}(\hat{\boldsymbol{r}}), \tag{2.1}$$

where $h_l^{(1)}$ is the spherical Hankel function of the first kind (we employ the definitions of Morse and Feshbach (1953) for the Bessel and Legendre functions), and $Y_{\sigma ml}$ is the normalized real spherical harmonic function.

$$Y_{\sigma ml}(\hat{\boldsymbol{r}}) = \left(\frac{\varepsilon_m}{2\pi}\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\right)^{1/2} P_l^m(\cos\theta)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}. \tag{2.2}$$

Here, P_l^m is the Legendre function and the Neumann factor $\varepsilon_m = 2 - \delta_{m0}$, $l = 0, 1, \ldots$, $m = 0, \ldots, l$, and $\sigma = \mathrm{e}, \mathrm{o}$ (even, odd). Furthermore, we need the regular function $\mathrm{Re}\,\psi_{\sigma ml}(\boldsymbol{r})$, which is obtained by taking the spherical Bessel rather than the spherical Hankel function in Eq. (2.1). For real wave numbers k the regular function is then just the real part of the outgoing one. In the following we usually employ a multi-index n instead of the three indices σ, m and l.

The cylindrical wave function is a product of a Bessel or Hankel function, a trigonometric function, and an exponential. We define the outgoing function (in the ρ direction)

$$\chi_{\sigma m}(\alpha; \boldsymbol{r}) = (\varepsilon_m/8\pi)^{1/2}\, H_m^{(1)}(k\rho\sin\alpha)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}\mathrm{e}^{\mathrm{i}kz\cos\alpha}, \tag{2.3}$$

where $H_m^{(1)}$ is the Hankel function of the first kind, $m = 0, 1, \ldots$ and $\sigma = \mathrm{e}, \mathrm{o}$. We also define the regular function $\mathrm{Re}\,\chi_{\sigma m}(\alpha; \boldsymbol{r})$ which contains a Bessel instead of a Hankel function, and we further need the functions $\chi_m^{\dagger}(\alpha; \boldsymbol{r})$ and $\mathrm{Re}\,\chi_m^{\dagger}(\alpha; \boldsymbol{r})$, which have a sign change in the exponential, which is thus $\exp(-\mathrm{i}kz\cos\alpha)$. We introduce the multi-index k instead of σ and m (there should be no confusion between this k and the wave number k).

The continuous real or complex parameter α belongs to the interval $[0, \pi]$ or to the contours C, C_+, C_- given in Fig. 1. According to Fig. 1, one has

$$\begin{aligned}
&\mathrm{Im}\sin\alpha \geqslant 0, && \cos\alpha\in(-\infty, \infty), && \alpha\in C,\\
&\sin\alpha\in[0, \infty), && \mathrm{Im}\cos\alpha \geqslant 0, && \alpha\in C_+,\\
&\sin\alpha\in[0, \infty), && \mathrm{Im}\cos\alpha \leqslant 0, && \alpha\in C_-.
\end{aligned} \tag{2.4}$$

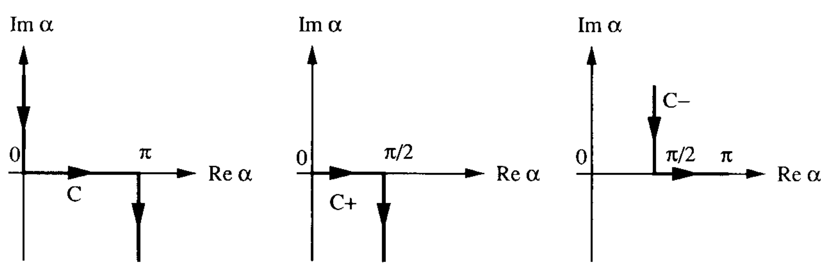

Fig. 1. Integration contours in the complex α plane.

When $\alpha \in C$, $k_z = k\cos\alpha$ is the Fourier transform variable corresponding to the z coordinate. On C_+ and C_-, $|k_x^2 + k_y^2|^{1/2} \equiv k\sin\alpha \in (0, \infty)$ is the radial Hankel transform variable corresponding to the radius ρ in the (x, y) plane. Since, $\operatorname{Im}\cos\alpha \geqslant 0$ on C_+ and $\operatorname{Im}\cos\alpha \leqslant 0$ on C_-, the function $\exp\{ikz\cos\alpha\}$ is bounded for a fixed α on C_+ or C_- when $z \geqslant 0$ or $z \leqslant 0$, respectively. Conversely, $\exp\{ikz\cos\alpha\}$ is bounded for a fixed $z > 0$ $(z < 0)$ when $\alpha \in C_+$ $(\alpha \in C_-)$.

In the rectangular coordinates the wave function is just a simple plane wave $\exp(i\boldsymbol{k}\cdot\boldsymbol{r})$. To bring out the similarities between the scalar and vector wave functions more clearly we define

$$\phi(\hat{\gamma}; \boldsymbol{r}) = (1/4\pi)e^{ik\hat{\gamma}\cdot\boldsymbol{r}}, \tag{2.5}$$

where $\hat{\gamma} = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$ is the direction of propagation of the plane wave, and α and β are the spherical angles of this direction. Here, α is the same as for the cylindrical case, i.e. it belongs to $[0, \pi]$, C, C_+, or C_-, and β belongs to the real interval $[0, 2\pi]$. The quantities $k_x = k\sin\alpha\cos\beta$ and $k_y = k\sin\alpha\sin\beta$ are the Fourier transform variables corresponding to the x and y coordinates, respectively. In analogy with the cylindrical case we also need the function $\phi^\dagger(\hat{\gamma}; \boldsymbol{r})$, which has a sign change in the exponential.

A scalar Green's function for the Helmholtz operator $(\nabla^2 + k^2)$ satisfies

$$\nabla^2 G(\boldsymbol{r}, \boldsymbol{r}) + k^2 G(\boldsymbol{r}, \boldsymbol{r}') = -\delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{2.6}$$

We shall consider the free-space Green's function which satisfies the radiation condition at infinity. It is given by

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{2.7}$$

[Note that a Green's function in three dimensions has dimension (length)$^{-1}$].

The expansion of the Green's function in the spherical wave functions is given, e.g. by Morse and Feshbach (1953). In our notation it is

$$G(\boldsymbol{r}, \boldsymbol{r}') = ik\sum_n \operatorname{Re}\psi_n(\boldsymbol{r}_<)\psi_n(\boldsymbol{r}_>), \tag{2.8}$$

where the summation over n, of course, indicates a triple sum over l, m, and σ. Further, $\boldsymbol{r}_<$ $(\boldsymbol{r}_>)$ denotes the radius vector with the smallest (greatest) value of r or r'.

For the cylindrical set we have two simple expansions of the Green's function, one when separating in $\rho_>$ and $\rho_<$, and one when separating in $z_>$ and $z_<$. Of course, we may also separate in $\phi_>$ and $\phi_<$ [as we may separate in $\theta_>$ and $\theta_<$ or $\phi_>$ and $\phi_<$ in the spherical system (Felsen and Marcuwitz 1973)], but this leads to a much more complicated expansion which is of less practical value. We thus give the following two expansions (which may be found in Felsen and Marcuwitz 1973, Morse and Feshbach 1953, respectively; cf. also Appendix A)

$$G(\boldsymbol{r}, \boldsymbol{r}') = ik\sum_k \int_C d\alpha \sin\alpha \operatorname{Re}\chi_k(\alpha; \boldsymbol{r}_<)\chi_k^\dagger(\alpha; \boldsymbol{r}_>), \tag{2.9}$$

$$G(\boldsymbol{r}, \boldsymbol{r}') = 2ik\sum_k \int_{C\pm} d\alpha \sin\alpha \operatorname{Re}\chi_k(\alpha; \boldsymbol{r}) \operatorname{Re}\chi_k^\dagger(\alpha; \boldsymbol{r}'), \quad z \gtrless z'. \tag{2.10}$$

Here a summation over k, of course, indicates a double summation over σ and m; while $\boldsymbol{r}_<$ ($\boldsymbol{r}_>$) as an argument in a cylindrical wave function denotes the radius vector with the smallest (greatest) value of ρ or ρ'. The dagger in Eq. (2.9) can be moved to the other function. The choice of C_+ or C_- in Eq. (2.10) is dependent on the values of the z coordinate as indicated. We note that Eq. (2.10) can be obtained from Eq. (2.9) (and vice versa) by a contour deformation and use of some properties of Hankel and Bessel functions (cf. Appendix A). Usually, Eq. (2.9) will be the more useful of the two expansions, as a separation into $z \gtrless z'$ can be effected more easily in the plane waves [cf. Eq. (2.11) below].

A commonly used expansion of the Green's function in terms of plane waves is given, e.g. by Devaney and Wolf (1974). This expansion is

$$G(\boldsymbol{r}, \boldsymbol{r}') = 2ik \int_{C_\pm} d\hat{\gamma}\, \phi(\hat{\gamma}; \boldsymbol{r})\, \phi^+(\hat{\gamma}; \boldsymbol{r}'), \quad z \gtrless z'. \tag{2.11}$$

Here we have introduced a short-hand notation for the double integral appearing in this expansion: $\int_{C_\pm} d\hat{\gamma} \equiv \int_0^{2\pi} d\beta \int_{C_\pm} \sin\alpha \, d\alpha$. We note the great similarities between Eqs. (2.11) and (2.10).

2.2. *Vector wave functions*

Turning to the vector wave functions, we emphasize the great similarities between the scalar and vector waves – the vector waves are obtained from the scalar ones essentially by applying gradient or curl operators. To emphasize the similarities we employ similar notations, thus writing $\boldsymbol{\psi}_n$, $\boldsymbol{\chi}_k(\alpha)$, etc.

We will employ a notation for the wave functions which is appropriate for elastodynamics. The vector wave functions must then satisfy the equation of motion for the displacement in elastodynamics, Eq. (1.2). By putting $k_p = k_s$ we obtain the vector Helmholtz equation and the corresponding wave functions. Further, by deleting all longitudinal terms we arrive at source-free electromagnetics: its wave equation, Eq. (1.3), and wave functions. Thus, we emphasize that the formulas given below contain all the necessary results for the electromagnetic case.

The outgoing spherical vector wave functions are defined as

$$\begin{aligned}
\boldsymbol{\psi}_{1\sigma ml}(\boldsymbol{r}) &= [l(l+1)]^{-1/2} \nabla \times \{\boldsymbol{r} h_l^{(1)}(k_s r)\, Y_{\sigma ml}(\hat{\boldsymbol{r}})\} \\
&= h_l^{(1)}(k_s r) \boldsymbol{A}_{1\sigma ml}(\hat{\boldsymbol{r}}), \\
\boldsymbol{\psi}_{2\sigma ml}(\boldsymbol{r}) &= [l(l+1)]^{-1/2} (1/k_s) \nabla \times \nabla \times \{\boldsymbol{r} h_l^{(1)}(k_s r)\, Y_{\sigma ml}(\hat{\boldsymbol{r}})\} \\
&= [(k_s r h_l^{(1)}(k_s r))'/k_s r] \boldsymbol{A}_{2\sigma ml}(\hat{\boldsymbol{r}}) + [l(l+1)]^{1/2} (h_l^{(1)}(k_s r)/k_s r) \boldsymbol{A}_{3\sigma ml}(\hat{\boldsymbol{r}}), \\
\boldsymbol{\psi}_{3\sigma ml}(\boldsymbol{r}) &= (k_p/k_s)^{3/2} (1/k_p) \nabla \{h_l^{(1)}(k_p r)\, Y_{\sigma ml}(\hat{\boldsymbol{r}})\} \\
&= (k_p/k_s)^{3/2} \{h_l^{(1)\prime}(k_p r) \boldsymbol{A}_{3\sigma ml}(\hat{\boldsymbol{r}}) + [l(l+1)]^{1/2} (h_l^{(1)}(k_p r)/k_p r) \boldsymbol{A}_{2\sigma ml}(\hat{\boldsymbol{r}})\}.
\end{aligned} \tag{2.12}$$

These wave functions will collectively be denoted $\boldsymbol{\psi}_n(\boldsymbol{r})$. The multi-index n now stands for τ, σ, m, l, where the first (mode) index τ (= 1, 2, 3) describes the three vector degrees of freedom. We also need the corresponding regular functions, denoted by Re $\boldsymbol{\psi}_n(\boldsymbol{r})$, in analogy with the scalar case. In Eq. (2.12) a prime denotes the derivative with respect to the whole argument. The $\boldsymbol{A}_n(\hat{\boldsymbol{r}})$ are the normalized real vector spherical harmonics:

$$\begin{aligned}
\boldsymbol{A}_{1\sigma ml}(\hat{\boldsymbol{r}}) &= [l(l+1)]^{-1/2}\nabla\times\{\boldsymbol{r}\,Y_{\sigma ml}(\hat{\boldsymbol{r}})\} \\
&= [l(l+1)]^{-1/2}\left\{-\hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta}+\hat{\boldsymbol{\theta}}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right\}Y_{\sigma ml}(\hat{\boldsymbol{r}}), \\
\boldsymbol{A}_{2\sigma ml}(\hat{\boldsymbol{r}}) &= [l(l+1)]^{-1/2}r\nabla Y_{\sigma ml}(\hat{\boldsymbol{r}}) \qquad (2.13)\\
&= [l(l+1)]^{-1/2}\left\{\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta}+\hat{\boldsymbol{\phi}}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right\}Y_{\sigma ml}(\hat{\boldsymbol{r}}), \\
\boldsymbol{A}_{3\sigma ml}(\hat{\boldsymbol{r}}) &= \hat{\boldsymbol{r}}\,Y_{\sigma ml}(\hat{\boldsymbol{r}}).
\end{aligned}$$

They constitute a complete orthonormal set on the unit sphere [the definition of $Y_{\sigma ml}(\hat{\boldsymbol{r}})$ is given in Eq. (2.2)]. We note that $\boldsymbol{\psi}_n$, $\tau = 1, 2, 3$ are essentially the wave functions denoted by $\boldsymbol{M}$, $\boldsymbol{N}$ and $\boldsymbol{L}$ in Morse and Feshbach (1953). Similarly, the $\boldsymbol{A}_n$ correspond to $\boldsymbol{C}$, $\boldsymbol{B}$ and $\boldsymbol{P}$ in Morse and Feshbach (1953).

We define the outgoing cylindrical vector wave functions as follows:

$$\begin{aligned}
\boldsymbol{\chi}_{1\sigma m}(\alpha;\boldsymbol{r}) &= (\varepsilon_m/8\pi)^{1/2}(1/k_s\sin\alpha)\nabla\times\left\{\hat{\boldsymbol{z}}H_m^{(1)}(k_s\rho\sin\alpha)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}e^{ik_sz\cos\alpha}\right\}, \\
\boldsymbol{\chi}_{2\sigma m}(\alpha;\boldsymbol{r}) &= (\varepsilon_m/8\pi)^{1/2}(1/k_s^2\sin\alpha)\nabla\times\nabla\times\left\{\hat{\boldsymbol{z}}H_m^{(1)}(k_s\rho\sin\alpha)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}e^{ik_sz\cos\alpha}\right\}, \\
\boldsymbol{\chi}_{3\sigma m}(\alpha;\boldsymbol{r}) &= (\varepsilon_m/8\pi)^{1/2}(k_p/k_s)^{3/2}(1/k_p)\nabla\left\{H_m^{(1)}(k_p\rho\sin\alpha)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}e^{ik_pz\cos\alpha}\right\},
\end{aligned} \qquad (2.14)$$

where α belongs to $[0,\pi], C, C_+$, or C_-. The mode index is denoted by τ, and the multi-index k now stands for τ, σ, m. The functions $\operatorname{Re}\boldsymbol{\chi}_k(\alpha)$, $\boldsymbol{\chi}_k^{\dagger}(\alpha)$, and $\operatorname{Re}\boldsymbol{\chi}_k^{\dagger}(\alpha)$ are defined in the obvious ways (note that the sign change in the exponential should be made before the differential operators are applied). More explicitly, the expressions (2.14) read

$$\begin{aligned}
\boldsymbol{\chi}_{1\sigma m}(\alpha;\boldsymbol{r}) = (\varepsilon_m/8\pi)^{1/2}\Bigg\{&\hat{\boldsymbol{\rho}}\frac{H_m^{(1)}(k_s\rho\sin\alpha)}{k_s\rho\sin\alpha}m\begin{pmatrix}-\sin m\phi\\ \cos m\phi\end{pmatrix}e^{ik_sz\cos\alpha} \\
&-\hat{\boldsymbol{\phi}}H_m^{(1)\prime}(k_s\rho\sin\alpha)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}e^{ik_sz\cos\alpha}\Bigg\}, \\
\boldsymbol{\chi}_{2\sigma m}(\alpha;\boldsymbol{r}) = (\varepsilon_m/8\pi)^{1/2}\Bigg\{&i\hat{\boldsymbol{\rho}}\cos\alpha H_m^{(1)\prime}(k_s\rho\sin\alpha)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}e^{ik_sz\cos\alpha} \\
&+i\hat{\boldsymbol{\phi}}\cos\alpha\frac{H_m^{(1)}(k_s\rho\sin\alpha)}{k_s\rho\sin\alpha}m\begin{pmatrix}-\sin m\phi\\ \cos m\phi\end{pmatrix}e^{ik_sz\cos\alpha} \\
&+\hat{\boldsymbol{z}}\sin\alpha H_m^{(1)}(k_s\rho\sin\alpha)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}e^{ik_sz\cos\alpha}\Bigg\}, \qquad (2.15)
\end{aligned}$$

$$\chi_{3\sigma m}(\alpha; \boldsymbol{r}) = (\varepsilon_m/8\pi)^{1/2}(k_p/k_s)^{3/2}$$
$$\times \left\{ \hat{\boldsymbol{\rho}} \sin\alpha H_m^{(1)\prime}(k_p \rho \sin\alpha) \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix} e^{ik_p z \cos\alpha} \right.$$
$$+ \hat{\boldsymbol{\phi}} \frac{m}{k_p \rho} H_m^{(1)}(k_p \rho \sin\alpha) \begin{pmatrix} -\sin m\phi \\ \cos m\phi \end{pmatrix} e^{ik_p z \cos\alpha}$$
$$\left. + i\hat{\boldsymbol{z}} \cos\alpha H_m^{(1)}(k_p \rho \sin\alpha) \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix} e^{ik_p z \cos\alpha} \right\}.$$

Finally, we define the plane vector wave functions

$$\begin{aligned} \boldsymbol{\phi}_1(\hat{\gamma}; \boldsymbol{r}) &= (1/4\pi k_s \sin\alpha) \nabla \times \{\hat{\boldsymbol{z}} e^{ik_s \hat{\gamma}\cdot\boldsymbol{r}}\} = -\hat{\boldsymbol{\beta}}(i/4\pi) e^{ik_s \hat{\gamma}\cdot\boldsymbol{r}}, \\ \boldsymbol{\phi}_2(\hat{\gamma}; \boldsymbol{r}) &= (1/4\pi k_s^2 \sin\alpha) \nabla \times \nabla \times \{\hat{\boldsymbol{z}} e^{ik_s \hat{\gamma}\cdot\boldsymbol{r}}\} = -\hat{\boldsymbol{\alpha}}(1/4\pi) e^{ik_s \hat{\gamma}\cdot\boldsymbol{r}}, \\ \boldsymbol{\phi}_3(\hat{\gamma}; \boldsymbol{r}) &= (k_p/k_s)^{3/2}(1/4\pi k_p) \nabla \{e^{ik_p \hat{\gamma}\cdot\boldsymbol{r}}\} = \hat{\gamma}(k_p/k_s)^{3/2}(i/4\pi) e^{ik_p \hat{\gamma}\cdot\boldsymbol{r}}, \end{aligned} \tag{2.16}$$

where $\hat{\gamma}$, $\hat{\boldsymbol{\alpha}}$, and $\hat{\boldsymbol{\beta}}$ are the spherical unit vectors given by

$$\begin{aligned} \hat{\gamma} &= (\sin\alpha \cos\beta, \sin\alpha \sin\beta, \cos\alpha), \\ \hat{\boldsymbol{\alpha}} &= (\cos\alpha \cos\beta, \cos\alpha \sin\beta, -\sin\alpha), \\ \hat{\boldsymbol{\beta}} &= (-\sin\beta, \cos\beta, 0). \end{aligned} \tag{2.17}$$

Here α and β belong to the same intervals as in the scalar case. We denote the mode index by j, and thus write $\boldsymbol{\phi}_j(\hat{\gamma})$. The function $\boldsymbol{\phi}_j^\dagger(\hat{\gamma})$ has a sign change in the exponential (made before the curl or gradient is applied).

