The final goal is to seek solution sets in a source-free region to

\[
\begin{align*}
\nabla^2 E(r, \omega) + k^2 E(r, \omega) &= 0 \\
\nabla \cdot E(r, \omega) &= 0
\end{align*}
\]

\[\leftrightarrow\]

\[
\nabla \times (\nabla \times E(r, \omega)) - k^2 E(r, \omega) = 0
\]

The set of solutions (modes) should be “rich” so that every well-behaved electric field can be written as a linear combination of these solutions.
Helmholtz’ theorem

Any well-behaved vector field $F(r)$ in a volume $V$ can be decomposed into a longitudinal (irrotational) and a transverse (solenoidal) part [9, 8]

$$F = F_\ell + F_t$$

$$F_\ell = -\nabla \Phi, \quad F_t = \nabla \times A$$

where

$$\Phi(r) = -\iiint_V F(r') \cdot \nabla' \left( \frac{1}{4\pi |r - r'|} \right) \, dv$$

$$A = \iiint_V F(r') \times \nabla' \left( \frac{1}{4\pi |r - r'|} \right) \, dv$$

Longitudinal and transverse fields

As a consequence, any solution to

$$\nabla^2 E(r, \omega) + k^2 E(r, \omega) = 0$$

can be separated into [8, page 1762]

$$E = E_\ell + E_t$$

$$E_\ell = -\nabla \Phi, \quad E_t = \nabla \times A$$

$$\nabla \times E_\ell = 0, \quad \nabla \cdot E_t = 0$$
 Scalar Helmholtz’ equation

Investigate in what curvilinear coordinate systems \((\xi_1, \xi_2, \xi_3)\) Helmholtz’ (scalar) equation

\[
\nabla^2 \psi(\mathbf{r}) + k^2 \psi(\mathbf{r}) = 0
\]

can be separated giving complete sets of solutions

\[
\psi(\xi_1, \xi_2, \xi_3) = X_1(\xi_1)X_2(\xi_2)X_3(\xi_3)
\]

There are eleven such systems [2, 7], e.g., Cartesian, spherical, cylindrical, elliptic, conical etc. (in general — confocal quadratic surfaces)

Spherical coordinate system \((r, \theta, \phi)\)

In the spherical coordinate system \((r, \theta, \phi)\)

\[
\psi(r, \theta, \phi) = R_l(kr)Y_{lm}(\theta, \phi)
\]

\(R_l(kr)\) is called the radial function and \(Y_{lm}(\theta, \phi)\) are called the spherical harmonics
The radial function satisfies the spherical Bessel differential equation [1, 3]

\[ z^2 \frac{d^2}{dz^2} R_l(z) + 2z \frac{d}{dz} R_l(z) + (z^2 - l(l + 1))R_l(z) = 0 \]

where \( l \) is assumed to be a non-negative integer and the argument \( z \) a complex number. Between the spherical Bessel function, \( R_l(z) \), and the cylindrical Bessel function, \( F_n(z) \), there is a relation

\[ R_l(z) = \left( \frac{\pi}{2z} \right)^{1/2} F_{l+1/2}(z) \]

Two independent solutions are:

\[
\begin{align*}
\{ j_l(z) \} & \quad \text{or} \quad \{ j_l(z) \} \\
\{ y_l(z) \} & \quad \text{or} \quad \{ h^{(1)}_l(z) \} \\
\{ h^{(2)}_l(z) \} & \quad \text{or} \quad \{ h^{(2)}_l(z) \}
\end{align*}
\]

where

- \( j_l(z) \) spherical Bessel function (regular at the origin)
- \( y_l(z) \) spherical Neumann function (singular at the origin)
- \( h^{(1)}_l(z) = j_l(z) + iy_l(z) \) spherical Hankel function of the first kind (singular at the origin)
- \( h^{(2)}_l(z) = j_l(z) - iy_l(z) \) spherical Hankel function of the second kind (singular at the origin)

All authors agree on the normalization!!
Spherical Bessel and Hankel functions III

Friedrich Wilhelm Bessel (1784–1846), German mathematician and astronomer

Carl Gottfried Neumann (1832–1925), German mathematician

Hermann Hankel (1839–1873), German mathematician

Spherical Bessel and Hankel functions IV

Spherical Bessel functions \( j_l(x) \)