We shall consider the free space Green's dyadic $\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')$ corresponding to Eq. (1.2), which satisfies ($\boldsymbol{I}$ is the unit dyadic):

$$(1/k_p^2)\nabla\nabla \cdot \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') - (1/k_s^2)\nabla \times \nabla \times \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') + \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') = -(1/k_s^3)\boldsymbol{I}\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{2.18}$$

plus the radiation conditions at infinity. An explicit expression for $\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')$ in Cartesian coordinates is

$$G_{ij}(\boldsymbol{r}, \boldsymbol{r}') = (1/4\pi) k_s^{-3} \left\{ (k_s^2 \delta_{ij} + \partial_i \partial_j) \frac{e^{ik_s|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} - \partial_i \partial_j \frac{e^{ik_p|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} \right\}. \tag{2.19}$$

The spherical wave expansion of $\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')$ is (see, e.g. Morse and Feshbach 1953)

$$\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') = i \sum_n \operatorname{Re} \boldsymbol{\psi}_n(\boldsymbol{r}_<) \boldsymbol{\psi}_n(\boldsymbol{r}_>), \tag{2.20}$$

where the summation over n now stands for a summation over τ, σ, m and l. The factor $(k_p/k_s)^{3/2}$ in $\boldsymbol{\psi}_{3\sigma ml}$ was included so as to make Eq. (2.20) simple.

The expansions of $\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')$ analogous to Eqs. (2.9) and (2.10) are (Boström and Olsson 1981, Ben-Menahem and Singh 1968)

$$\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') = i \sum_k \int_C d\alpha \sin\alpha \operatorname{Re} \boldsymbol{\chi}_k(\alpha; \boldsymbol{r}_<) \boldsymbol{\chi}_k^\dagger(\alpha; \boldsymbol{r}_>), \tag{2.21}$$

$$\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') = 2\mathrm{i}\sum_k \int_{C_\pm} \mathrm{d}\alpha \sin\alpha \,\mathrm{Re}\,\boldsymbol{\chi}_k(\alpha; \boldsymbol{r})\,\mathrm{Re}\,\boldsymbol{\chi}_k^\dagger(\alpha; \boldsymbol{r}'), \quad z \gtrless z', \tag{2.22}$$

and an expansion in terms of plane waves is (Boström and Kristensson 1980)

$$\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') = 2\mathrm{i}\sum_j \int_{C_\pm} \mathrm{d}\hat{\gamma}\, \boldsymbol{\phi}_j(\hat{\gamma}; \boldsymbol{r})\boldsymbol{\phi}_j^\dagger(\hat{\gamma}; \boldsymbol{r}'), \quad z \gtrless z'. \tag{2.23}$$

3. Definition of two-dimensional wave functions; expansions of the Green's functions

As two-dimensional models are often considered, we define in this section cylindrical and plane scalar and vector wave functions in two dimensions. We can then rely heavily on the previous section as the discussion about the time factor, the choice of angular functions, etc., is valid here also.

The two-dimensional wave functions can, in fact, be obtained from the corresponding three-dimensional ones by simply putting $\alpha = \frac{1}{2}\pi$ and multiplying by $(2\pi)^{1/2}$. The factor $(2\pi)^{-1/2}\exp(\mathrm{i}kz\cos\alpha)$ then becomes unity, and the z dependence is suppressed in this way. We also get $\sin\alpha = 1$, which means that the wave vector lies entirely in the xy plane. It is convenient to remove the factor $(2\pi)^{-1/2}$, since this leads to very simple expansions of the two-dimensional Green's function and dyadic. We use the same notations for the two-dimensional wave functions as for the three-dimensional ones, with the obvious change that the α and z dependence is absent.

From what has just been stated, it follows that all the relevant transformation formulas in the following sections – which are written for the three-dimensional case – can be used also for the two-dimensional wave functions. With a slight change of notation, the formulas giving the transformations between cylindrical and plane wave functions, and their translational and rotational properties are still applicable. In the following sections all the equations that have two-dimensional versions are marked with an asterisk (*).

3.1. Scalar wave functions

We consider the cylindrical system first. The wave function is then a product of a Bessel or Hankel function and a trigonometric function. The outgoing function is defined as

$$\chi_{\sigma m}(\boldsymbol{r}) = (\varepsilon_m/4)^{1/2} H_m^{(1)}(k\rho)\begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix}, \tag{3.1}$$

and the corresponding regular function $\mathrm{Re}\,\chi_{\sigma m}(\boldsymbol{r})$ contains a Bessel instead of a Hankel function. We introduce the multi-index k instead of σ and m (there should be no confusion between this k and the wave number k).

In the rectangular coordinates the wave function is simply a plane wave, which we normalize according to

$$\phi(\beta; \boldsymbol{r}) = (1/8\pi)^{1/2} \mathrm{e}^{\mathrm{i}k\hat{\gamma}\cdot\boldsymbol{r}}, \tag{3.2}$$

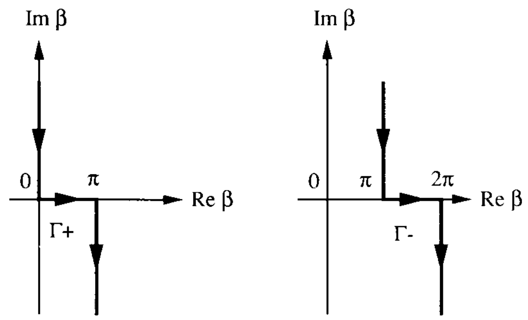

Fig. 2. Integration contours in the complex β plane.

where $\hat{\gamma} = (\cos\beta, \sin\beta)$ is the direction of propagation (which may be complex). The angle β belongs to Γ_+ or Γ_- (see Fig. 2), where $\cos\beta \in (-\infty, \infty)$ and $\operatorname{Im}\sin\beta \geqslant 0$ when $\beta \in \Gamma_+$, and $\operatorname{Im}\sin\beta \leqslant 0$ when $\beta \in \Gamma_-$. This means that $k_x = k\cos\beta$ is the Fourier transform variable corresponding to the x coordinate. It also means that the function $\exp(iky\sin\beta)$ is bounded for a fixed β on Γ_+ or Γ_- when $y \geqslant 0$ or $y \leqslant 0$, respectively. Conversely, $\exp(iky\sin\beta)$ is bounded for a fixed $y \geqslant 0$ ($y \leqslant 0$) when $\beta \in \Gamma_+$ ($\beta \in \Gamma_-$). As in the three-dimensional case we also need the function $\phi^\dagger(\beta; \boldsymbol{r})$ which has a sign change in the exponential.

The free-space Green's function for the Helmholtz equation satisfies

$$\nabla^2 G(\boldsymbol{r}, \boldsymbol{r}') + k^2 G(\boldsymbol{r}, \boldsymbol{r}') = -\delta(\boldsymbol{r} - \boldsymbol{r}'), \tag{3.3}$$

and the radiation condition. Its explicit form is

$$G(\boldsymbol{r}, \boldsymbol{r}') = \tfrac{1}{4}\mathrm{i}H_0^{(1)}(k|\boldsymbol{r} - \boldsymbol{r}'|), \tag{3.4}$$

and we note that it is dimensionless. In cylindrical coordinates we have the expansion (Morse and Feshbach 1953)

$$G(\boldsymbol{r}, \boldsymbol{r}') = \mathrm{i}\sum_k \operatorname{Re}\chi_k(\boldsymbol{r}_<)\chi_k(\boldsymbol{r}_>), \tag{3.5}$$

where $\boldsymbol{r}_<$ ($\boldsymbol{r}_>$) denotes the radius vector with the smallest (largest) value of ρ or ρ', and where summation over k indicates a summation over both σ and m. In rectangular coordinates we have the expansion (Morse and Feshbach 1953)

$$G(\boldsymbol{r}, \boldsymbol{r}') = 2\mathrm{i}\int_{\Gamma\pm} \mathrm{d}\beta\, \phi(\beta; \boldsymbol{r})\phi^\dagger(\beta; \boldsymbol{r}'), \quad y \gtrless y', \tag{3.6}$$

where Γ_+ (Γ_-) is chosen when $y > y'$ ($y < y'$) as indicated.

3.2. Vector wave functions

We first note that two-dimensional vector wave functions are relevant only in elastodynamics – in electrodynamics a two-dimensional problem always reduces to scalar problems. In two dimensions we only have two vector degrees of freedom, so we have only two different modes, one transverse and one longitudinal (the remaining transverse mode, in the $\hat{z}$ direction, decouples from these two).

The outgoing cylindrical vector wave functions are

$$\begin{aligned}\boldsymbol{\chi}_{1\sigma m}(\boldsymbol{r}) &= (\varepsilon_m/4)^{1/2}(1/k_s)\nabla\times\left[\hat{z}H_m^{(1)}(k_s\rho)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}\right]\\ &= (\varepsilon_m/4)^{1/2}\left\{\hat{\boldsymbol{\rho}}\frac{m}{k_s\rho}H_m^{(1)}(k_s\rho)\begin{pmatrix}-\sin m\phi\\ \cos m\phi\end{pmatrix} - \hat{\boldsymbol{\phi}}H_m^{(1)\prime}(k_s\rho)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}\right\},\\ \boldsymbol{\chi}_{2\sigma m}(\boldsymbol{r}) &= (\varepsilon_m/4)^{1/2}(1/k_s)\nabla\left[H_m^{(1)}(k_p\rho)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}\right]\\ &= (\varepsilon_m/4)^{1/2}(k_p/k_s)\left\{\hat{\boldsymbol{\rho}}H_m^{(1)\prime}(k_p\rho)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}\right.\\ &\qquad\left.+\hat{\boldsymbol{\phi}}\frac{m}{k_p\rho}H_m^{(1)}(k_p\rho)\begin{pmatrix}-\sin m\phi\\ \cos m\phi\end{pmatrix}\right\},\end{aligned} \tag{3.7}$$

and the corresponding regular function $\mathrm{Re}\,\boldsymbol{\chi}_k(\boldsymbol{r})$ contains a Bessel instead of a Hankel function. Now the multi-index k contains also the mode index τ $(=1, 2)$. The factor k_p/k_s is included in the longitudinal wave function so as to give the expansion of the Green's dyadic a simple form.

The plane vector wave functions are defined as

$$\begin{aligned}\boldsymbol{\phi}_1(\beta;\boldsymbol{r}) &= (1/8\pi)^{1/2}(1/k_s)\nabla\times[\hat{z}e^{ik_s\hat{\gamma}\cdot\boldsymbol{r}}] = -i\hat{\boldsymbol{\beta}}(1/8\pi)^{1/2}e^{ik_s\hat{\gamma}\cdot\boldsymbol{r}},\\ \boldsymbol{\phi}_2(\beta;\boldsymbol{r}) &= (1/8\pi)^{1/2}(1/k_s)\nabla[e^{ik_p\hat{\gamma}\cdot\boldsymbol{r}}] = i\hat{\gamma}(k_p/k_s)(1/8\pi)^{1/2}e^{ik_p\hat{\gamma}\cdot\boldsymbol{r}},\end{aligned} \tag{3.8}$$

where

$$\hat{\gamma} = (\cos\beta, \sin\beta), \qquad \hat{\boldsymbol{\beta}} = (-\sin\beta, \cos\beta). \tag{3.9}$$

The angle β belongs to Γ_+ or Γ_- as in the scalar case. We denote the mode index j $(=1, 2)$, and introduce the function $\boldsymbol{\phi}_j^\dagger(\beta)$ which has a sign change in the exponential (made before the curl or gradient is applied).

The two-dimensional free-space Green's dyadic associated with the reduced wave equation of elastodynamics satisfies

$$k_p^{-2}\nabla\nabla\cdot\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}') - k_s^{-2}\nabla\times\nabla\times\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}') + \boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}') = -(1/k_s^2)\boldsymbol{I}\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{3.10}$$

and the radiation condition. Its explicit form in rectangular coordinates is

$$G_{ij}(\boldsymbol{r},\boldsymbol{r}') = (i/4)k_s^{-2}\{(k_s^2\delta_{ij} + \partial_i\partial_j)H_0^{(1)}(k_s|\boldsymbol{r}-\boldsymbol{r}'|) - \partial_i\partial_j H_0^{(1)}(k_p|\boldsymbol{r}-\boldsymbol{r}'|)\}. \tag{3.11}$$

In analogy with the scalar case we then have the expansions [which can be obtained from the corresponding scalar expansions and Eq. (3.11)]

$$\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}') = i\sum_k \mathrm{Re}\,\boldsymbol{\chi}_k(\boldsymbol{r}_<)\boldsymbol{\chi}_k(\boldsymbol{r}_>), \tag{3.12}$$

$$\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}') = 2i\sum_j\int_{\Gamma\pm} d\beta\,\boldsymbol{\phi}_j(\beta;\boldsymbol{r})\boldsymbol{\phi}_j^\dagger(\beta;\boldsymbol{r}'), \quad y \gtrless y'. \tag{3.13}$$

4. Transformations between the wave functions

Having defined the wave functions, we now turn to the transformations between the different sets of wave functions. We can always transform back and forth between the regular waves without restrictions. But, when the outgoing waves are involved, there are either geometrical limitations or else the expansion does not seem to exist at all. Thus, it is believed that there is no expansion of a plane wave in the outgoing spherical ones, at least not in any region of interest.

We will not give any proofs of the transformations considered here. However, as some of them do not seem to be found in the literature, we discuss their derivations in Appendix A, especially those involving the irregular scalar wave functions.

4.1. Scalar wave functions

We start with the transformations between the spherical and cylindrical scalar waves. The transformation function is essentially a normalized Legendre function

$$C_{nk'}(\alpha) = \mathrm{i}^{m-l}\delta_{\sigma\sigma'}\delta_{mm'}\left[\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\alpha). \tag{4.1}$$

The transformations between the regular functions are then (Stratton 1941)

$$\operatorname{Re}\psi_n(\boldsymbol{r}) = \sum_{k'}\int_0^{\pi} \mathrm{d}\alpha \sin\alpha\, C_{nk'}(\alpha) \operatorname{Re}\chi_{k'}(\alpha; \boldsymbol{r}), \tag{4.2}$$

$$\operatorname{Re}\chi_k(\alpha; \boldsymbol{r}) = \sum_{n'} C^{\dagger}_{n'k}(\alpha) \operatorname{Re}\psi_{n'}(\boldsymbol{r}). \tag{4.3}$$

The summation over n' is essentially over $l' = m, m+1, \ldots$ $C^{\dagger}_{nk'}(\alpha)$ is obtained from $C_{nk'}(\alpha)$ simply by changing the first factor to i^{l-m} (for real α this is then the same as taking the complex conjugate). Furthermore, we can expand the outgoing spherical waves in the regular or the outgoing cylindrical waves (Boström 1980b)

$$\psi_n(\boldsymbol{r}) = 2\sum_{k'}\int_{C_\pm} \mathrm{d}\alpha \sin\alpha\, C_{nk'}(\alpha) \operatorname{Re}\chi_{k'}(\alpha; \boldsymbol{r}), \quad z \gtrless 0, \tag{4.4}$$

$$\psi_n(\boldsymbol{r}) = \sum_{k'}\int_{C} \mathrm{d}\alpha \sin\alpha\, C_{nk'}(\alpha) \chi_{k'}(\alpha; \boldsymbol{r}), \quad \rho > 0. \tag{4.5}$$

As mentioned above we have geometrical restrictions on these expansions. The restriction in Eq. (4.4) is due to the conditions at infinity (boundedness and radiation conditions). In Eq. (4.5), on the other hand, it is the location of the singularities that put a natural limit on the region of validity.

The transformation function for the spherical and plane waves is essentially a normalized spherical harmonic

$$B_n(\hat{\gamma}) = \mathrm{i}^{-l}\left[\frac{\varepsilon_m}{2\pi}\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\alpha)\begin{pmatrix}\cos m\beta\\ \sin m\beta\end{pmatrix}. \tag{4.6}$$

We then have the transformations (Stratton 1941, Morse and Feshbach 1953, Danos and Maximon 1965)

$$\operatorname{Re}\psi_n(\boldsymbol{r}) = \int_0^{\pi} \mathrm{d}\hat{\gamma} B_n(\hat{\gamma})\phi(\hat{\gamma}; \boldsymbol{r}), \tag{4.7}$$

$$\phi(\hat{\gamma}; \boldsymbol{r}) = \sum_n B_n^{\dagger}(\hat{\gamma})\operatorname{Re}\psi_n(\boldsymbol{r}), \tag{4.8}$$

$$\psi_n(\boldsymbol{r}) = 2\int_{C_\pm} \mathrm{d}\hat{\gamma} B_n(\hat{\gamma})\phi(\hat{\gamma}; \boldsymbol{r}), \quad z \gtrless 0. \tag{4.9}$$

The short-hand notation for the integrations was explained following Eq. (2.11). $B_n^{\dagger}(\hat{\gamma})$ is obtained from $B_n(\hat{\gamma})$ by changing i^{-l} to i^l.

The transformations between the cylindrical and plane waves are the simplest ones. The transformation function is just a trigonometric function

$$D_k(\beta) = \mathrm{i}^{-m}(\varepsilon_m/2\pi)^{1/2}\begin{pmatrix}\cos m\beta \\ \sin m\beta\end{pmatrix}. \tag{4.10}$$

The transformations are (Stratton 1941) (cf. also Appendix A)

$$\operatorname{Re}\chi_k(\alpha; \boldsymbol{r}) = \int_0^{2\pi} \mathrm{d}\beta\, D_k(\beta)\phi(\hat{\gamma}; \boldsymbol{r}), \tag{4.11)*$$

$$\phi(\hat{\gamma}; \boldsymbol{r}) = \sum_k D_k^{\dagger}(\beta)\operatorname{Re}\chi_k(\alpha; \boldsymbol{r}), \tag{4.12)*$$

$$\chi_k(\alpha; \boldsymbol{r}) = 2\int_{\Gamma_\pm} \mathrm{d}\beta\, D_k(\beta)\,\phi(\hat{\gamma}; \boldsymbol{r}), \quad y \gtrless 0; \quad \alpha \in [0, \pi]. \tag{4.13)*$$

$D_k^{\dagger}(\beta)$ is obtained from $D_k(\beta)$ by changing i^{-m} to i^m. By changing the contour in Eq. (4.13) we may obtain a transformation valid, e.g. for $x \gtrless 0$.

Using the relations between the regular wave functions twice we obtain the orthogonality and completeness relations

$$\sum_{k''}\int_0^{\pi} \mathrm{d}\alpha \sin\alpha\, C_{nk''}(\alpha)C_{n'k''}^{\dagger}(\alpha) = \delta_{nn'}, \tag{4.14}$$

$$\sum_{n''} C_{n''k}(\alpha)C_{n''k'}^{\dagger}(\alpha') = \delta_{kk'}\delta(\cos\alpha - \cos\alpha'), \quad \alpha, \alpha' \in [0, \pi], \tag{4.15}$$

$$\int_0^{\pi} \mathrm{d}\hat{\gamma} B_n(\hat{\gamma})B_{n'}^{\dagger}(\hat{\gamma}) = \delta_{nn'}, \tag{4.16}$$

$$\sum_n B_n(\hat{\gamma})B_n^{\dagger}(\hat{\gamma}') = \delta(\cos\alpha - \cos\alpha')\,\delta(\beta - \beta'), \quad \alpha, \alpha' \in [0, \pi]; \quad \beta, \beta' \in [0, 2\pi], \tag{4.17}$$

$$\int_0^{2\pi} \mathrm{d}\beta\, D_k(\beta)D_{k'}^{\dagger}(\beta) = \delta_{kk'}, \tag{4.18}$$

$$\sum_k D_k(\beta)D_k^{\dagger}(\beta') = \delta(\beta - \beta'), \quad \beta, \beta' \in [0, 2\pi]. \tag{4.19}$$

In fact, these relations are simply the orthogonality and completeness relations for the Legendre and trigonometric functions. Furthermore, we get relations between the transformation functions:

$$B_n(\hat{\gamma}) = \sum_{k'} C_{nk'}(\alpha) D_{k'}(\beta), \tag{4.20}$$

$$C_{nk'}(\alpha) = \int_0^{2\pi} \mathrm{d}\beta \, B_n(\hat{\gamma}) D_{k'}^{\dagger}(\beta), \tag{4.21}$$

$$D_k(\beta) = \sum_{n'} \int_0^{\pi} \mathrm{d}\alpha \sin\alpha \, B_{n'}(\hat{\gamma}) \, C_{n'k}^{\dagger}(\alpha). \tag{4.22}$$

It is now natural to ask if there are any transformations involving the outgoing functions in addition to those already given. Apart from the possibility of obtaining additional expansions of the outgoing spherical and cylindrical waves, with other geometrical restrictions, we strongly believe that we have listed all possible transformations. Indications of this are obtained by considering the radiation condition or by assuming an expansion and looking at the implications of this. Let us, e.g. assume an expansion of a plane wave in the outgoing spherical waves, and then expand the plane wave in the regular spherical waves employing Eq. (4.8). Multiplying by the transformation function $B_n(\hat{\gamma})$, integrating, and using Eq. (4.16), we would then obtain an expansion of a regular spherical wave in the outgoing ones. But this would be a very peculiar expansion, which seems impossible to realize in any domain of interest.

The wave functions can in fact be said to be ordered in a definite hierarchy, so that a function with a more discrete (less continuous) spectrum can be expanded in functions with a less discrete (more continuous) spectrum. This may also be expressed by saying that a more singular function can be expanded in less singular ones (under suitable geometrical restrictions). The hierarchy is illustrated in Fig. 3, where an arrow from one wave function to another means that the former can be expanded in terms of the latter.

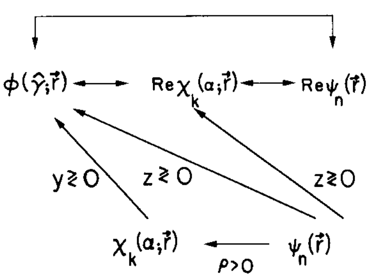

Fig. 3. Existing transformations between the wave functions; an arrow from one function to another means that the former can be expanded in terms of the latter.

4.2. Vector wave functions

All the remarks concerning the existence of transformations involving the outgoing wave functions are valid for the vector case as well. The completeness relations will also have the same appearance, so we will not reproduce them.

The transformation function for the spherical and cylindrical waves is

$$
\begin{aligned}
C_{nk'}(\alpha) &= C_{\tau\sigma ml,\,\tau'\sigma'm'}(\alpha) \\
&= \delta_{mm'} \mathrm{i}^{m-l}\left[\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\right]^{1/2}
\end{aligned}
$$

$$
\times \left|\begin{array}{cccccc}
-\mathrm{i}\Delta_l^m(\alpha) & 0 & 0 & \pi_l^m(\alpha) & 0 & 0 \\
0 & -\mathrm{i}\Delta_l^m(\alpha) & -\pi_l^m(\alpha) & 0 & 0 & 0 \\
0 & \pi_l^m(\alpha) & -\mathrm{i}\Delta_l^m(\alpha) & 0 & 0 & 0 \\
-\pi_l^m(\alpha) & 0 & 0 & -\mathrm{i}\Delta_l^m(\alpha) & 0 & 0 \\
0 & 0 & 0 & 0 & P_l^m(\cos\alpha) & 0 \\
0 & 0 & 0 & 0 & 0 & P_l^m(\cos\alpha)
\end{array}\right|
\begin{array}{c}
\tau\sigma = \\ 1\mathrm{e} \\ 1\mathrm{o} \\ 2\mathrm{e} \\ 2\mathrm{o} \\ 3\mathrm{e} \\ 3\mathrm{o}
\end{array}
$$

$$
\tau'\sigma' = \quad 1\mathrm{e} \quad 1\mathrm{o} \quad 2\mathrm{e} \quad 2\mathrm{o} \quad 3\mathrm{e} \quad 3\mathrm{o} \tag{4.23}
$$

where

$$
\begin{aligned}
\Delta_l^m(\alpha) &= [l(l+1)]^{-1/2}\frac{\mathrm{d}}{\mathrm{d}\alpha}P_l^m(\cos\alpha), \\
\pi_l^m(\alpha) &= [l(l+1)]^{-1/2}\, mP_l^m(\cos\alpha)/\sin\alpha.
\end{aligned} \tag{4.24}
$$

$C^{\dagger}_{nk'}(\alpha)$ has all the explicit "i" changed to " − i". The transformations analogous to Eqs. (4.2)–(4.5) are (Pogorzelski and Lun 1976, Boström and Olsson 1981)

$$
\operatorname{Re}\boldsymbol{\psi}_n(\boldsymbol{r}) = \sum_{k'}\int_0^{\pi} \mathrm{d}\alpha \sin\alpha\, C_{nk'}(\alpha) \operatorname{Re}\boldsymbol{\chi}_{k'}(\alpha;\boldsymbol{r}), \tag{4.25}
$$

$$
\operatorname{Re}\boldsymbol{\chi}_k(\alpha;\boldsymbol{r}) = \sum_{n'} C^{\dagger}_{n'k}(\alpha)\operatorname{Re}\boldsymbol{\psi}_{n'}(\boldsymbol{r}), \tag{4.26}
$$

$$
\boldsymbol{\psi}_n(\boldsymbol{r}) = 2\sum_{k'}\int_{C_\pm} \mathrm{d}\alpha\sin\alpha\, C_{nk'}(\alpha)\operatorname{Re}\boldsymbol{\chi}_{k'}(\alpha;\boldsymbol{r}), \quad z \gtrless 0, \tag{4.27}
$$

$$
\boldsymbol{\psi}_n(\boldsymbol{r}) = \sum_{k'}\int_{C} \mathrm{d}\alpha\sin\alpha\, C_{nk'}(\alpha)\boldsymbol{\chi}_{k'}(\alpha;\boldsymbol{r}), \quad \rho > 0. \tag{4.28}
$$

For the transformations between the spherical and plane waves the transformation function is

$$B_{nj}(\hat{\gamma}) = B_{\tau\sigma ml,\, j}(\alpha, \beta)$$
$$= \mathrm{i}^{-l}\left[\frac{\varepsilon_m}{2\pi}\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\right]^{1/2}$$
$$\times \begin{bmatrix} -\mathrm{i}\Delta_l^m(\alpha)\begin{pmatrix}\cos m\beta\\ \sin m\beta\end{pmatrix} & -\pi_l^m(\alpha)\begin{pmatrix}-\sin m\beta\\ \cos m\beta\end{pmatrix} & 0 \\ -\pi_l^m(\alpha)\begin{pmatrix}-\sin m\beta\\ \cos m\beta\end{pmatrix} & -\mathrm{i}\Delta_l^m(\alpha)\begin{pmatrix}\cos m\beta\\ \sin m\beta\end{pmatrix} & 0 \\ 0 & 0 & P_l^m(\cos\alpha)\begin{pmatrix}\cos m\beta\\ \sin m\beta\end{pmatrix}\end{bmatrix} \begin{matrix} \tau\sigma = \\ 1^{\mathrm{e}}_{\mathrm{o}} \\ 2^{\mathrm{e}}_{\mathrm{o}}, \\ 3^{\mathrm{e}}_{\mathrm{o}} \end{matrix}$$
$$j = \qquad 1 \qquad\qquad 2 \qquad\qquad 3 \tag{4.29}$$

where $\Delta_l^m(\alpha)$ and $\pi_l^m(\alpha)$ are given in Eq. (4.24). $B_{nj}^{\dagger}(\hat{\gamma})$ has all the explicit "i" changed to " − i". We note that $B_{nj}(\hat{\gamma})$ is essentially one of the spherical components of the vector spherical harmonic $\boldsymbol{A}_n(\hat{\gamma})$ [cf. Eq. (2.13)]. The transformations analogous to Eqs. (4.7)–(4.9) are (Morse and Feshbach 1953, Devaney and Wolf 1974)

$$\mathrm{Re}\,\boldsymbol{\psi}_n(\boldsymbol{r}) = \sum_j \int_0^{\pi} \mathrm{d}\hat{\gamma}\, B_{nj}(\hat{\gamma})\boldsymbol{\phi}_j(\hat{\gamma}; \boldsymbol{r}), \tag{4.30}$$

$$\boldsymbol{\phi}_j(\hat{\gamma}; \boldsymbol{r}) = \sum_n B_{nj}^{\dagger}(\hat{\gamma})\,\mathrm{Re}\,\boldsymbol{\psi}_n(\boldsymbol{r}), \tag{4.31}$$

$$\boldsymbol{\psi}_n(\boldsymbol{r}) = 2\sum_j \int_{C_{\pm}} \mathrm{d}\hat{\gamma}\, B_{nj}(\hat{\gamma})\boldsymbol{\phi}_j(\hat{\gamma}; \boldsymbol{r}), \quad z \gtrless 0. \tag{4.32}$$

The transformations between the cylindrical and plane waves are the same for the calar and vector cases, since both these vector waves are constructed in exactly the same manner from the corresponding scalar waves. Thus, the transformation function is

$$D_{kj}(\beta) = D_{\tau\sigma m,\, j}(\beta) = \delta_{\tau j}\mathrm{i}^{-m}[\varepsilon_m/2\pi]^{1/2}\begin{pmatrix}\cos m\beta\\ \sin m\beta\end{pmatrix}, \tag{4.33}$$

and analogous to Eqs. (4.11)–(4.13) we have

$$\mathrm{Re}\,\boldsymbol{\chi}_k(\alpha; \boldsymbol{r}) = \sum_j \int_0^{2\pi} \mathrm{d}\beta\, D_{kj}(\beta)\boldsymbol{\phi}_j(\hat{\gamma}; \boldsymbol{r}), \tag{4.34*}$$

$$\boldsymbol{\phi}_j(\hat{\gamma}; \boldsymbol{r}) = \sum_k D_{kj}^{\dagger}(\beta)\,\mathrm{Re}\,\boldsymbol{\chi}_k(\alpha; \boldsymbol{r}), \tag{4.35*}$$

$$\boldsymbol{\chi}_k(\alpha; \boldsymbol{r}) = 2\sum_j \int_{\Gamma_{\pm}} \mathrm{d}\beta\, D_{kj}(\beta)\boldsymbol{\phi}_j(\hat{\gamma}; \boldsymbol{r}) \quad y \gtrless 0; \quad \alpha \in [0, \pi]. \tag{4.36*}$$

In conclusion, we illustrate with one example how these three-dimensional transformation formulas are used to construct their two-dimensional counterparts. Consider, e.g. Eq. (4.34). According to the prescription given in Section 3, we use $\alpha = \frac{1}{2}\pi$ and disregard the $\hat{z}$ components. Thus, on the left-hand side of Eq. (4.34) there remains $\operatorname{Re}\chi_{1\sigma m}(\frac{1}{2}\pi; x, y, 0)$ and $\operatorname{Re}\chi_{3\sigma m}(\frac{1}{2}\pi; x, y, 0)$. After applying the same prescription to the right-hand side, there remains $\boldsymbol{\phi}_1(\hat{\gamma}(\alpha = \frac{1}{2}\pi, \beta);\ x,\ y,\ 0)$ and $\boldsymbol{\phi}_3(\hat{\gamma}(\alpha = \frac{1}{2}\pi, \beta);\ x, y, 0)$. Note that these two modes are enumerated by $\tau = 1, 2$ in the two-dimensional wave functions. However, the only τ and j dependence in the transformation function $D_{kj}(\beta)$ is a factor $\delta_{\tau j}$ [cf. Eq. (4.33)]. Therefore, since the change in the normalization is the same, i.e.

$$\operatorname{Re}\chi_{3\sigma m}(\tfrac{1}{2}\pi; x, y, 0) = (2\pi)^{-1/2}(k_{\mathrm{p}}/k_{\mathrm{s}})^{1/2}\operatorname{Re}\chi_{2\sigma m}(x, y) \tag{4.37}$$

and

$$\boldsymbol{\phi}_3(\hat{\gamma}(\alpha = \tfrac{1}{2}\pi, \beta); x, y, 0) = (2\pi)^{-1/2}(k_{\mathrm{p}}/k_{\mathrm{s}})^{1/2}\boldsymbol{\phi}_2(\beta; x, y), \tag{4.38}$$

we get

$$\operatorname{Re}\chi_k(\boldsymbol{r}) = \sum_{j=1}^{2}\int_0^{2\pi} \mathrm{d}\beta\, D_{kj}(\beta)\boldsymbol{\phi}_j(\beta; \boldsymbol{r}), \tag{4.39}$$

which is the desired two-dimensional version of Eq. (4.34).

5. Translations

In this section we will review the translation properties of both the scalar and vector wave functions defined in Section 2. A general translation vector is denoted $\boldsymbol{d}$ and we write $\boldsymbol{r}' = \boldsymbol{d} + \boldsymbol{r}$. Here, we do not consider any rotation of the coordinate axes, i.e. they are pointing in the same direction in both coordinate systems. A more general transformation can be performed by introducing rotations.

5.1. Scalar wave functions

The plane waves have trivial translation properties. We have

$$\phi(\hat{\gamma}; \boldsymbol{r}') = \mathrm{e}^{\mathrm{i}k\hat{\gamma}\cdot \boldsymbol{d}}\,\phi(\hat{\gamma}; \boldsymbol{r}). \tag{5.1}^*$$

This is just a phase adjustment for the plane wave, due to the shift of origin.

The translation properties of the spherical wave functions are well known and can be found, e.g. in Danos and Maximon (1965), Miller (1964) and Peterson and Ström (1974). Let (r, θ, ϕ), (d, η, ψ) and (r', θ', ϕ') be the spherical coordinates of $\boldsymbol{r}$, $\boldsymbol{d}$, and $\boldsymbol{r}'$, respectively, where as above $\boldsymbol{r}' = \boldsymbol{d} + \boldsymbol{r}$. We then have

$$\operatorname{Re}\psi_n(\boldsymbol{r}') = \sum_{n'} R_{nn'}(\boldsymbol{d})\operatorname{Re}\psi_{n'}(\boldsymbol{r}), \tag{5.2}$$

$$\psi_n(\boldsymbol{r}') = \sum_{n'} R_{nn'}(\boldsymbol{d})\psi_{n'}(\boldsymbol{r}), \quad r > d \tag{5.3}$$

$$\psi_n(\boldsymbol{r}') = \sum_{n'} P_{nn'}(\boldsymbol{d})\operatorname{Re}\psi_{n'}(\boldsymbol{r}), \quad r < d. \tag{5.4}$$

The matrices $R_{nn'}(\boldsymbol{d})$ and $P_{nn'}(\boldsymbol{d})$ are given by

$$R_{nn'}(\boldsymbol{d}) = S_{nn'}(\boldsymbol{d}; j_\lambda), \tag{5.5}$$

$$P_{nn'}(\boldsymbol{d}) = S_{nn'}(\boldsymbol{d}; h_\lambda^{(1)}). \tag{5.6}$$

The matrix $S_{nn'}(\boldsymbol{d}; z_\lambda)$ may be written as

$$S_{\sigma ml,\,\sigma m'l'}(\boldsymbol{d}; z_\lambda) = (-1)^{m'} B_{ml,\,m'l'}(d, \eta; z_\lambda)\cos(m-m')\psi + (-1)^{\sigma} B_{ml,\,-m'l'}(d, \eta; z_\lambda)\cos(m+m')\psi, \tag{5.7}$$

$$S_{\sigma ml,\,\sigma' m'l'}(\boldsymbol{d}; z_\lambda) = (-1)^{m'+\sigma'} B_{ml,\,m'l'}(d, \eta; z_\lambda)\sin(m-m')\psi + B_{ml,\,-m'l'}(d, \eta; z_\lambda)\sin(m+m')\psi, \quad \sigma \neq \sigma', \tag{5.8}$$

where

$$(-1)^{\sigma} = \begin{cases} 1 & \sigma = \mathrm{e}, \\ -1 & \sigma = \mathrm{o}, \end{cases}$$

and

$$B_{ml,\,m'l'}(d, \eta, z_\lambda) = (-1)^{m+m'}(\varepsilon_m \varepsilon_{m'}/4)^{1/2} \times \sum_{\lambda=|l-l'|}^{l+l'} (-1)^{(l'-l+\lambda)/2}(2\lambda+1)\left[\frac{(2l+1)(2l'+1)(\lambda-(m-m'))!}{(\lambda+(m-m'))!}\right]^{1/2} \times \begin{pmatrix} l & l' & \lambda \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l' & \lambda \\ m & -m' & m'-m \end{pmatrix} z_\lambda(kd) P_\lambda^{m-m'}(\cos\eta) \tag{5.9}$$

In Eqs. (5.7)–(5.9) z_λ stands for j_λ or $h_\lambda^{(1)}$. The $3-j$ symbol $(:\,:\,:)$ can be found, e.g. in Edmonds (1957). Since $\left(\begin{smallmatrix} l & l' & \lambda \\ 0 & 0 & 0 \end{smallmatrix}\right)$ is zero when $l + l' + \lambda$ is an odd integer, the factor $(-1)^{(l'-l+\lambda)/2}$ is real. A translation along the z axis leads to considerable simplifications (Peterson and Ström 1974). In this case $\eta = 0, \pi$ and $S_{nn'}(\pm d\hat{z}; z_\lambda)$ is diagonal in the σ and m indices.