- \( l = 0 \)
- \( l = 10 \)
- \( l = 20 \)
The first spherical Bessel Neumann, and Hankel functions are

\[
\begin{align*}
  j_0(z) &= \frac{\sin z}{z} \\
  y_0(z) &= -\frac{\cos z}{z} \\
  j_1(z) &= \frac{\sin z}{z^2} - \frac{\cos z}{z} \\
  y_1(z) &= -\frac{\cos z}{z^2} - \frac{\sin z}{z} \\
  h_0^{(1)}(z) &= -\frac{i}{z}e^{iz} \\
  h_0^{(2)}(z) &= \frac{i}{z}e^{-iz} \\
  h_1^{(1)}(z) &= e^{iz}\left(-\frac{1}{z} - \frac{i}{z^2}\right) \\
  h_1^{(2)}(z) &= e^{-iz}\left(-\frac{1}{z} + \frac{i}{z^2}\right)
\end{align*}
\]
Legendre polynomials I

The Legendre polynomials, \( P_l(x) \), defined for non-negative integers, \( l \geq 0 \), by the Rodrigues’ formula\(^\dagger\) [1, 6]

\[
P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l
\]

The first polynomials are:

\[
\begin{align*}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= (3x^2 - 1)/2 \\
P_3(x) &= (5x^3 - 3x)/2
\end{align*}
\]

All authors agree on the normalization!!

\(^\dagger\) Olinde Rodrigues (1795–1851), French mathematician
Legendre polynomials II

The Legendre polynomials are orthogonal on the interval $[-1, 1]$, i.e.,

$$\int_{-1}^{1} P_l(x)P_{l'}(x) \, dx = \frac{2\delta_{ll'}}{2l + 1}$$

and they satisfy the differential equation [1, 3, 6]

$$(1 - x^2)P''_l(x) - 2xP'_l(x) + l(l + 1)P_l(x) = 0$$

Legendre polynomials III

The set $\{P_l(x)\}_{l=0}^{\infty}$ is a complete orthogonal system on the interval $[-1, 1]$. Every well-behaved function on the interval $[-1, 1]$ has a convergent Fourier series (in norm or weaker, pointwise)

$$f(x) = \sum_{l=0}^{\infty} a_l P_l(x), \quad x \in [-1, 1]$$

where the (Fourier) coefficients $a_l$ are

$$a_l = \frac{2l + 1}{2} \int_{-1}^{1} f(x)P_l(x) \, dx$$
The associated Legendre functions, defined for non-negative integers, \( l, m \geq 0 \), are defined by [1, 6]

\[
P^m_l(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad x \in [-1, 1]
\]

For negative integers \( m \) \((m = -1, -2, \ldots, -l + 1, -l)\), we have

\[
P^m_l(x) = (-1)^{|m|} \frac{(l + |m|)!}{(l - |m|)!} P_l^{|m|}(x), \quad x \in [-1, 1]
\]

Introduce an angle \( \theta \in [0, \pi] \), \( x = \cos \theta \), \((m \geq 0)\)

\[
P^m_l(\cos \theta) = \sin^m \theta \frac{d^m}{d(\cos \theta)^m} P_l(\cos \theta), \quad \theta \in [0, \pi]
\]

### Normalization — Associated Legendre functions

<table>
<thead>
<tr>
<th>Source</th>
<th>Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abramowitz &amp; Stegun [1]</td>
<td>((-1)^{m})</td>
</tr>
<tr>
<td>Arfken [3]</td>
<td>1</td>
</tr>
<tr>
<td>Hansen [5]</td>
<td>1</td>
</tr>
<tr>
<td>Morse &amp; Feshbach [7]</td>
<td>1</td>
</tr>
<tr>
<td>Mathematica</td>
<td>((-1)^{m})</td>
</tr>
<tr>
<td>MATLAB</td>
<td>((-1)^{m})</td>
</tr>
<tr>
<td>Python</td>
<td>?</td>
</tr>
</tbody>
</table>
Spherical harmonics I

The spherical harmonics are denoted \( Y_{lm}(\theta, \phi) \) \[3\]

\[
Y_{lm}(\theta, \phi) = C_{lm} P_l^m(\cos \theta) e^{im\phi}
\]

where the indices \( l, m \) take the following values:

\[
l = 0, 1, 2, \ldots, \quad m = -l, -l + 1, \ldots, -1, 0, 1, \ldots, l - 1, l
\]