The translation of the cylindrical wave functions in the z direction is a trivial phase shift, while a translation in the x–y plane is more complicated but well known. Let (ρ, ϕ, z), (λ, ψ, ζ) and (ρ', ϕ', z') be the cylindrical coordinates of $\boldsymbol{r}$, $\boldsymbol{d}$, and $\boldsymbol{r}'$, respectively, where, as above, $\boldsymbol{r}' = \boldsymbol{d} + \boldsymbol{r}$. We then have (see, e.g. Danos and Maximon 1965, Miller 1968, Peterson and Ström 1974)

$$\mathrm{Re}\,\chi_k(\alpha; \boldsymbol{r}') = \sum_{k'} R_{kk'}(\alpha; \boldsymbol{d})\,\mathrm{Re}\,\chi_{k'}(\alpha; \boldsymbol{r}), \tag{5.10*}$$

$$\chi_k(\alpha; \boldsymbol{r}') = \sum_{k'} R_{kk'}(\alpha; \boldsymbol{d})\chi_{k'}(\alpha; \boldsymbol{r}), \qquad \rho > \lambda, \tag{5.11*}$$

$$\chi_k(\alpha; \boldsymbol{r}') = \sum_{k'} P_{kk'}(\alpha; \boldsymbol{d})\,\mathrm{Re}\,\chi_{k'}(\alpha; \boldsymbol{r}), \qquad \rho < \lambda, \tag{5.12*}$$

where

$$R_{kk'}(\alpha; \boldsymbol{d}) = S_{kk'}(\alpha; \boldsymbol{d}; J_\nu), \tag{5.13}$$

$$P_{kk'}(\alpha; \boldsymbol{d}) = S_{kk'}(\alpha; \boldsymbol{d}; H_\nu^{(1)}). \tag{5.14}$$

The matrix $S_{kk'}(\alpha; \boldsymbol{d}; Z_\nu)$ may be written

$$S_{\sigma m,\sigma m'}(\alpha; \boldsymbol{d}; Z_\nu) = (\varepsilon_m \varepsilon_{m'}/4)^{1/2} \{ Z_{m-m'}(k\lambda \sin\alpha)\cos(m - m')\psi + (-1)^{m'+\sigma} Z_{m+m'}(k\lambda \sin\alpha)\cos(m + m')\psi \} e^{ik\zeta\cos\alpha} \quad (5.15)$$

$$S_{\sigma m,\sigma' m'}(\alpha; \boldsymbol{d}; Z_\nu) = (\varepsilon_m \varepsilon_{m'}/4)^{1/2} \{ (-1)^{\sigma'} Z_{m-m'}(k\lambda \sin\alpha)\sin(m - m')\psi + (-1)^{m'} Z_{m+m'}(k\lambda \sin\alpha)\sin(m + m')\psi \} e^{ik\zeta\cos\alpha}, \quad \sigma \neq \sigma'. \quad (5.16)$$

In Eqs. (5.15) and (5.16) Z_ν stands for J_ν or $H_\nu^{(1)}$.

The simple translation properties of the plane waves can be used in translating both the spherical and cylindrical wave functions using the interrelating formulas from Section 3. The translation of the regular spherical waves may serve as an example. Thus, we get from Eqs. (4.7), (5.1) and (4.8) (cf. also Appendix B)

$$\mathrm{Re}\,\psi_n(\boldsymbol{r}') = \int_0^\pi d\hat{\gamma}\, B_n(\hat{\gamma})\phi(\hat{\gamma}; \boldsymbol{r}') = \int_0^\pi d\hat{\gamma}\, B_n(\hat{\gamma}) e^{ik\hat{\gamma}\cdot\boldsymbol{d}} \phi(\hat{\gamma}; \boldsymbol{r}) = \sum_{n'} R_{nn'}(\boldsymbol{d})\mathrm{Re}\,\psi_{n'}(\boldsymbol{r}), \quad (5.17)$$

where

$$R_{nn'}(\boldsymbol{d}) = \int_0^\pi d\hat{\gamma}\, B_n(\hat{\gamma}) e^{ik\hat{\gamma}\cdot\boldsymbol{d}} B_{n'}^\dagger(\hat{\gamma}), \quad (5.18)$$

provided the interchange of summation and integration can be justified. That Eq. (5.18) is identical to Eqs. (5.7)–(5.9) can be shown by reducing the product of the spherical harmonics in $B_n(\hat{\gamma})B_{n'}^\dagger(\hat{\gamma})$ to a single spherical harmonic (note that this reduction involves a finite sum), and then by using Eq. (4.7). This justifies the interchange of summation and intergration in Eq. (5.17) since we know that Eq. (5.2) is convergent everywhere. In the same manner we can obtain

$$P_{nn'}(\boldsymbol{d}) = 2\int_{C_\pm} d\hat{\gamma}\, B_n(\hat{\gamma}) e^{ik\hat{\gamma}\cdot\boldsymbol{d}} B_{n'}^\dagger(\hat{\gamma}), \quad \hat{\boldsymbol{z}}\cdot\boldsymbol{d} \gtrless 0, \quad (5.19)$$

$$R_{kk'}(\alpha; \boldsymbol{d}) = \int_0^{2\pi} d\beta\, D_k(\beta) e^{ik\hat{\gamma}\cdot\boldsymbol{d}} D_{k'}^\dagger(\beta), \quad (5.20)$$

$$P_{kk'}(\alpha; \boldsymbol{d}) = 2\int_{\Gamma_\pm} d\beta\, D_k(\beta) e^{ik\hat{\gamma}\cdot\boldsymbol{d}} D_{k'}^\dagger(\beta), \quad \hat{\boldsymbol{y}}\cdot\boldsymbol{d} \gtrless 0; \quad \alpha \in [0, \pi]. \quad (5.21)$$

5.2. Vector wave functions

The formulas found in the preceding subsection for the scalar waves can be extended to the vector case in a rather straightforward way. In this subsection we use notations which are analogous to those for the scalar case.

The plane vector waves have translation properties similar to Eq. (5.1):

$$\boldsymbol{\phi}_j(\hat{\gamma}; \boldsymbol{r}') = e^{ik_s\hat{\gamma}\cdot\boldsymbol{d}} \boldsymbol{\phi}_j(\hat{\gamma}; \boldsymbol{r}), \quad j = 1, 2,$$
$$\boldsymbol{\phi}_3(\hat{\gamma}; \boldsymbol{r}') = e^{ik_p\hat{\gamma}\cdot\boldsymbol{d}} \boldsymbol{\phi}_3(\hat{\gamma}; \boldsymbol{r}). \quad (5.22)^*$$

The translation properties of the spherical vector wave functions, as defined in Eq. (2.12), are similar to those for the scalar waves. However, in the vector case a coupling in the mode index τ occurs for the transverse modes ($\tau = 1, 2$). Viewed as a matrix in the mode indices, the translation matrix has the structure

$$(R_{\tau\tau'}) = \begin{pmatrix} R_{11} & R_{12} & 0 \\ R_{21} & R_{22} & 0 \\ 0 & 0 & R_{33} \end{pmatrix} \tag{5.23}$$

and similarly for $(P_{\tau\tau'})$. We observe that the longitudinal mode ($\tau = 3$) does not couple to the transverse modes ($\tau = 1, 2$). In analogy to Eqs. (5.2)–(5.4) we have

$$\operatorname{Re}\boldsymbol{\psi}_n(\boldsymbol{r}') = \sum_{n'} R_{nn'}(\boldsymbol{d}) \operatorname{Re}\boldsymbol{\psi}_{n'}(\boldsymbol{r}), \tag{5.24}$$

$$\boldsymbol{\psi}_n(\boldsymbol{r}') = \sum_{n'} R_{nn'}(\boldsymbol{d}) \boldsymbol{\psi}_{n'}(\boldsymbol{r}), \qquad r > d, \tag{5.25}$$

$$\boldsymbol{\psi}_n(\boldsymbol{r}') = \sum_{n'} P_{nn'}(\boldsymbol{d}) \operatorname{Re}\boldsymbol{\psi}_{n'}(\boldsymbol{r}), \qquad r < d, \tag{5.26}$$

and, as before,

$$R_{nn'}(\boldsymbol{d}) = S_{nn'}(\boldsymbol{d}; j_\lambda), \tag{5.27}$$

$$P_{nn'}(\boldsymbol{d}) = S_{nn'}(\boldsymbol{d}; h_\lambda^{(1)}). \tag{5.28}$$

However, $S_{nn'}(\boldsymbol{d}; z_\lambda)$ is now more complicated than the scalar case and we have (Boström 1980a, Peterson and Ström 1973, Danos and Maximon 1965)

$$\begin{aligned} S_{1\sigma ml,\, 1\sigma m'l'}(\boldsymbol{d}; z_\lambda) = {} & (-1)^{m'} C_{ml,\, m'l'}(d, \eta; z_\lambda) \cos(m - m')\psi \\ & + (-1)^\sigma C_{ml,\, -m'l'}(d, \eta; z_\lambda) \cos(m + m')\psi, \end{aligned} \tag{5.29}$$

$$\begin{aligned} S_{1\sigma ml,\, 1\sigma' m'l'}(\boldsymbol{d}; z_\lambda) = {} & (-1)^{m'+\sigma'} C_{ml,\, m'l'}(d, \eta; z_\lambda) \sin(m - m')\psi \\ & + C_{ml,\, -m'l'}(d, \eta; z_\lambda) \sin(m + m')\psi, \quad \sigma \neq \sigma', \end{aligned} \tag{5.30}$$

$$\begin{aligned} S_{1\sigma ml,\, 2\sigma' m'l'}(\boldsymbol{d}; z_\lambda) = {} & (-1)^{m'+\sigma} D_{ml,\, m'l'}(d, \eta; z_\lambda) \cos(m - m')\psi \\ & - D_{ml,\, -m'l'}(d, \eta; z_\lambda) \cos(m + m')\psi, \quad \sigma \neq \sigma', \end{aligned} \tag{5.31}$$

$$\begin{aligned} S_{1\sigma ml,\, 2\sigma m'l'}(\boldsymbol{d}; z_\lambda) = {} & (-1)^{m'} D_{ml,\, m'l'}(d, \eta; z_\lambda) \sin(m - m')\psi \\ & + (-1)^\sigma D_{ml,\, -m'l'}(d, \eta; z_\lambda) \sin(m + m')\psi, \end{aligned} \tag{5.32}$$

$$S_{2\sigma ml,\, \tau'\sigma' m'l'}(\boldsymbol{d}; z_\lambda) = S_{1\sigma ml,\, \tau\sigma' m'l'}(\boldsymbol{d}; z_\lambda), \quad \tau \neq \tau' = 1, 2, \tag{5.33}$$

$$\begin{aligned} S_{3\sigma ml,\, 3\sigma m'l'}(\boldsymbol{d}; z_\lambda) = {} & (-1)^{m'} B_{ml,\, m'l'}(d, \eta; z_\lambda) \cos(m - m')\psi \\ & + (-1)^\sigma B_{ml,\, -m'l'}(d, \eta; z_\lambda) \cos(m + m')\psi, \end{aligned} \tag{5.34}$$

$$\begin{aligned} S_{3\sigma ml,\, 3\sigma' m'l'}(\boldsymbol{d}; z_\lambda) = {} & (-1)^{m'+\sigma'} B_{ml,\, m'l'}(d, \eta; z_\lambda) \sin(m - m')\psi \\ & + B_{ml,\, -m'l'}(d, \eta; z_\lambda) \sin(m + m')\psi, \quad \sigma \neq \sigma', \end{aligned} \tag{5.35}$$

where

$$C_{ml,m'l'}(d,\eta;z_\lambda) = (-1)^{m+m'}(\varepsilon_m\varepsilon_{m'}/4)^{1/2}$$
$$\times \tfrac{1}{2}\sum_{\lambda=|l-l'|}^{l+l'} (-1)^{(l'-l+\lambda)/2}(2\lambda+1)\left(\frac{(2l+1)(2l'+1)(\lambda-(m-m'))!}{l(l+1)l'(l'+1)(\lambda+m-m')!}\right)^{1/2}$$
$$\times\begin{pmatrix} l & l' & \lambda \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l' & \lambda \\ m & -m' & m'-m\end{pmatrix}[l(l+1)+l'(l'+1)-\lambda(\lambda+1)]$$
$$\times z_\lambda(k_s d)P_\lambda^{m-m'}(\cos\eta), \tag{5.36}$$

$$D_{ml,m'l'}(d,\eta;z_\lambda) = (-1)^{m+m'}(\varepsilon_m\varepsilon_{m'}/4)^{1/2}$$
$$\times \tfrac{1}{2}\sum_{\lambda=|l-l'|+1}^{l+l'} \mathrm{i}^{l'-l+\lambda+1}(2\lambda+1)\left(\frac{(2l+1)(2l'+1)(\lambda-(m-m'))!}{l(l+1)l'(l'+1)(\lambda+m-m')!}\right)^{1/2}$$
$$\times\begin{pmatrix} l & l' & \lambda-1 \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l' & \lambda \\ m & -m' & m'-m\end{pmatrix}[\lambda^2-(l-l')^2]^{1/2}$$
$$\times[(l+l'+1)^2-\lambda^2]^{1/2}z_\lambda(k_s d)P_\lambda^{m-m'}(\cos\eta), \tag{5.37}$$

$$B_{ml,m'l'}(d,\eta;z_\lambda) = (-1)^{m+m'}(\varepsilon_m\varepsilon_{m'}/4)^{1/2}$$
$$\times \sum_{\lambda=|l-l'|}^{l+l'} (-1)^{(l'-l+\lambda)/2}(2\lambda+1)\left[\frac{(2l+1)(2l'+1)(\lambda-(m-m'))!}{(\lambda+m-m')!}\right]^{1/2}$$
$$\times\begin{pmatrix} l & l' & \lambda \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l' & \lambda \\ m & -m' & m'-m\end{pmatrix}z_\lambda(k_p d)P_\lambda^{m-m'}(\cos\eta). \tag{5.38}$$

As in the preceding subsection, $(:::)$ is the $3-j$ symbol. A translation along the z axis will lead to simplifications. Thus, $S_{nn'}(\pm d\hat{z};z_\lambda)$ is diagonal in the m index, and in the $(\tau\sigma,\tau'\sigma')$ indices only the terms $(\tau\sigma,\tau\sigma)$, $\tau=1,2,3$, $\sigma=\mathrm{e},\mathrm{o}$ and (1e, 2o), (1o, 2e), (2e, 1o) and (2o, 1e) are different from zero.

Finally, we consider the cylindrical vector wave functions, whose translation properties are almost identical to those for the scalar ones. This can be seen from their definitions in Eq. (2.14). The transverse wave functions are obtained by taking the curl (once or twice) of the scalar wave function times the constant vector $\hat{z}$, and the longitudinal wave function is just the gradient of the scalar wave function. Since the gradient, the curl and the $\hat{z}$ vector are the same in both the original and the translated coordinate systems we have

$$\mathrm{Re}\,\boldsymbol{\chi}_k(\alpha;\boldsymbol{r}') = \sum_{k'} R_{kk'}(\alpha;\boldsymbol{d})\mathrm{Re}\,\boldsymbol{\chi}_{k'}(\alpha;\boldsymbol{r}), \tag{5.39*}$$

$$\boldsymbol{\chi}_k(\alpha;\boldsymbol{r}') = \sum_{k'} R_{kk'}(\alpha;\boldsymbol{d})\boldsymbol{\chi}_{k'}(\alpha;\boldsymbol{r}), \qquad \rho>\lambda, \tag{5.40*}$$

$$\boldsymbol{\chi}_k(\alpha;\boldsymbol{r}') = \sum_{k'} P_{kk'}(\alpha;\boldsymbol{d})\mathrm{Re}\,\boldsymbol{\chi}_{k'}(\alpha;\boldsymbol{r}), \qquad \rho<\lambda. \tag{5.41*}$$

The definitions of $R_{kk'}(\alpha; \boldsymbol{d})$ and $P_{kk'}(\alpha; \boldsymbol{d})$ are almost identical to those for the scalar case, cf. Eqs. (5.13)–(5.16):

$$R_{kk'}(\alpha; \boldsymbol{d}) = S_{kk'}(\alpha; \boldsymbol{d}; J_\nu), \tag{5.42}$$

$$P_{kk'}(\alpha; \boldsymbol{d}) = S_{kk'}(\alpha; \boldsymbol{d}; H_\nu^{(1)}), \tag{5.43}$$

and

$$S_{\tau\sigma m,\,\tau'\sigma m'}(\alpha; \boldsymbol{d}; Z_\nu) = (\varepsilon_m \varepsilon_{m'}/4)^{1/2} \delta_{\tau\tau'} \{ Z_{m-m'}(k_s \lambda \sin\alpha) \cos(m-m')\psi + (-1)^{m'+\sigma} Z_{m+m'}(k_s \lambda \sin\alpha) \cos(m+m')\psi \} \mathrm{e}^{\mathrm{i}k_s \zeta \cos\alpha}, \tag{5.44}$$

$$S_{\tau\sigma m,\,\tau'\sigma' m'}(\alpha; \boldsymbol{d}; Z_\nu) = (\varepsilon_m \varepsilon_{m'}/4)^{1/2} \delta_{\tau\tau'} \{ (-1)^{\sigma'} Z_{m-m'}(k_s \lambda \sin\alpha) \sin(m-m')\psi + (-1)^{m'} Z_{m+m'}(k_s \lambda \sin\alpha) \sin(m+m')\psi \} \mathrm{e}^{\mathrm{i}k_s \zeta \cos\alpha}, \tag{5.45}$$

where $\sigma \neq \sigma'$ and $\tau, \tau' = 1, 2$ and

$$S_{3\sigma m,\,3\sigma m'}(\alpha; \boldsymbol{d}; Z_\nu) = (\varepsilon_m \varepsilon_{m'}/4)^{1/2} \{ Z_{m-m'}(k_p \lambda \sin\alpha) \cos(m-m')\psi + (-1)^{m'+\sigma} Z_{m+m'}(k_p \lambda \sin\alpha) \cos(m+m')\psi \} \mathrm{e}^{\mathrm{i}k_p \zeta \cos\alpha}, \tag{5.46}$$

$$S_{3\sigma m,\,3\sigma' m'}(\alpha; \boldsymbol{d}; Z_\nu) = (\varepsilon_m \varepsilon_{m'}/4)^{1/2} \{ (-1)^{\sigma'} Z_{m-m'}(k_p \lambda \sin\alpha) \sin(m-m')\psi + (-1)^{m'} Z_{m+m'}(k_p \lambda \sin\alpha) \sin(m+m')\psi \} \mathrm{e}^{\mathrm{i}k_p \zeta \cos\alpha}, \tag{5.47}$$

where $\sigma \neq \sigma'$.

Integral representations of the translation matrices $R_{nn'}$ and $P_{nn'}$ can be obtained in analogy to the scalar case [cf. Eqs. (5.18) and (5.19)]. The integral representations of the translation matrices for the cylindrical vector waves are essentially the same as in the scalar case and are not repeated here. For the spherical vector waves we have

$$\begin{aligned} R_{nn'}(\boldsymbol{d}) &= \sum_j \int_0^\pi \mathrm{d}\hat{\gamma}\, B_{nj}(\hat{\gamma})\, \mathrm{e}^{\mathrm{i}k_s \hat{\gamma}\cdot \boldsymbol{d}} B^\dagger_{n'j}(\hat{\gamma}), \qquad \tau, \tau' = 1, 2, \\ R_{nn'}(\boldsymbol{d}) &= \sum_j \int_0^\pi \mathrm{d}\hat{\gamma}\, B_{nj}(\hat{\gamma})\, \mathrm{e}^{\mathrm{i}k_p \hat{\gamma}\cdot \boldsymbol{d}} B^\dagger_{n'j}(\hat{\gamma}), \qquad \tau = \tau' = 3, \end{aligned} \tag{5.48}$$

$$\begin{aligned} P_{nn'}(\boldsymbol{d}) &= 2\sum_j \int_{C_\pm} \mathrm{d}\hat{\gamma}\, B_{nj}(\hat{\gamma})\, \mathrm{e}^{\mathrm{i}k_s \hat{\gamma}\cdot \boldsymbol{d}} B^\dagger_{n'j}(\hat{\gamma}), \qquad \tau, \tau' = 1, 2;\ \hat{z}\cdot\boldsymbol{d} \gtrless 0, \\ P_{nn'}(\boldsymbol{d}) &= 2\sum_j \int_{C_\pm} \mathrm{d}\hat{\gamma}\, B_{nj}(\hat{\gamma})\, \mathrm{e}^{\mathrm{i}k_p \hat{\gamma}\cdot \boldsymbol{d}} B^\dagger_{n'j}(\hat{\gamma}), \qquad \tau = \tau' = 3;\ \hat{z}\cdot\boldsymbol{d} \gtrless 0. \end{aligned} \tag{5.49}$$

6. Rotations

We now turn to the relation between the wave functions in an original (x, y, z) and a rotated (x', y', z') coordinate system. The rotation can be parametrized by the conventional Euler angles $(\psi, \eta, 0)$ (i.e. an azimuthal rotation ψ about the z axis, succeeded by a rotation η about the rotated y' axis). The most general rotation also includes a final azimuthal rotation about the rotated y' axis, but since this rotation is a trivial shift in the azimuthal angle no generality is lost by putting it equal to zero. The spherical angles of both $\boldsymbol{r}$ and $\hat{\boldsymbol{y}}$ (the direction of the plane wave) in the rotated system are denoted by a prime (i.e. (θ', ϕ') and (α', β'), respectively) (see Fig. 4). The relation between the original and rotated coordinate systems can be found e.g. in Edmonds (1957):

$$\boldsymbol{r}' = R\boldsymbol{r}, \qquad \hat{\boldsymbol{y}}' = R\hat{\boldsymbol{y}}. \tag{6.1}$$

Here $\boldsymbol{r}, \boldsymbol{r}', \hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{y}}'$ are interpreted as column vectors and R as a matrix:

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \boldsymbol{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, \quad \hat{\boldsymbol{y}} = \begin{pmatrix} \sin\alpha\cos\beta \\ \sin\alpha\sin\beta \\ \cos\alpha \end{pmatrix}, \quad \hat{\boldsymbol{y}}' = \begin{pmatrix} \sin\alpha'\cos\beta' \\ \sin\alpha'\sin\beta' \\ \cos\alpha' \end{pmatrix},$$
$$R = \begin{pmatrix} \cos\eta & 0 & -\sin\eta \\ 0 & 1 & 0 \\ \sin\eta & 0 & \cos\eta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.2}$$

6.1. Scalar wave functions

First we consider the plane wave functions, since a rotation of these is more or less trivial. This is a consequence of the fact that the argument in the exponential is

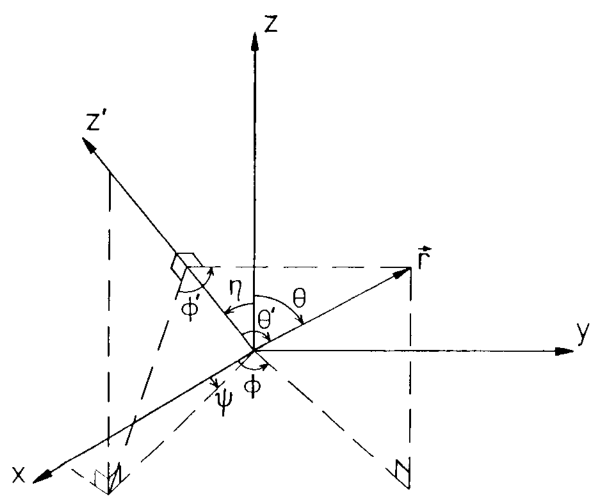

Fig. 4. Angles involved in a rotation of the coordinate system.

invariant under a rotation of the coordinate system. More explicitly, we obtain [for the definition of the plane waves, see Eq. (2.5)]

$$\phi(\hat{\gamma}'; \boldsymbol{r}') = \phi(\hat{\gamma}'; R\boldsymbol{r}) = \phi(R^{t}\hat{\gamma}'; \boldsymbol{r}) = \phi(\hat{\gamma}; \boldsymbol{r}), \tag{6.3}^*$$

where R^t is the transpose of R.

The relation between the spherical waves in the rotated and the original system is well known. The rotation does not affect the absolute value of $\boldsymbol{r}$ and this implies no coupling in the l index. We refer, e.g. to Rose (1957) and Edmonds (1957) for more details. We have

$$\operatorname{Re}\psi_n(\boldsymbol{r}') = \sum_{n'} \mathscr{D}_{nn'}(\eta, \psi) \operatorname{Re}\psi_{n'}(\boldsymbol{r}), \tag{6.4}$$

$$\psi_n(\boldsymbol{r}') = \sum_{n'} \mathscr{D}_{nn'}(\eta, \psi)\psi_{n'}(\boldsymbol{r}). \tag{6.5}$$

The rotation matrix $\mathscr{D}_{nn'}(\eta, \psi)$ is diagonal in the l index and can be expressed in a number of ways. We use the form (with matrices in the σ index)

$$\mathscr{D}_{nn'}(\eta, \psi) = \delta_{ll'}(\varepsilon_m \varepsilon_{m'}/4)^{1/2}(-1)^{m+m'} \begin{pmatrix} A^l_{mm'}(\eta) & 0 \\ 0 & B^l_{mm'}(\eta) \end{pmatrix} \begin{pmatrix} \cos m'\psi & \sin m'\psi \\ -\sin m'\psi & \cos m'\psi \end{pmatrix}, \tag{6.6}$$

where

$$\begin{aligned} A^l_{mm'}(\eta) &= d^l_{mm'}(\eta) + (-1)^{m'} d^l_{m-m'}(\eta), \\ B^l_{mm'}(\eta) &= d^l_{mm'}(\eta) - (-1)^{m'} d^l_{m-m'}(\eta), \end{aligned} \tag{6.7}$$

with

$$\begin{aligned} d^l_{mm'}(\eta) &= \left(\frac{(l+m)!(l-m)!}{(l+m')!(l-m')!} \right)^{1/2} \cos^{m+m'} \eta/2 \sin^{m-m'} \eta/2 \times P^{(m-m',\, m+m')}_{l-m}(\cos\eta), \\ d^l_{m\,-m'}(\eta) &= (-1)^{l+m} d^l_{mm'}(\pi - \eta). \end{aligned} \tag{6.8}$$

Here $P^{(\alpha,\beta)}_n(\cos\eta)$ is the Jacobi polynomial [for a definition see Edmonds (1957)]. It should be noted that we use a slightly different normalization of the spherical harmonics as compared to Edmonds (1957) [cf. Eq. (2.2)].

It may also be illustrative to consider another way of finding the rotation matrices $\mathscr{D}_{nn'}(\eta, \psi)$. In a preceding paragraph we noted the simple rotation properties of the plane waves, and from Section 4.1 we know the transformations between the spherical and plane waves. Thus, we obtain from Eqs. (4.7), (6.3) and (4.8),

$$\begin{aligned} \operatorname{Re}\psi_n(\boldsymbol{r}') &= \int_0^\pi \mathrm{d}\hat{\gamma}'\, B_n(\hat{\gamma}')\phi(\hat{\gamma}'; \boldsymbol{r}') = \int_0^\pi \mathrm{d}\hat{\gamma}'\, B_n(\hat{\gamma}')\phi(R^t\hat{\gamma}'; \boldsymbol{r}) \\ &= \int_0^\pi \mathrm{d}\hat{\gamma}'\, B_n(\hat{\gamma}') \sum_{n'} B^\dagger_{n'}(R^t\hat{\gamma}') \operatorname{Re}\psi_{n'}(\boldsymbol{r}) \\ &= \sum_{n'} \mathscr{D}_{nn'}(\eta, \psi) \operatorname{Re}\psi_{n'}(\boldsymbol{r}), \end{aligned} \tag{6.9}$$

where

$$\mathscr{D}_{nn'}(\eta, \psi) = \int_0^\pi d\hat{\gamma}' B_n(\hat{\gamma}') B_{n'}^\dagger(R^t\hat{\gamma}'). \tag{6.10}$$

This expression is an integral representation of the rotation matrix and can be verified by means of group-theoretical results and the orthogonality of the spherical harmonics.

We observe that the singularity of the outgoing spherical wave function ψ_n at the origin is invariant under a rotation (η, ψ). This is not the case for the outgoing cylindrical wavefunction $\chi_k(\alpha)$ (except for $\eta = 0$). The singular z axis does not coincide with the rotated z' axis, and this property suggests that no simple expression for the rotation of the outgoing cylindrical wave functions exists (no cylinder exists which circumscribes both the z and z' axes). In this chapter we will not pursue these aspects any further, but concentrate on the regular cylindrical waves, for which these problems do not appear.

The rotation of $\operatorname{Re}\chi_k(\alpha)$ can be obtained by means of an intermediate transformation either to the plane or spherical waves. Thus by using Eqs. (4.11), (6.3) and (4.12) we find

$$\operatorname{Re}\chi_k(\alpha'; \boldsymbol{r}') = \int_0^{2\pi} d\beta' D_k(\beta')\phi(\hat{\gamma}'; \boldsymbol{r}') = \int_0^{2\pi} d\beta' D_k(\beta') \sum_{k'} D_{k'}^\dagger(\beta) \operatorname{Re}\chi_{k'}(\alpha; \boldsymbol{r}). \tag{6.11*}$$

In this expression α, β are functions of the angles η, ψ and α', β' [cf., Eqs. (6.1) and (6.2)]. In the alternative way of expressing the rotation we employ the spherical wave functions. Equations (4.3), (6.3) and (4.2) yield

$$\begin{aligned}
\operatorname{Re}\chi_k(\alpha'; \boldsymbol{r}') &= \sum_{n_1} C_{n_1k}^\dagger(\alpha') \operatorname{Re}\psi_{n_1}(\boldsymbol{r}') \\
&= \sum_{n_1 n_2} C_{n_1k}^\dagger(\alpha')\, \mathscr{D}_{n_1n_2}(\eta, \psi) \sum_{k'} \int_0^\pi d\alpha \sin\alpha\, C_{n_2k'}(\alpha) \operatorname{Re}\chi_{k'}(\alpha; \boldsymbol{r}) \\
&= \sum_{k'} \int_0^\pi d\alpha \sin\alpha\, \mathscr{D}_{kk'}(\eta, \psi; \alpha, \alpha') \operatorname{Re}\chi_{k'}(\alpha; \boldsymbol{r}).
\end{aligned} \tag{6.12}$$

Here we have introduced

$$\mathscr{D}_{kk'}(\eta, \psi; \alpha, \alpha') = \sum_{n_1 n_2} C_{n_1k}^\dagger(\alpha')\, \mathscr{D}_{n_1n_2}(\eta, \psi)\, C_{n_2k'}(\alpha). \tag{6.13}$$

Equations (6.11) and (6.12) are believed to be of limited value in applications, and they are included mainly for completeness.

6.2. *Vector wave functions*

The relations found in the preceding subsection for the scalar wave functions can be used to obtain the corresponding formulas for the vector case. However, we have to pay proper attention to the vector character of the wave functions.

For convenience we treat the spherical waves first in this subsection, since in some respects this set has the simplest rotation properties. We notice that the definition of the spherical vector wave functions [see Eq. (2.12)] contains the curl of the position vector $\boldsymbol{r}$ times the scalar wave function or just the gradient of the scalar wave function. Since the gradient, the curl and the position vector transform as vectors under a rotation, we obtain

$$\begin{aligned}
\boldsymbol{\psi}'_{\tau n}(\boldsymbol{r}') &= [l(l+1)]^{-1/2}(1/k_s\nabla'\times)^{\tau}(k_s\boldsymbol{r}'\psi_{\sigma ml}(\boldsymbol{r}')) \\
&= [l(l+1)]^{-1/2}(1/k_s\nabla\times)^{\tau}\left(k_s\boldsymbol{r}\sum_{n'}\mathscr{D}_{nn'}(\eta,\psi)\psi_{n'}(\boldsymbol{r})\right) \\
&= \sum_{n'}\mathscr{D}_{nn'}(\eta,\psi)\boldsymbol{\psi}_{\tau n'}(\boldsymbol{r}), \quad \tau = 1,2, \\
\boldsymbol{\psi}'_{\tau n}(\boldsymbol{r}') &= (k_p/k_s)^{3/2}(1/k_p)\nabla'\psi_{\sigma ml}(\boldsymbol{r}') \\
&= (k_p/k_s)^{3/2}(1/k_p)\nabla\sum_{n'}\mathscr{D}_{nn'}(\eta,\psi)\psi_{n'}(\boldsymbol{r}) \\
&= \sum_{n'}\mathscr{D}_{nn'}(\eta,\psi)\boldsymbol{\psi}_{\tau n'}(\boldsymbol{r}), \quad \tau = 3.
\end{aligned} \tag{6.14}$$

Here the prime on the vector wave functions on the left-hand side reminds us that the vector components are with respect to the primed spherical unit vectors $\hat{\boldsymbol{r}}'$, $\hat{\boldsymbol{\theta}}'$, $\hat{\boldsymbol{\phi}}'$. The definition of the rotation matrix $\mathscr{D}_{nn'}(\eta, \psi)$ is found in Eqs. (6.6)–(6.8).

As a consequence of the vector operations used in the definition of the spherical vector waves we have obtained fairly simple transformation properties under a rotation of the coordinate system. For the plane vector waves the situation is different, since a specific direction $\hat{\boldsymbol{z}}$ appears in their definition. Therefore, they possess more complicated transformation properties under a rotation. Nevertheless, it is most convenient to work with the definition found in Eq. (2.16) (a definition analogous to the spherical one inevitably leads to complications in various respects).

The rectangular coordinates of $\hat{\boldsymbol{\gamma}}$ are transformed according to Eq. (6.1), but this is not the case for the vectors $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$. The two right-handed triplets $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}})$ and $(\hat{\boldsymbol{\alpha}}', \hat{\boldsymbol{\beta}}', \hat{\boldsymbol{\gamma}}')$ are both orthogonal and they are linearly dependent; i.e. $(\hat{\boldsymbol{\alpha}}', \hat{\boldsymbol{\beta}}', \hat{\boldsymbol{\gamma}}')$ can be expressed in terms of the triplet $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}})$ and vice versa. Due to the fact that the vector $\hat{\boldsymbol{\gamma}}$ is not changed under the rotation of the coordinate system only $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\alpha}}'$, $\hat{\boldsymbol{\beta}}'$ couple, and this transformation is a rotation with an angle Ω around the vector $\hat{\boldsymbol{\gamma}}$. We thus have

$$\boldsymbol{\phi}'_j(\hat{\boldsymbol{\gamma}}';\boldsymbol{r}') = \sum_{j'}\mathscr{R}_{jj'}\boldsymbol{\phi}_{j'}(\hat{\boldsymbol{\gamma}};\boldsymbol{r}), \quad (\mathscr{R}_{jj'}) = \begin{pmatrix} \cos\Omega & -\mathrm{i}\sin\Omega & 0 \\ -\mathrm{i}\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.15}$$

Note that the matrix $\mathscr{R}$ describes the coupling in the mode index j, and should not be confused with the matrix R, cf. Eq. (6.2), which effects the transformation of the rectangular coordinates under the rotation. As above, the prime on the left-hand side in Eq. (6.15) reminds us that the components of the plane vector wave are with respect to the primed unit vectors. The matrix $\mathscr{R}_{jj'}$ that performs the rotation of the unit vectors is a function of the angles of rotation (η, ψ) and the angles (α, β), and we prefer to write the elements of $\mathscr{R}$ as

$$\cos\Omega = \frac{\sin\alpha\cos\eta - \cos\alpha\sin\eta\cos(\beta-\psi)}{\sin\alpha'}, \qquad \sin\Omega = \frac{\sin\eta\sin(\beta-\psi)}{\sin\alpha'}, \tag{6.16}$$

where $\sin\alpha'$ is determined from $\cos\alpha' = \sin\alpha\sin\eta\cos(\beta-\psi) + \cos\alpha\cos\eta$.