The normalization factor \( C_{lm} \) is

\[
C_{lm} = (-1)^m \sqrt{\frac{2l + 1 (l - m)!}{4\pi (l + m)!}}
\]

The extra factor \((-1)^m\) is called the Condon-Shortley phase
Spherical harmonics II

Alternative definition of the spherical harmonics

\[ Y_{lm}(\theta, \phi) = \left( -\frac{m}{|m|} \right)^m \sqrt{\frac{2l + 1}{4\pi} \frac{(l - |m|)!}{(l + |m|)!}} P_0^{|m|} \cos \theta \, e^{im\phi} \]

where

\[ \left( -\frac{m}{|m|} \right)^m = \begin{cases} (-1)^m, & m > 0 \\ 1, & m = 0 \\ 1, & m < 0 \end{cases} \]

This definition employs only associated Legendre functions with non-negative \( m \)-values

Note that (real \( \theta \) and \( \phi \))

\[ Y_{l-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi) \]

Spherical harmonics III

Normalization — Spherical harmonics

<table>
<thead>
<tr>
<th>Source</th>
<th>Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abramowitz &amp; Stegun [1]</td>
<td>1</td>
</tr>
<tr>
<td>Arfken [3]</td>
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<td>((-1)^m)</td>
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<tr>
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<td>MATLAB</td>
<td>?</td>
</tr>
<tr>
<td>Python</td>
<td>?</td>
</tr>
</tbody>
</table>
Real vs complex notation

In the literature, there are several ways of defining the spherical harmonics, each with its own pros and cons. Commonly used are the complex combinations [3]

\[ Y_{lm}(\theta, \phi) = C_{lm} P_l^m(\cos \theta)e^{im\phi} \]

The real combinations of the azimuth functions are [7]

\[ Y_{\sigma ml}(\theta, \phi) = C_{\sigma ml}' P_l^m(\cos \theta) \begin{cases} \cos m\phi \\ \sin m\phi \end{cases} \]

where \( m = 0, 1, \ldots, l \) and \( \sigma = e, o \) (even or odd in \( \phi \)).

The transformations between the two are \((m \geq 0)\)

\[
\begin{align*}
Y_{l \pm m} &= \frac{C_{l \pm m}'}{C_{lm}'} \left( Y_{eml} \pm iY_{oml} \right) \\
Y_{\sigma ml} &= \frac{C_{\sigma ml}'}{2C_{lm}'} \left\{ Y_{lm} + \frac{C_{lm}^{-m}}{C_{l^{-m}m}} Y_{l^{-m}} \right. \\
& \quad \left. - i(Y_{lm} - \frac{C_{lm}^{-m}}{C_{l^{-m}m}} Y_{l^{-m}}) \right\}
\end{align*}
\]
Instead of the angles $\theta$ and $\phi$ as arguments, use the unit vector $\hat{r} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta$, i.e.,

$$Y_{lm}(\hat{r}) = Y_{lm}(\theta, \phi)$$

Special values

Special argument (verify yourself)

$$Y_{lm}(\hat{z}) = \delta_{m0} \sqrt{\frac{2l + 1}{4\pi}}$$

The lowest order spherical harmonics are (verify yourself)

$$Y_{00}(\hat{r}) = \sqrt{\frac{1}{4\pi}}$$

$$\left\{ \begin{array}{l}
Y_{10}(\hat{r}) = \sqrt{\frac{3}{4\pi}} \cos \theta \\
Y_{1\pm1}(\hat{r}) = \pm \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}
\end{array} \right.$$
The dagger operation

Frequently, we need to change the sign in the exponential

\[ \tilde{Y}_{lm}^\dagger(\theta, \phi) = C_{lm} P_l^m(\cos \theta)e^{-im\phi} \]

The dagger operation is identical to the complex conjugate, denoted \((*)\), of the spherical harmonics, provided the angles \(\theta\) and \(\phi\) are real numbers, i.e.,

\[ Y_{lm}^\dagger(\theta, \phi) = Y_{lm}^*(\theta, \phi), \quad \theta, \phi \in \mathbb{R} \]

We also have

\[ Y_{l-m}(\theta, \phi) = (-1)^m Y_{lm}^\dagger(\theta, \phi) \]