The cylindrical vector waves are defined in analogy to the plane vector waves, cf. Eqs. (2.14) and (2.16). In the scalar case we only discussed the regular cylindrical waves and this is also the case in this subsection. Thus we obtain, either by transforming to the plane waves or to the spherical waves, the following formulas, cf. Eqs. (4.34), (6.15), (4.35) and (4.26), (6.14), (4.25):

$$\operatorname{Re}\chi'_k(\alpha'; \boldsymbol{r}') = \sum_{jj'}\int_0^{2\pi} d\beta'\, \mathscr{D}_{kj}(\beta')\mathscr{R}_{jj'}\sum_{k'}\mathscr{D}^{\dagger}_{k'j'}(\beta)\operatorname{Re}\chi_{k'}(\alpha; \boldsymbol{r}), \tag{6.17*}$$

$$\operatorname{Re}\chi'_k(\alpha'; \boldsymbol{r}') = \sum_{k'}\int_0^{\pi} d\alpha \sin\alpha\, \mathscr{D}_{kk'}(\eta, \psi; \alpha', \alpha)\operatorname{Re}\chi_{k'}(\alpha; \boldsymbol{r}). \tag{6.18}$$

Here α, β are functions of η, ψ and α', β', and analogously to the scalar case we have defined

$$\mathscr{D}_{kk'}(\eta, \psi; \alpha', \alpha) = \sum_{n_1 n_2} C^{\dagger}_{n_1 k}(\alpha')\mathscr{D}_{n_1 n_2}(\eta, \psi)\, C_{n_2 k'}(\alpha). \tag{6.19}$$

7. An illustrative problem

As already pointed out in the introduction, the transformation properties discussed in this chapter are relevant in solving a large number of scattering problems with more than one diffracting surface. This is true for most of the approaches usually employed, exceptions being many approximate and purely numerical methods. Some examples of multiple-scattering problems where transformation properties have been applied include two and several bounded obstacles (see, e.g. Peterson and Ström 1973, 1974, Boström 1980a, Twersky 1967, Van Buren and King 1972, Cooray and Ciric 1989), a plane interface and a bounded obstacle (see, e.g. Boström and Kristensson 1980, Yakonov 1959, Höllinger and Ziegler 1979, Kristensson and Ström 1978, Kristensson 1980, Boström and Karlsson 1984, 1987), and a sphere or spheroid in a cylindrical waveguide (Boström 1980b, Boström and Olsson 1981, Morita et al. 1979). Transformation properties are also needed, e.g. when considering scattering of a beam (Pogorzelski and Lun 1976) and multiple-scattering in quantum mechanics (Agassi and Gal 1973).

To further exemplify the use of the transformation properties we will consider in this section a complex scattering geometry, namely an infinite cylinder and a bounded obstacle outside the cylinder. This will also give us an opportunity to discuss another aspect. Namely, if we have reason to expand, e.g. a cylindrical wave function in the spherical waves about a different origin, which should be performed first, the translation or the transformation between the two sets of wave functions?

The method we will employ in the above-mentioned problem is the null-field approach (the T-matrix method), introduced by Waterman (1969, 1971, 1976). Consider the geometry as depicted in Fig. 5. Requirements on the cylinder S_0 and the obstacle S_1 are that there exist inscribed and circumscribed circular cylinders and spheres, respectively. Our coordinate systems are then chosen accordingly, see Fig. 5 (the separation vector $\boldsymbol{d}$ between the origins is pointing towards O_1). Furthermore, the cylinder and the obstacle must not be too close. We remark that it is not necessary in principle that the cylinder S_0 has a constant cross section (although in practice this will probably be the only manageable case). Our starting point is the following integral representation for the scalar field u:

$$u^{i}(\boldsymbol{r}) + \int_{S_0}\left[u_+(\boldsymbol{r}_0)\frac{\partial}{\partial n_0}G(\boldsymbol{r},\boldsymbol{r}_0) - G(\boldsymbol{r},\boldsymbol{r}_0)\left(\frac{\partial}{\partial n_0}u(\boldsymbol{r}_0)\right)_+\right]\mathrm{d}S_0$$
$$+ \int_{S_1}\left[u_+(\boldsymbol{r}_1)\frac{\partial}{\partial n_1}G(\boldsymbol{r},\boldsymbol{r}_1) - G(\boldsymbol{r},\boldsymbol{r}_1)\left(\frac{\partial}{\partial n_1}u(\boldsymbol{r}_1)\right)_+\right]\mathrm{d}S_1$$
$$= \begin{cases} u(\boldsymbol{r}) & \boldsymbol{r} \text{ outside } S_0 \text{ and } S_1, \\ 0 & \boldsymbol{r} \text{ inside } S_0 \text{ or } S_1, \end{cases} \tag{7.1}$$

where u^{i} is the prescribed incoming field with sources outside S_0 and S_1, $\partial/\partial n$ is the normal derivative (outward-pointing normal) and G is the free-space Green's function defined in Section 2.1. For simplicity we take the boundary conditions on S_0 and S_1 to be the homogeneous Neumann conditions so that the second term in the integrals vanishes.

We first employ Eq. (7.1) for $\boldsymbol{r}$ inside the inscribed cylinder of S_0. In the first integral we expand the Green's function in cylindrical wave functions about O [Eq. (2.9)]:

$$G(\boldsymbol{r}_0, \boldsymbol{r}'_0) = \mathrm{i}k\sum_k \int_C \mathrm{d}\alpha \sin\alpha \operatorname{Re}\chi_k(\alpha; \boldsymbol{r}_<)\chi_k^{\dagger}(\alpha; \boldsymbol{r}_>), \tag{7.2}$$

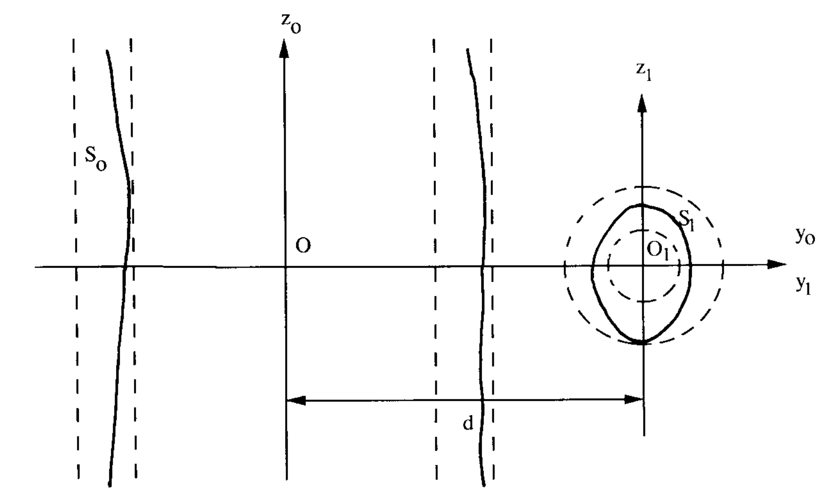

Fig. 5. Geometry for a cylinder-like and a bounded obstacle.

and in the second integral we expand in spherical wave functions about O_1 [Eq. (2.8)]:

$$G(\boldsymbol{r}_1, \boldsymbol{r}_1') = \mathrm{i}k \sum_n \mathrm{Re}\,\psi_n(\boldsymbol{r}_<)\psi_n(\boldsymbol{r}_>). \tag{7.3}$$

Inside the inscribed cylinder of S_0 we can further expand the incoming field in regular cylindrical wave functions about O:

$$u^{\mathrm{i}}(\boldsymbol{r}_0) = \sum_k \int_C \mathrm{d}\alpha \sin\alpha\, a_k(\alpha) \mathrm{Re}\,\chi_k(\alpha; \boldsymbol{r}_0). \tag{7.4}$$

From Eq. (7.1) we then have (here we must demand that $\max_{r_1 \in S_1} r_1 < d$)

$$\begin{aligned} &\sum_k \int_C \mathrm{d}\alpha \sin\alpha\, a_k(\alpha) \mathrm{Re}\,\chi_k(\alpha; \boldsymbol{r}_0) \\ &\quad + \mathrm{i}k \sum_k \int_C \mathrm{d}\alpha \sin\alpha\, \mathrm{Re}\,\chi_k(\alpha; \boldsymbol{r}_0) \int_{S_0} u_+(\boldsymbol{r}_0') \frac{\partial}{\partial n_0'} \chi_k^\dagger(\alpha; \boldsymbol{r}_0') \,\mathrm{d}S_0' \\ &\quad + \mathrm{i}k \sum_n \psi_n(\boldsymbol{r}_1) \int_{S_1} u_+(\boldsymbol{r}_1') \frac{\partial}{\partial n_1'} \mathrm{Re}\,\psi_n(\boldsymbol{r}_1') \,\mathrm{d}S_1' = 0. \end{aligned} \tag{7.5}$$

As the regular cylindrical wave functions with $\alpha \in C$ form a complete and linearly independent set inside a cylinder, we clearly want to expand $\psi_n(\boldsymbol{r}_1)$ in $\mathrm{Re}\,\chi_k(\alpha; \boldsymbol{r}_0)$ in Eq. (7.5). This can be accomplished, e.g. by using Eqs. (4.5) and (5.12);

$$\begin{aligned} \psi_n(\boldsymbol{r}_1) &= \sum_{k'} \int_C \mathrm{d}\alpha \sin\alpha\, C_{nk'}(\alpha) \chi_{k'}(\alpha; \boldsymbol{r}_1) \\ &= \sum_{k'k''} \int_C \mathrm{d}\alpha \sin\alpha\, C_{nk'}(\alpha) P_{k'k''}(\alpha; -\boldsymbol{d}) \mathrm{Re}\,\chi_{k''}(\alpha; \boldsymbol{r}_0), \quad \rho_0 < d. \end{aligned} \tag{7.6}$$

Employing Eq. (7.6) in Eq. (7.5) and equating coefficients of $\mathrm{Re}\,\chi_k(\alpha; \boldsymbol{r}_0)$ yields

$$\begin{aligned} &a_k(\alpha) + \mathrm{i}k \int_{S_0} u_+(\boldsymbol{r}_0) \frac{\partial}{\partial n_0} \chi_k^\dagger(\alpha; \boldsymbol{r}_0) \,\mathrm{d}S_0 \\ &\quad + \mathrm{i}k \sum_{n''k'} C_{n''k'}(\alpha) P_{k'k}(\alpha; -\boldsymbol{d}) \int_{S_1} u_+(\boldsymbol{r}_1) \frac{\partial}{\partial n_1} \mathrm{Re}\,\psi_{n''}(\boldsymbol{r}_1) \,\mathrm{d}S_1 = 0. \end{aligned} \tag{7.7}$$

Next we apply Eq. (7.1) for $\boldsymbol{r}$ inside the inscribed sphere of S_1. We still use the expansions of the Green's function Eqs. (7.2) and (7.3) (with the dagger on $\mathrm{Re}\,\chi_k(\alpha)$ in Eq. (7.2)), but this time we expand the incoming field in regular spherical waves

$$u^{\mathrm{i}}(\boldsymbol{r}_1) = \sum_n a_n \mathrm{Re}\,\psi_n(\boldsymbol{r}_1). \tag{7.8}$$

Equation (7.1) then gives (here we must demand that $\max_{r_0 \in S_0} \rho_0 < d$)

$$\begin{aligned} &\sum_n a_n \mathrm{Re}\,\psi_n(\boldsymbol{r}_1) + \mathrm{i}k \sum_k \int_C \mathrm{d}\alpha \sin\alpha\, \chi_k(\alpha; \boldsymbol{r}_0) \int_{S_0} u_+(\boldsymbol{r}_0') \frac{\partial}{\partial n_0'} \mathrm{Re}\,\chi_k^\dagger(\alpha; \boldsymbol{r}_0') \,\mathrm{d}S_0' \\ &\quad + \mathrm{i}k \sum_n \mathrm{Re}\,\psi_n(\boldsymbol{r}_1) \int_{S_1} u_+(\boldsymbol{r}_1') \frac{\partial}{\partial n_1'} \psi_n(\boldsymbol{r}_1') \,\mathrm{d}S_1' = 0. \end{aligned} \tag{7.9}$$

We now want to expand $\chi_k(\alpha; \boldsymbol{r}_0)$ in $\operatorname{Re}\psi_n(\boldsymbol{r}_1)$. This is done by means of Eqs. (5.12) and (4.3):

$$\chi_k(\alpha; \boldsymbol{r}_0) = \sum_{k'} P_{kk'}(\alpha; \boldsymbol{d}) \operatorname{Re}\chi_{k'}(\alpha; \boldsymbol{r}_1)$$

$$= \sum_{k'n''} P_{kk'}(\alpha; \boldsymbol{d}) C^{\dagger}_{n''k'}(\alpha) \operatorname{Re}\psi_{n''}(\boldsymbol{r}_1), \quad \rho_1 < d. \tag{7.10}$$

Using Eq. (7.10) in Eq. (7.9) and equating coefficients of $\operatorname{Re}\psi_n(\boldsymbol{r}_1)$ we get

$$a_n + ik \sum_{k'k''} \int_C \mathrm{d}\alpha \sin\alpha \, P_{k'k''}(\alpha; \boldsymbol{d}) \, C^{\dagger}_{nk''}(\alpha) \int_{S_0} u_+(\boldsymbol{r}_0) \frac{\partial}{\partial n_0} \operatorname{Re}\chi^{\dagger}_{k'}(\alpha; \boldsymbol{r}_0) \, \mathrm{d}S_0$$

$$+ ik \int_{S_1} u_+(\boldsymbol{r}_1) \frac{\partial}{\partial n_1} \psi_n(\boldsymbol{r}_1) \, \mathrm{d}S_1 = 0. \tag{7.11}$$

Finally, we employ Eq. (7.1) to obtain the scattered field. Outside a cylinder centered at O and circumscribing both S_0 and the circumscribing sphere of S_1, we can expand the scattered field in the outgoing cylindrical wave functions

$$u^{\mathrm{s}}(\boldsymbol{r}_0) = \sum_k \int_C \mathrm{d}\alpha \sin\alpha \, f_k(\alpha) \chi_k(\alpha; \boldsymbol{r}_0). \tag{7.12}$$

Inserting the expansions of the Green's function in Eq. (7.1) then gives

$$ik \sum_k \int_C \mathrm{d}\alpha \sin\alpha \, \chi_k(\alpha; \boldsymbol{r}_0) \int_{S_0} u_+(\boldsymbol{r}'_0) \frac{\partial}{\partial n'_0} \operatorname{Re}\chi^{\dagger}_k(\alpha; \boldsymbol{r}'_0) \, \mathrm{d}S'_0$$

$$+ ik \sum_n \psi_n(\boldsymbol{r}_1) \int_{S_1} u_+(\boldsymbol{r}'_1) \frac{\partial}{\partial n'_1} \operatorname{Re}\psi_n(\boldsymbol{r}'_1) \, \mathrm{d}S'_1 = \sum_k \int_C \mathrm{d}\alpha \sin\alpha \, f_k(\alpha) \chi_k(\alpha; \boldsymbol{r}_0). \tag{7.13}$$

Expanding $\psi_n(\boldsymbol{r}_1)$ in $\chi_k(\alpha; \boldsymbol{r}_0)$ by means of Eqs. (4.5) and (5.11),

$$\psi_n(\boldsymbol{r}_1) = \sum_{k'} \int_C \mathrm{d}\alpha \sin\alpha \, C_{nk'}(\alpha) \chi_{k'}(\alpha; \boldsymbol{r}_1)$$

$$= \sum_{k'k''} \int_C \mathrm{d}\alpha \sin\alpha \, C_{nk'}(\alpha) R_{k'k''}(\alpha; -\boldsymbol{d}) \chi_{k''}(\alpha; \boldsymbol{r}_0), \quad \rho_0 > d. \tag{7.14}$$

Equation (7.13) becomes

$$ik \int_{S_0} u_+(\boldsymbol{r}_0) \frac{\partial}{\partial n_0} \operatorname{Re}\chi_k(\alpha; \boldsymbol{r}_0) \, \mathrm{d}S_0$$

$$+ ik \sum_{n''k'} C_{n''k'}(\alpha) R_{k'k}(\alpha; -\boldsymbol{d}) \int_{S_1} u_+(\boldsymbol{r}_1) \frac{\partial}{\partial n_1} \operatorname{Re}\psi_{n''}(\boldsymbol{r}_1) \, \mathrm{d}S_1 = f_k(\alpha). \tag{7.15}$$

In some suitable manner we now have to eliminate the two surface fields between Eqs. (7.7), (7.11) and (7.15). As the expansion coefficients $a_k(\alpha)$ and a_n of the incoming field are assumed to be known, this then determines the expansion coefficients $f_k(\alpha)$ of the scattered field. The procedure usually employed in the null-field approach is to expand the surface fields in an appropriate set of wave functions, e.g. to expand the field on S_1 in the regular spherical waves. However, as we are mainly interested in exemplifying the use of the various transformations from the preceding sections, we will not pursue the present scattering problem any further here.

Instead, we turn to a short discussion of the transformations used above. The first transformation we used was Eq. (7.6), an expansion of an outgoing spherical wave in the origin-shifted regular cylindrical waves. The region of validity of this formula is $\rho_0 < d$, which follows immediately from Eqs. (4.5) and (5.12). Since we only needed the formula inside the inscribed cylinder, this was quite adequate. Another similar transformation is obtained by first translating the outgoing spherical wave function and then transforming to the regular cylindrical ones. Thus, Eqs. (5.4) and (4.2) yield

$$\psi_n(\boldsymbol{r}_1) = \sum_{n'k''} \int_0^{\pi} \mathrm{d}\alpha \sin\alpha \, P_{nn'}(-\boldsymbol{d}) \, C_{n'k''}(\alpha) \operatorname{Re} \chi_{k''}(\alpha; \boldsymbol{r}_0), \quad r_0 < d. \tag{7.16}$$

From the derivation we have the more stringent restriction $r_0 < d$ on this transformation, which means that we cannot equate the coefficients of $\operatorname{Re} \chi_k(\alpha; \boldsymbol{r}_0)$ in order to obtain a coefficient relation corresponding to Eq. (7.7). In fact, the requirement that one shall be able to extract equations for the coefficients of members of a complete and linearly independent set usually restricts the available choices of transformations. Indeed, it is nontrivial to show (except by just retracing the derivations above) that Eqs. (7.6) and (7.16) agree in their common region of validity, the most apparent difficulty being perhaps the different ranges of integration.

The second transformation we employed in the example above was the expansion of an outgoing cylindrical wave in the origin-shifted regular spherical waves, Eq. (7.10). However, it is now impossible to perform the transformation to the spherical system before the translation, since there is no expansion of an outgoing cylindrical wave in the (regular or outgoing) spherical waves.

In the above we have encountered many examples of the use of the transformations between the wave functions and of their translations, but we have not employed any rotations of the coordinate systems. Rotations will, in fact, mostly enter as a useful computational tool. As an example, let us consider again the scattering by a cylinder and a bounded obstacle, and let us assume that the obstacle is rotationally symmetric about the y_1 axis (see Fig. 5). If the scattering from the bounded obstacle is expressed in terms of its transition matrix, it is definitely advantageous to employ a rotated system about O_1 with the new z_1 axis in the direction of the old y_1 axis.

As is illustrated by the discussion above, different paths may be used in order to reach the solution of a given multiple-scattering problem. In practice, it is normally not too difficult to realize what the available alternatives and their relative merits are.

8. Conclusion

It has been our aim to present the basic formulas concerning the transformations between spherical, cylindrical, and plane wave functions, as well as the translations and rotations of these functions in a systematic and readily accessible form. In particular, it should be possible for the reader to get an insight into the structure of the formulas that exist, before going into the full details of the various functions that are involved. The formulas that have been given are relevant tools for the treatment of wide classes of multiple scattering problems in acoustics, electrodynamics and elastodynamics. They usually provide succinct and efficient ways of expressing the salient features of the multiple-scattering problem, as is illustrated by the example in Section 7. Further examples can be found in the cited literature.

Acknowledgement

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Appendix A: Mathematical comments on the transformations of the outgoing scalar waves

This appendix contains some mathematical remarks on the transformations given in Section 4 between the different sets of wave functions. We discuss only the equations involving the irregular wave functions, since the regular counterparts are assumed to be more familiar. Furthermore, only the scalar transformations are given in some detail, as the proofs in the vector case require essentially no new analytic tools (Devaney and Wolf 1974).

The simplest of the transformations treated in this appendix is the expansion of an irregular cylindrical wave function in terms of the plane waves, see Eq. (4.13). Let $\alpha \in [0, \pi]$ and consider the following expression for $\phi \in (0, \pi)$:

$$\int_{\Gamma_+} d\beta\, e^{im\beta + ik\hat{\gamma}\cdot r} = e^{ikz\cos\alpha} \int_{i\infty}^{\pi - i\infty} e^{im\beta + ik\rho\sin\alpha\cos(\beta-\phi)}\, d\beta$$

$$= e^{ikz\cos\alpha + im\phi} \int_{-\phi + i\infty}^{\pi - \phi - i\infty} e^{im\beta + ik\rho\sin\alpha\cos\beta}\, d\beta. \tag{A.1}$$

The contour Γ_+ is defined in Section 3 (see Fig. 2). The change of integration variable in Eq. (A.1) is a simple rotation in two dimensions, such that the new azimuthal angle β is measured with respect to the $\hat{\rho}$ direction of $\boldsymbol{r}$. The following integral representation of the Hankel function of the first kind (Watson 1966) will be useful [cf., Eq. (A.1)]:

$$H_m^{(1)}(z) = (1/\pi)\mathrm{i}^{-m} \int_{-\pi/2 + i\infty}^{\pi/2 - i\infty} e^{im\beta + iz\cos\beta}\, d\beta, \quad |\arg z| < \tfrac{1}{2}\pi. \tag{A.2}$$

Since the integrand in the integral on the right-hand side of Eq. (A.1) is an analytic function, we can deform the contour arbitrarily provided the contour starts and ends in the region where the integrand is small. For $\alpha \in [0, \pi]$ this region is the hatched region in Fig. A.1. We conclude that for $\phi \in (0, \pi)$ the contour on the right-hand side of

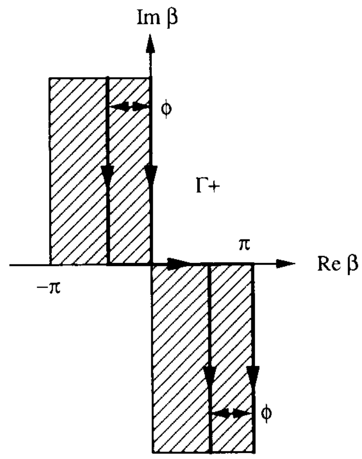

Fig. A.1. Integration contours in the complex β plane; for the hatched region: $\operatorname{Im}\cos\beta > 0$.

Eq. (A.1) can be deformed to the contour given in Eq. (A.2), and we thus have

$$\int_{\Gamma_+} d\beta\, e^{im\beta + ik\hat{\gamma}\cdot r} = e^{ikz\cos\alpha + im\phi}\, \pi i^m\, H_m^{(1)}(k\rho\sin\alpha). \tag{A.3}$$

The case when $\phi \in (\pi, 2\pi)$ can be obtained from the one considered above. We make an auxiliary transformation $\phi' = \phi - \pi$, combined with $\beta' = \beta - \pi$. We then have $\phi' \in (0, \pi)$, and we easily obtain

$$\int_{\Gamma_-} d\beta\, e^{im\beta + ik\hat{\gamma}\cdot r} = e^{ikz\cos\alpha} \int_{\Gamma_+} d\beta'\, e^{im\beta' + im\pi + ik\rho\sin\alpha\cos(\beta' - \phi')}$$
$$= \pi i^m H_m^{(1)}(k\rho\sin\alpha)\, e^{im\phi' + im\pi + ikz\cos\alpha}. \tag{A.4}$$

The contour Γ_- is defined in Section 3, and furthermore, we have used Eq. (A.3) in Eq. (A.4). Thus, we can conclude that

$$\int_{\Gamma_\pm} d\beta\, e^{im\beta + ik\hat{\gamma}\cdot r} = i^m \pi H_m^{(1)}(k\rho\sin\alpha)\, e^{im\phi + ikz\cos\alpha}, \quad y \gtrless 0. \tag{A.5}$$

By, respectively, adding and subtracting this equation for positive and negative m values we finally obtain Eq. (4.13):

$$\chi_k(\alpha; r) = 2 \int_{\Gamma_\pm} d\beta\, D_k(\beta)\phi(\hat{\gamma}; r), \quad y \gtrless 0. \tag{A.6}$$

The expansion of an outgoing spherical wave in terms of the plane waves, Eq. (4.9), can be derived in an analogous way. In this case we rely on a rotation in three dimensions. Thus consider, for $z > 0$, the following expression (for notations see Sections 2 and 4):

$$\int_0^{2\pi} d\beta \int_{C_+} Y_n(\hat{\gamma})\, e^{ik\hat{\gamma}\cdot r} \sin\alpha\, d\alpha. \tag{A.7}$$

In this equation we can put $\phi = 0$ without loss of generality. The case $\phi \neq 0$ contains an extra azimuthal rotation, and this can be treated in analogy to the derivation of Eq. (4.13), but we omit the details. In Eq. (A.7) we now make a coordinate transformation (rotation):

$$\begin{aligned} \cos\alpha' &= \sin\theta\sin\alpha\cos\beta + \cos\theta\cos\alpha, \\ \sin\alpha'\cos\beta' &= \cos\theta\sin\alpha\cos\beta - \sin\theta\cos\alpha, \\ \sin\alpha'\sin\beta' &= \sin\alpha\sin\beta. \end{aligned} \tag{A.8}$$

This transformation implies that the new z' axis is pointing in the $\hat{r}'$ direction, and this simplifies the evaluation of the integral considerably. We also have (Edmonds 1957)

$$Y_n(\hat{\gamma}) = \sum_{n'} \mathscr{D}_{n'n}(\theta, 0)\, Y_{n'}(\hat{\gamma}'), \tag{A.9}$$

where $\mathscr{D}_{nn'}(\theta, \phi)$ is defined in Eqs. (6.6)–(6.8). Thus, we can write

$$\begin{aligned} &\int_0^{2\pi} \mathrm{d}\beta \int_{C_+} Y_n(\hat{\gamma})\, \mathrm{e}^{ik\hat{\gamma}\cdot r} \sin\alpha\, \mathrm{d}\alpha \\ &\quad = \sum_{n'} \mathscr{D}_{n'n}(\theta, 0) \int_{\gamma'} \mathrm{d}\beta' \int_{C'_+} Y_{n'}(\hat{\gamma}')\, \mathrm{e}^{ikr\cos\alpha'} \sin\alpha'\, \mathrm{d}\alpha'. \end{aligned} \tag{A.10}$$

Here γ' and C'_+ are the transformed contours of $[0, 2\pi]$ and C_+, respectively. Note that the measure on the unit sphere is invariant under the transformation (A.8). An analysis of γ' and C'_+ is found in Danos and Maximon (1965), and that paper should be consulted for further details. The next step in the proof is a deformation of the contours γ' and C'_+ back to the original ones ($\beta' \in [0, 2\pi]$ and $\alpha' \in C_+$). We again refer here to Danos and Maximon (1965), who proved that it is possible to deform the contours γ' and C'_+ back to the original ones, and that no singularities are crossed, provided $z > 0$. Thus, we can write

$$\begin{aligned} &\int_0^{2\pi} \mathrm{d}\beta \int_{C_+} Y_n(\hat{\gamma})\, \mathrm{e}^{ik\hat{\gamma}\cdot r} \sin\alpha\, \mathrm{d}\alpha \\ &\quad = \sum_{n'} \mathscr{D}_{n'n}(\theta, \phi) \int_0^{2\pi} \mathrm{d}\beta' \int_{C_+} Y_{n'}(\hat{\gamma}')\, \mathrm{e}^{ikr\cos\alpha'} \sin\alpha'\, \mathrm{d}\alpha'. \end{aligned} \tag{A.11}$$

In this last equation we have returned to the more general situation when $\phi \neq 0$. By performing the integrations we get

$$\begin{aligned} &\int_0^{2\pi} \mathrm{d}\beta \int_{C_+} Y_n(\hat{\gamma})\, \mathrm{e}^{ik\hat{\gamma}\cdot r} \sin\alpha\, \mathrm{d}\alpha \\ &\quad = \mathscr{D}_{e0l,\sigma ml}(\theta, \phi)[(2l+1)\pi]^{1/2} \int_{i\infty}^{1} P_l(t)\, \mathrm{e}^{ikrt}\, \mathrm{d}t \\ &\quad = 2\pi \left[\frac{\varepsilon_m}{2\pi}\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta) \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix} \mathrm{i}^l h_l^{(1)}(kr) = 2\pi \mathrm{i}^l \psi_n(\boldsymbol{r}). \end{aligned} \tag{A.12}$$

In the above equation we have used the following expression, which can be obtained from Eqs. (6.6)–(6.8):

$$\mathscr{D}_{e0l,\,\sigma ml}(\theta,\phi)=\left[\varepsilon_m\frac{(l-m)!}{(l+m)!}\right]^{1/2}P_l^m(\cos\theta)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}. \tag{A.13}$$

Furthermore, we have used the integral representation of $h_l^{(1)}(z)$ (Watson 1966):

$$h_l^{(1)}(z)=\mathrm{i}^{-l}\int_{\infty\,\exp\mathrm{i}v}^{1}P_l(t)\,\mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t,\quad v\in(\arg z,\pi-\arg z). \tag{A.14}$$

We consider now the case when $z<0$. As for the two-dimensional case, we make a combined transformation, i.e. $\theta'=\pi-\theta$ and $\alpha'=\pi-\alpha$. Denote the transformed arguments of $\hat{\gamma}$ and $\boldsymbol{r}$ by $\hat{\gamma}'$ and $\boldsymbol{r}'$, respectively, and we have from Eq. (A.12),

$$\begin{aligned}\int_0^{2\pi}\mathrm{d}\beta\int_{C_-}\mathrm{d}\alpha\sin\alpha\,Y_n(\hat{\gamma})\,\mathrm{e}^{\mathrm{i}k\hat{\gamma}\cdot\boldsymbol{r}}&=\int_0^{2\pi}\mathrm{d}\beta\int_{C_+}\mathrm{d}\alpha'\sin\alpha'(-1)^{m+l}\,Y_n(\hat{\gamma}')\,\mathrm{e}^{\mathrm{i}k\hat{\gamma}'\cdot\boldsymbol{r}'}\\&=(-1)^{l+m}2\pi\mathrm{i}^l\psi_n(\boldsymbol{r}')=2\pi\mathrm{i}^l\psi_n(\boldsymbol{r}).\end{aligned} \tag{A.15}$$

We have thus verified the transformation between the scalar irregular spherical and plane waves, Eq. (4.9),

$$\psi_n(\boldsymbol{r})=2\int_{C_\pm}\mathrm{d}\hat{\gamma}\,B_n(\hat{\gamma})\,\phi(\hat{\gamma};\boldsymbol{r}),\quad z\gtrless 0. \tag{A.16}$$

Equation (4.4), the expansion of an outgoing spherical wave in the regular cylindrical ones, can be verified immediately from Eq. (A.16) by using the following integral representation (Watson 1966)

$$J_m(z)=(1/2\pi)\,\mathrm{i}^{-m}\int_0^{2\pi}\mathrm{d}\beta\,\mathrm{e}^{\mathrm{i}m\beta+\mathrm{i}z\cos\beta}. \tag{A.17}$$

By performing the β integration we obtain from Eq. (A.16)

$$\begin{aligned}\psi_n(\boldsymbol{r})&=(1/2\pi)\,\mathrm{i}^{-l}\left[\frac{\varepsilon_m}{2\pi}\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\right]^{1/2}\int_0^{2\pi}\mathrm{d}\beta\int_{C_\pm}\mathrm{d}\alpha\sin\alpha\,P_l^m(\cos\alpha)\begin{pmatrix}\cos m\beta\\ \sin m\beta\end{pmatrix}\mathrm{e}^{\mathrm{i}k\hat{\gamma}\cdot\boldsymbol{r}}\\&=\mathrm{i}^{m-l}\left[\frac{\varepsilon_m}{2\pi}\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\right]^{1/2}\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}\\&\quad\times\int_{C_\pm}\mathrm{d}\alpha\sin\alpha\,P_l^m(\cos\alpha)\,J_m(k\rho\sin\alpha)\,\mathrm{e}^{\mathrm{i}kz\cos\alpha}\\&=2\sum_{k'}\int_{C_\pm}\mathrm{d}\alpha\sin\alpha\,C_{nk'}(\alpha)\,\mathrm{Re}\,\chi_{k'}(\alpha;\boldsymbol{r}),\quad z\gtrless 0,\end{aligned} \tag{A.18}$$

which is just Eq. (4.4).

Consider next the corresponding relation with the irregular cylindrical waves, Eq. (4.5). We start with Eq. (A.18) and deform the contours $C_\pm$ to the contour C. In order to do this we have to study the singularities of the integrand in greater detail.

Consider the following expression for $z > 0$ and $\rho > 0$:

$$\int_{C_+} \mathrm{d}\alpha \sin\alpha\, P_l^m(\cos\alpha)\, J_m(k\rho\sin\alpha)\, \mathrm{e}^{ikz\cos\alpha}$$

$$= \frac{1}{2}\int_{C_+} \mathrm{d}\alpha \sin\alpha\, P_l^m(\cos\alpha)\,[H_m^{(1)}(k\rho\sin\alpha) + H_m^{(2)}(k\rho\sin\alpha)]\, \mathrm{e}^{ikz\cos\alpha}. \quad \text{(A.19)}$$

The second term in the sum is now transformed by changing α to $-\alpha$. This transformation gives $H_m^{(2)} \to -H_m^{(1)}$ and we get

$$\int_{C_+} \mathrm{d}\alpha \sin\alpha\, P_l^m(\cos\alpha)\, J_m(k\rho\sin\alpha)\, \mathrm{e}^{ikz\cos\alpha}$$

$$= \frac{1}{2}\int_{C'} \mathrm{d}\alpha \sin\alpha\, P_l^m(\cos\alpha)\, H_m^{(1)}(k\rho\sin\alpha)\, \mathrm{e}^{ikz\cos\alpha}. \quad \text{(A.20)}$$

The contour C' in Eq. (A.20) is depicted in Fig. A.2. This contour can be deformed to any contour in the complex α plane, provided the endpoints of the contour remain within the region where the integrand is small, and provided no singularities are crossed. The region where the integrand is small is crosshatched in Fig. A.2. This region is the one where $\operatorname{Im}\cos\alpha > 0$ and $\operatorname{Im}\sin\alpha > 0$ hold. The branch cut of $H_m^{(1)}$ for negative real values of $\sin\alpha$ is also shown in Fig. A.2. Due to the construction, the contour C' passes the cut on the right and upper side, respectively. Furthermore, the integrand in Eq. (A.20) has two more branch points, viz. $\cos\alpha = \pm 1$ from $P_l^m(\cos\alpha)$. We observe that a deformation of the contour C' to the contour C defined in Section 2 can be made without crossing any singularities, and that the integrand stays small at the endpoints. Thus, we have

$$\psi_n(\mathbf{r}) = 2\sum_{k'} \int_{C_+} \mathrm{d}\alpha \sin\alpha\, C_{nk'}(\alpha) \operatorname{Re}\chi_{k'}(\alpha; \mathbf{r})$$

$$= \sum_{k'} \int_C \mathrm{d}\alpha \sin\alpha\, C_{nk'}(\alpha)\chi_{k'}(\alpha; \mathbf{r}), \quad \rho > 0,\ z > 0. \quad \text{(A.21)}$$

For $z < 0$ and $\rho > 0$ we can go through the same analysis as above or we can make a combined transformation, $\theta \to \pi - \theta$ and $\alpha \to \pi - \alpha$ (the proper contour to start with in this case is, of course, C_-). In this case we will end up with the same expression as in Eq. (A.21) and we thus have

$$\psi_n(\mathbf{r}) = \sum_{k'} \int_C \mathrm{d}\alpha \sin\alpha\, C_{nk'}(\alpha)\chi_{k'}(\alpha; \mathbf{r}), \quad \rho > 0. \quad \text{(A.22)}$$

This expression thus holds for $z \gtrless 0$, and furthermore, since the limits $z \to 0^{\pm}$ in Eq. (A.22) from both sides of zero give the same finite expression, we can conclude that Eq. (A.22) holds for all z and $\rho > 0$. This completes the proof of Eq. (4.5).