Orthogonality

Orthonormal over the unit sphere \(\Omega\) [3], i.e.,

\[
\int_{\Omega} \int \tilde{Y}_{lm}(\mathbf{r}) \tilde{Y}_{l'm'}^\dagger(\mathbf{r}) \, d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta \, d\theta \, Y_{lm}(\theta, \phi) Y_{l'm'}^\dagger(\theta, \phi) = \delta_{ll'} \delta_{mm'}
\]

where the surface measure on the unit sphere \(\Omega\) is

\[ d\Omega = \sin \theta \, d\theta \, d\phi \]
Completeness I

The system \( Y_{lm}(\hat{r}) \), \( l = 0, 1, \ldots, m = -l, \ldots, 0, \ldots, l \), is a complete orthonormal system over the unit sphere \( \Omega \) [4, page 24]

A square integrable function \( f(\theta, \phi) \) has a convergent expansion in the spherical harmonics \( Y_{lm}(\theta, \phi) \)

\[
f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi), \quad \theta \in [0, \pi], \phi \in [0, 2\pi)
\]

where the generalized Fourier-coefficients \( a_{lm} \) are determined by the integrals

\[
a_{lm} = \int_{0}^{2\pi} d\phi \int_{0}^{\pi} \sin \theta d\theta f(\theta, \phi) Y_{lm}^{\dagger}(\theta, \phi)
\]

Completeness II

The spherical harmonics \( Y_{lm}(\hat{r}) \) are eigenfunctions to the Laplace operator \( \nabla_{\Omega}^2 \) on the unit sphere with eigenvalues \(-l(l + 1)\), i.e.,

\[
\nabla_{\Omega}^2 Y_{lm}(\hat{r}) = -l(l + 1)Y_{lm}(\hat{r})
\]

where the Laplace operator on the unit sphere \( \Omega \) is

\[
\nabla_{\Omega}^2 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}
\]
The index set \( l = 0, 1, 2, \ldots \), and 
\( m = -l, \ldots, -1, 0, 1, \ldots, l \) is most conveniently mapped to a single index (countable set) by the formula

\[
j = l(l+1) + m + 1
\]

where \( J = (L+1)^2 \)

\[
\sum_{l=0}^{L} \sum_{m=-l}^{l} = \sum_{j=1}^{J}
\]
Provided these conditions are met [8]

$$\nabla \times F = \nabla \left( \frac{\partial w\psi}{\partial \xi_1} \right) + k^2 \hat{a}_1 w\psi$$

and consequently

$$\nabla \times (\nabla \times F) = k^2 F$$

Repeated curl gives

$$G = \frac{1}{k} \nabla \times F \quad \Rightarrow \quad \nabla \times (\nabla \times G) = k^2 G$$

**Conclusion:** $F$ and $G$ are two transverse vector fields

We augment this set with a longitudinal part $\nabla \psi$ which satisfies the vector Helmholtz’ equation

---

**Vector Helmholtz’ equation — Separation of variables III**

Only six coordinate systems satisfy these conditions

<table>
<thead>
<tr>
<th>System</th>
<th>$\hat{a}_1$</th>
<th>Scale factors</th>
<th>$w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian ($x, y, z$)</td>
<td>$\hat{x}$</td>
<td>$h_1 = h_2 = h_3 = 1$</td>
<td>1</td>
</tr>
<tr>
<td>Cartesian ($x, y, z$)</td>
<td>$\hat{y}$</td>
<td>$h_1 = h_2 = h_3 = 1$</td>
<td>1</td>
</tr>
<tr>
<td>Cartesian ($x, y, z$)</td>
<td>$\hat{z}$</td>
<td>$h_1 = h_2 = h_3 = 1$</td>
<td>1</td>
</tr>
<tr>
<td>Spherical ($r, \theta, \phi$)</td>
<td>$\hat{r}$</td>
<td>$h_1 = 1$ $h_2 = r$ $h_3 = r \sin \theta$</td>
<td>$\xi_1 = r$</td>
</tr>
<tr>
<td>Cylindrical ($\rho, \phi, z$)</td>
<td>$\hat{\rho}$</td>
<td>$h_1 = 1$ $h_2 = \rho$ $h_3 = 1$</td>
<td>1</td>
</tr>
<tr>
<td>Conical ($\xi_1, \xi_2, \xi_3$)</td>
<td>$\hat{\xi}_1$</td>
<td>$h_1 = 1$ $h_2 = \xi_1 \frac{\xi_2^2 + \xi_3^2}{(\alpha^2 - \xi_2^2)(\beta^2 + \xi_2^2)}$ $h_3 = \xi_1 \frac{\xi_2 + \xi_3}{(\alpha^2 + \xi_3^2)(\beta^2 - \xi_2^2)}$</td>
<td>$\xi_1$</td>
</tr>
</tbody>
</table>
Spherical vector waves