In this context it is also illustrative to discuss the various expansions of the Green's function (see Section 2). The scalar Green's function is, apart from normalization,

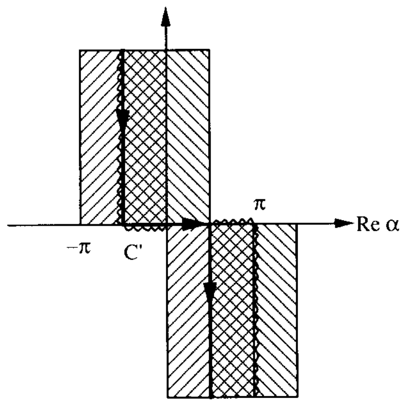

Fig. A.2. The integration contour C' in the complex α plane; \\\: $\operatorname{Im}\sin\alpha > 0$; ///: $\operatorname{Im}\cos\alpha > 0$.

a spherical wave ($l = 0$, $m = 0$, $\sigma = \mathrm{e}$) and this fact can be used to derive some of the expansions found in Section 2. Thus, Eq. (2.11) is obtained immediately from Eq. (A.16) with a position vector $\boldsymbol{r} - \boldsymbol{r}'$. Furthermore, Eqs. (2.9) and (2.10) can be derived from Eqs. (A.22) and (A.18), respectively, and by using the translation properties of the cylindrical waves, see Eqs. (5.10)–(5.12).

The vector wave transformations can all be derived from the corresponding scalar ones. As a consequence of the way in which the vector waves are defined in terms of vector operations on the scalar waves, the derivations are quite analogous to the scalar case, and we do not give any details here (cf., e.g. Devaney and Wolf 1974).

Appendix B: Some remarks on the connection to the representation theory for E(3)

In this appendix we have collected some remarks on the connection between the transformation properties of the three-dimensional functions treated in this chapter and the theory of the unitary irreducible representations of the group E(3) [there is a similar connection between the transformation properties of the two-dimensional wave functions and the unitary irreducible representations of the two-dimensional Euclidean group E(2)]. These remarks are included mainly in order to provide the reader with a glimpse of this fundamental and very useful background to the relations concerning, in the first place, the sets of regular scalar wave functions. The extension of these relations to the irregular case and to the vector case can, to a large extent, also be treated using the group-theoretical concepts (although considerations of analytic continuation also plays an important role), but we will not go into these aspects here. For further details concerning the facts touched upon here we refer to books (Miller 1968, Talman 1968, Vilenkin 1968) as well as to several articles (Miller 1964, Kalnins et al. 1973, Boyer et al. 1976) from which the vast literature in this field can be traced. Some familiarity with the basic concepts of the theory of Lie algebras, Lie groups, and their representations will be helpful in the following.

Very briefly, the relation between the sets of wave functions treated in this chapter and the representation theory of E(3) can be described in the following way: the regular wave functions can be identified with specific matrix elements of a general translation $\boldsymbol{r}$ in a unitary representation. The same translation $\boldsymbol{r}$ can be represented by its matrix elements in different bases in the representation space and, in this way, the relations between different sets of wave functions can be expressed in terms of similarity transformations between these bases. Some of the bases which we shall consider will be eigenfunctions belonging to eigenvalues in a continuous spectrum, i.e. they are nonnormalizable (generalized) eigenfunctions which, strictly speaking, lie outside the representation space. However, the relevant results can be presented formally in a simplified fashion. Furthermore, general translations and rotations of the wave functions are determined as soon as they have been identified with specific matrix elements.

Let $\boldsymbol{L} = (L_1, L_2, L_3)$ and $\boldsymbol{P} = (P_1, P_2, P_3)$ be the generators of rotations and translations, respectively. $\boldsymbol{L}$ and $\boldsymbol{P}$ form a basis of the Lie algebra of E(3) (Miller 1964, 1968). Two invariant second-order operators are $\boldsymbol{P}^2 = P_1^2 + P_2^2 + P_3^2$ and $\boldsymbol{P}\cdot\boldsymbol{L} = P_1L_1 + P_2L_2 + P_3L_3 = \boldsymbol{L}\cdot\boldsymbol{P}$. In a unitary irreducible representation of E(3), these operators take the values k^2 and kl_0, respectively, where $k > 0$ and $l_0 = 0, 1, 2, \ldots$ The unitary representations which correspond to $l_0 = 0$ admit a particularly simple realization as operators on functions of a vector $\boldsymbol{k}$ with fixed absolute value k, i.e. functions of $\hat{\boldsymbol{k}}$. In order to describe the relevant properties of the regular scalar wave functions it suffices to consider the case $l_0 = 0$. As a starting point, we consider a proper basis of square-integrable functions of $\hat{\boldsymbol{k}}$. The representation space can be constructed as a direct sum of finite-dimensional spaces which are spanned by the common eigenvectors of $\boldsymbol{L}^2$ and L_3, with eigenvalues $l(l+1)$ and m, respectively (see, e.g. Miller 1964, 1968). Since $\boldsymbol{P}^2$ and $\boldsymbol{P}\cdot\boldsymbol{L}$ have the eigenvalues k^2 and 0, respectively, a suitable notation for a member of this angular momentum basis is $|l, m; k, 0\rangle$. However, since the eigenvalues of $\boldsymbol{P}^2$ and $\boldsymbol{P}\cdot\boldsymbol{L}$ will not change often, we shall also use the abbreviated notation $|l, m\rangle$.

If $\boldsymbol{d}_i$ and R_i, $i = 1, 2$ (cf. Sections 5 and 6) denote two general translations and rotations, respectively, in three-dimensional space, the composition law for successive E(3) transformations $\{\boldsymbol{d}; R\}$ is (Miller 1964, 1968)

$$\{\boldsymbol{d}_1; R_1\}\{\boldsymbol{d}_2; R_2\} = \{\boldsymbol{d}_1 + R_1\boldsymbol{d}_2; R_1R_2\}, \tag{B.1}$$

i.e. the operators $U(\boldsymbol{d}; R)$ in a unitary representation of E(3) satisfy

$$U(\boldsymbol{d}_1; R_1)U(\boldsymbol{d}_2; R_2) = U(\boldsymbol{d}_1 + R_1\boldsymbol{d}_2; R_1R_2). \tag{B.2}$$

According to the definition of the basis $|l, m\rangle$, the matrix elements of the operator $U(R) \equiv U(\boldsymbol{0}, R)$ representing a general rotation $R \equiv R(\psi, \eta, \chi)$ are

$$\langle l', m'|U(R)|l, m\rangle = \delta_{ll'}D^l_{m'm}(\psi, \eta, \chi), \tag{B.3}$$

where

$$D^l_{m'm}(\psi, \eta, \chi) = e^{im'\chi}d^l_{m'm}(\eta)e^{im\psi},$$

[cf. Eq. (6.6); in order to conform with the literature referred to in this appendix, and since only formal developments will be presented, we will, for convenience, use the exponential notation for the azimuthal angles]. An operator $U(\boldsymbol{d}) \equiv U(\boldsymbol{d}; 1)$ representing a general translation $\boldsymbol{d}$ has nonzero matrix elements for $l \neq l'$, several methods for the computation of $D^l_{mm'}$ (see, e.g. Gelfand et al. 1963), as well as the general translation matrix elements $\langle l', m'|U(\boldsymbol{d})|l, m\rangle$ (see, e.g. Miller 1964). In particular, one obtains integral representations for these functions from the theory of induced representations, and a host of difference–differential relations from the Lie algebra representations.

The connection with the regular wave functions considered in this chapter is obtained by noting the fact that they can be identified with particular matrix elements of the operator $U(\boldsymbol{r})$. For instance, the special matrix elements $\langle 0, 0|U(\boldsymbol{r})|l, m\rangle$ are nothing but the regular spherical waves (Miller 1964, 1968) (apart from a normalization constant). It follows immediately that the translation properties of the regular spherical waves are determined by the functions $\langle l, m|U(\boldsymbol{d})|l', m'\rangle$. If $\boldsymbol{r}_1 = \boldsymbol{r} + \boldsymbol{d}$, we have in general

$$\langle l', m'|U(\boldsymbol{r}_1)|l, m\rangle = \sum_{l''=0}^{\infty} \sum_{m''=-l''}^{l''} \langle l', m'|U(\boldsymbol{r})|l'', m''\rangle \langle l'', m''|U(\boldsymbol{d})|l, m\rangle, \quad \text{(B.4)}$$

i.e. by specializing to $l' = m' = 0$, we get

$$\langle 0, 0|U(\boldsymbol{r}_1)|l, m\rangle = \sum_{l''m''} \langle 0, 0|U(\boldsymbol{r})|l'', m''\rangle \langle l'', m''|U(\boldsymbol{d})|l, m\rangle, \quad \text{(B.5)}$$

which is nothing but Eq. (5.2) in a different notation.

In the same way as the properties of the spherical waves can be derived by referring to properties of matrix elements in an angular momentum basis of a representation $(k, 0)$, the properties of the plane and cylindrical waves can be discussed in terms of properties of matrix elements in two other (generalized) bases. Consider first the linear momentum basis $|\boldsymbol{k}; k, 0\rangle \equiv |\boldsymbol{k}\rangle$, where the operators P_i, $i = 1, 2, 3$, have the eigenvalues $k_i \in (-\infty, \infty)$. The orthonormality relation for this basis reads $\langle \boldsymbol{k}|\boldsymbol{k}'\rangle = \delta(\boldsymbol{k} - \boldsymbol{k}')$ and we have

$$\langle \boldsymbol{k}|U(\boldsymbol{r})|\boldsymbol{k}'\rangle = \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\langle \boldsymbol{k}|\boldsymbol{k}'\rangle = \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\delta(\boldsymbol{k} - \boldsymbol{k}'). \quad \text{(B.6)}$$

Note that since $k = |\boldsymbol{k}|$ is taken to be fixed, the basis $|\boldsymbol{k}\rangle$ is determined by two parameters, but it is sometimes convenient to use the redundant notation $|\boldsymbol{k}\rangle$. The two parameters are naturally chosen as the spherical angles of $\boldsymbol{k}$ and they are referred to by the symbol $\hat{\boldsymbol{k}}$. Thus, we write $\langle \boldsymbol{k}'|\boldsymbol{k}\rangle = k^{-2}\ \delta(k - k')\langle \hat{\boldsymbol{k}}'|\hat{\boldsymbol{k}}\rangle = k^{-2} \times \delta(k - k')\delta(\hat{k} - \hat{k}')$. Translations as well as rotations are simple to express in this basis [cf. Eqs. (5.1) and (6.3)], but this advantage is partially off-set by the fact that one has to deal with continuous variables, which is a disadvantage from a computational point of view.

The transformations between the plane and spherical waves appear naturally when we consider transitions back and forth between the bases $|l, m\rangle$ and $|\boldsymbol{k}\rangle$. The transformation coefficients are $\langle \boldsymbol{k}'; k', 0|l, m; k, 0\rangle = k^{-2}\delta(k - k')\langle \hat{\boldsymbol{k}}'|l, m\rangle$, where $\langle \hat{\boldsymbol{k}}'|l, m\rangle = Y_{lm}(\hat{\boldsymbol{k}}')$ (see, e.g. Messiah 1961). Consider first the effect of introducing the angular momentum basis into $\langle \hat{k}'|U(r)|\hat{k}\rangle$. One has

$$\int d\hat{k}' \langle \hat{k}'|U(r)|\hat{k}\rangle = \int d\hat{k}'\, \delta(\hat{k} - \hat{k}')\, e^{ik\cdot r} = e^{ik\cdot r} \tag{B.7}$$

($\int d\hat{k}'$ implies integration over the unit sphere $\hat{k}'$). On the other hand,

$$\begin{aligned}\int d\hat{k}' \langle \hat{k}'|U(r)|\hat{k}\rangle &= \sum_{\substack{lm\\l'm'}} \int d\hat{k}' \langle \hat{k}'|l', m'\rangle \langle l', m'|U(r)|l, m\rangle \langle l, m|\hat{k}\rangle \\ &= \sum_{\substack{lm\\l'm'}} \int d\hat{k}'\, Y_{l'm'}(\hat{k}') \langle l', m'|U(r)|l, m\rangle\, \overline{Y_{lm}(\hat{k})} \\ &= \sum_{\substack{lm\\l'm'}} \delta_{0l'}\delta_{0m'} \langle l', m'|U(r)|l, m\rangle\, \overline{Y_{lm}(\hat{k})} \\ &= \sum_{lm} \langle 0, 0|U(r)|l, m\rangle\, \overline{Y_{lm}(\hat{k})},\end{aligned}$$

(where the bar denotes complex conjugate) which gives

$$e^{ik\cdot r} = \sum_{lm} \langle 0, 0|U(r)|l, m\rangle\, \overline{Y_{lm}(\hat{k})}, \tag{B.8}$$

i.e. the expansion of a plane wave in spherical waves, Eq. (4.8).

The inverse of this transformation is obtained simply by introducing the linear momentum basis into the matrix elements $\langle 0, 0|U(r)|l, m\rangle$. We get

$$\begin{aligned}\langle 0, 0|U(r)|l, m\rangle &= \int d\hat{k} \int d\hat{k}' \langle 0, 0|\hat{k}\rangle \langle \hat{k}|U(r)|\hat{k}'\rangle \langle \hat{k}'|l, m\rangle \\ &= \int d\hat{k} \int d\hat{k}'\, \delta(\hat{k} - \hat{k}')\, e^{ik\cdot r}\, Y_{lm}(\hat{k}') = \int d\hat{k}\, e^{ik\cdot r}\, Y_{lm}(\hat{k}),\end{aligned} \tag{B.9}$$

which corresponds to Eq. (4.7).

By reference to the linear momentum basis we obtain an integral representation for the general translation matrix element $\langle l', m'|U(r)|l, m\rangle$, which occurs in Eqs. (B.4) and (B.5) [i.e. in Eq. (5.2)]. We get

$$\begin{aligned}\langle l', m'|U(d)|l, m\rangle &= \int d\hat{k} \int d\hat{k}' \langle l', m'|\hat{k}\rangle \langle \hat{k}|U(d)|\hat{k}'\rangle \langle \hat{k}'|l, m\rangle \\ &= \int d\hat{k}\, \overline{Y_{l'm'}(\hat{k})}\, Y_{lm}(\hat{k})\, e^{ik\cdot d},\end{aligned} \tag{B.10}$$

which corresponds to Eq. (5.18).

In order to obtain corresponding relations for the regular cylindrical wave functions, we consider a new basis which is connected to the two-dimensional Euclidean subgroup formed by translations and rotations in the x–y plane. The Lie algebra of this subgroup is spanned by L_3, P_1 and P_2. Let $|q, m; k, 0\rangle = |q, m\rangle$ be a (generalized) basis of eigenfunctions of $P_1^2 + P_2^2$ and L_3, with eigenvalues q^2 and m, respectively. Sometimes we shall write the basis functions as $|q, m; h\rangle$, where h is the eigenvalue of P_3, in order to emphasize that, since they are eigenfunctions of both $\boldsymbol{P}^2$ and $P_1^2 + P_2^2$, they are in fact also eigenfunctions of P_3. According to $(P_1^2 + P_2^2)|q, m; h\rangle = q^2|q, m; h\rangle$ and $P_3|q, m; h\rangle = h|q, m; h\rangle$, and with $\boldsymbol{k} = k(\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$ we then have $(k\cos\alpha - h)|q, m; h\rangle = 0$. Here $|q, m; h\rangle$ is realized as a function of α and β, i.e. of $\hat{\boldsymbol{k}}$ and to emphasize this fact we write the functions $|q, m; h\rangle$, in the symbolic notation $\langle\hat{\boldsymbol{k}}|q, m; h\rangle$ (Messiah 1961). From $(k\cos\alpha - h)\langle\hat{\boldsymbol{k}}|q, m; h\rangle = 0$ we then have (with a suitable normalization)

$$\langle\hat{\boldsymbol{k}}|q, m; h\rangle = \delta(\cos\alpha - h/k)\,e^{im\beta} \tag{B.11}$$

[i.e. using this notation, it becomes evident that these functions at the same time give the transformation coefficients between the bases $|\hat{\boldsymbol{k}}\rangle$ and $|q, m; h\rangle$, cf. Eq. (4.10)]. The transformation coefficients between the bases $|l, m\rangle$ and $|q, m\rangle$ are

$$\begin{aligned}\langle l, m|q, m'\rangle &= \int d\hat{\boldsymbol{k}}'\,\langle l, m|\hat{\boldsymbol{k}}'\rangle\langle\hat{\boldsymbol{k}}'|q, m'\rangle = \int d\hat{\boldsymbol{k}}'\,Y_{lm}(\hat{\boldsymbol{k}}')\delta(\cos\alpha - h/k)\,e^{im'\beta'} \\ &= \delta_{mm'}N_1(l, m)P_l^m(h/k),\end{aligned} \tag{B.12}$$

where $N_1(l, m)$ is a normalization constant which is of no concern to us here. Thus, Eq. (B.12) corresponds to Eq. (4.1).

In the basis $|q, m\rangle$ the matrix elements of a translation $\boldsymbol{d}_1$ in the x–y plane, with polar coordinates (ρ_1, ϕ_1) is given by (Miller 1968)

$$\langle q, m|U(\boldsymbol{d}_1)|q, m'\rangle = i^{m-m'}J_{m-m'}(q\rho_1)\,e^{-i(m-m')\phi_1}. \tag{B.13}$$

For a translation z_1 in the z direction we have

$$\langle q, m; h|U(z_1\hat{\boldsymbol{z}})|q, m; h'\rangle = \delta(h - h')\,e^{ihz_1}. \tag{B.14}$$

In complete analogy with what was found for the spherical and plane waves, we now get the regular cylindrical waves as special matrix elements in the $|q, m; h\rangle$ basis of a general translation $\boldsymbol{r}$, with cylinder coordinates (ρ, ϕ, z). From Eqs. (B.13) and (B.14) we have

$$\langle q, 0; h|U(\boldsymbol{r})|q, m; h'\rangle = i^{-m}\delta(h - h')J_m(q\rho)\,e^{im\phi}\,e^{ihz}. \tag{B.15}$$

The transition from spherical to cylindrical waves is obtained as

$$\begin{aligned}&\langle 0, 0|U(\boldsymbol{r})|l, m\rangle \\ &= \sum_{m'm''}\int dh\int dh'\langle 0, 0|q, m'; h\rangle\langle q, m'; h|U(\boldsymbol{r})|q, m''; h'\rangle\langle q, m''; h'|l, m\rangle \\ &= N_2(l, m)\int dh\,J_m(q\rho)\,e^{im\phi}\,e^{ihz}\,P_l^m(h/k),\end{aligned} \tag{B.16}$$

which corresponds to Eq. (4.2). The inverse relation is derived in an analogous way.

This concludes our brief exposition of how the transformation properties of the wave functions can be related to group-theoretical concepts and results. We have seen that the relevant transformation properties of the regular scalar wave sets can be obtained by considering matrix elements in various bases of unitary operators representing general E(3) transformations. Additional wave functions have been considered in the literature (see, e.g. Boyer et al. 1976 for an extensive discussion) but, as was remarked in the introduction, they will, on the whole, have more complicated transformation properties.

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