In the spherical coordinate system \((r, \theta, \phi)\)

\[
F(r, \theta, \phi) = \nabla \times \left( \hat{r} r R_l(kr) Y_{lm}(\theta, \phi) \right) = \psi(r, \theta, \phi)
\]

Rewrite using

\[
\nabla \times (\varphi \mathbf{a}) = \varphi (\nabla \times \mathbf{a}) + (\nabla \varphi) \times \mathbf{a}
\]

and \(\nabla R_l(kr) = \hat{r} k R'_l(kr)\)

\[
F(r, \theta, \phi) = \nabla \times (R_l(kr) r Y_{lm}(\theta, \phi)) = R_l(kr) \nabla \times (r Y_{lm}(\theta, \phi))
\]

First vector spherical harmonics \(\nabla \times (r Y_{lm}(\hat{r}))\)

Vector spherical harmonics I

Definition

\[
\begin{aligned}
A_{1lm}(\hat{r}) &= \frac{1}{\sqrt{l(l+1)}} \nabla \times (r Y_{lm}(\hat{r})) = \frac{1}{\sqrt{l(l+1)}} \nabla Y_{lm}(\hat{r}) \times r \\
A_{2lm}(\hat{r}) &= \frac{1}{\sqrt{l(l+1)}} r \nabla Y_{lm}(\hat{r}) \\
A_{3lm}(\hat{r}) &= \hat{r} Y_{lm}(\hat{r})
\end{aligned}
\]

The integer indices are \(l = 0, 1, 2, 3, \ldots\) and \(m = -l, \ldots, 0, 1, \ldots, l\)

For \(l = 0\), define \(A_{100}(\hat{r}) = A_{200}(\hat{r}) = 0\)
The vector spherical harmonics have the following properties:

\[
\begin{align*}
\hat{r} \cdot A_{\tau lm}(\hat{r}) &= 0, \quad \tau = 1, 2 \\
\hat{r} \times A_{3 lm}(\hat{r}) &= 0
\end{align*}
\]

and

\[
\begin{align*}
A_{1 lm}(\hat{r}) &= A_{2 lm}(\hat{r}) \times \hat{r} \\
A_{2 lm}(\hat{r}) &= \hat{r} \times A_{1 lm}(\hat{r})
\end{align*}
\]

which we can summarize as

\[
\hat{r} \times A_{n}(\hat{r}) = (-1)^{\tau+1} A_{\tilde{n}}(\hat{r}), \quad \tau = 1, 2
\]

where \( n = \{\tau lm\} \) and \( \tilde{n} = \{\tilde{\tau} lm\} \) are a multiindices (the dual index to \( \tau \) is defined by \( \tilde{1} = 2, \tilde{2} = 1 \))
Alternatively, we can express the vector spherical harmonics in the spherical basis vectors \( \{ \hat{r}, \hat{\theta}, \hat{\phi} \} \) (verify this yourself)

\[
\begin{align*}
A_{1lm}(\hat{r}) &= \frac{1}{\sqrt{l(l+1)}} \left( \hat{\theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} Y_{lm}(\hat{r}) - \hat{\phi} \frac{\partial}{\partial \theta} Y_{lm}(\hat{r}) \right) \\
A_{2lm}(\hat{r}) &= \frac{1}{\sqrt{l(l+1)}} \left( \hat{\theta} \frac{\partial}{\partial \theta} Y_{lm}(\hat{r}) + \hat{\phi} \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} Y_{lm}(\hat{r}) \right) \\
A_{3lm}(\hat{r}) &= \hat{r} Y_{lm}(\hat{r})
\end{align*}
\]

In analogy with the spherical harmonics, we frequently need to change the sign in the exponential in the vector spherical harmonics, i.e.,

\[
\begin{align*}
A_{1lm}^\dagger(\hat{r}) &= \frac{1}{\sqrt{l(l+1)}} \nabla \times \left( r Y_{lm}^\dagger(\hat{r}) \right) = \frac{1}{\sqrt{l(l+1)}} \nabla Y_{lm}^\dagger(\hat{r}) \times r \\
A_{2lm}^\dagger(\hat{r}) &= \frac{1}{\sqrt{l(l+1)}} r \nabla Y_{lm}^\dagger(\hat{r}) \\
A_{3lm}^\dagger(\hat{r}) &= \hat{r} Y_{lm}^\dagger(\hat{r})
\end{align*}
\]

We have

\[ A_n^\dagger(\theta, \phi) = A_n^*(\theta, \phi), \quad \theta, \phi \in \mathbb{R} \]

and

\[ A_{\tau l-m}(\theta, \phi) = (-1)^m A_{\tau lm}^\dagger(\theta, \phi) \]
Vector spherical harmonics VI

With an argument \( \hat{r} = \hat{z} (\theta = 0) \), the vector spherical harmonics have specific values

\[
\begin{align*}
A_{1lm}(\hat{z}) &= \mp \delta_{m\pm 1} \sqrt{\frac{2l+1}{16\pi}} (\pm i\hat{x} - \hat{y}) \\
A_{2lm}(\hat{z}) &= \mp \delta_{m\pm 1} \sqrt{\frac{2l+1}{16\pi}} (\hat{x} \pm i\hat{y}) \\
A_{3lm}(\hat{z}) &= \delta_{m0} \sqrt{\frac{2l+1}{4\pi}} \hat{z}
\end{align*}
\]

The lowest order vector spherical harmonics for \( m = 0 \)

\[
\begin{align*}
A_{100}(\hat{r}) &= 0 \\
A_{200}(\hat{r}) &= 0 \\
A_{300}(\hat{r}) &= \sqrt{\frac{1}{4\pi}} \hat{r}
\end{align*}
\]

Vector spherical harmonics VII

\[
\begin{align*}
A_{110}(\hat{r}) &= \sqrt{\frac{3}{8\pi}} \phi \sin \theta = -\sqrt{\frac{3}{8\pi}} \hat{r} \times \hat{z} \\
A_{210}(\hat{r}) &= -\sqrt{\frac{3}{8\pi}} \theta \sin \theta = \sqrt{\frac{3}{8\pi}} \hat{r} \times (\hat{z} \times \hat{r}) \\
A_{310}(\hat{r}) &= \sqrt{\frac{3}{4\pi}} \cos \theta = \sqrt{\frac{3}{4\pi}} \hat{r} (\hat{z} \cdot \hat{r})
\end{align*}
\]

The lowest order vector spherical harmonics for \( m = \pm 1 \)

\[
\begin{align*}
A_{11\pm 1}(\hat{r}) &= \mp \sqrt{\frac{3}{16\pi}} (\hat{x} \pm i\hat{y}) \times \hat{r} \\
A_{21\pm 1}(\hat{r}) &= \hat{r} \times A_{11\pm 1}(\hat{r}) = \mp \sqrt{\frac{3}{16\pi}} \hat{r} \times ((\hat{x} \pm i\hat{y}) \times \hat{r}) \\
A_{31\pm 1}(\hat{r}) &= \mp \sqrt{\frac{3}{8\pi}} \hat{r} \sin \theta e^{\pm i\phi} = \sqrt{\frac{3}{8\pi}} \hat{r} ((\hat{x} \pm i\hat{y}) \cdot \hat{r})
\end{align*}
\]
The vector spherical harmonics is a orthonormal set on the unit sphere $\Omega$

$$\iint_{\Omega} A_n(\hat{r}) \cdot A_{n'}^\dagger(\hat{r}) \, d\Omega$$

$$= \int_{0}^{2\pi} d\phi \int_{0}^{\pi} \sin \theta \, d\theta \, A_{\tau l m}(\theta, \phi) \cdot A_{\tau' l' m'}^\dagger(\theta, \phi)$$

$$= \delta_{\tau \tau'} \delta_{ll'} \delta_{mm'} = \delta_{nn'}$$

Every well-behaved vector field on the unit sphere has a convergent expansion in vector spherical harmonics [4, page 170]

$$F(\theta, \phi) = \sum_{\tau=1}^{3} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{\tau l m} A_{\tau l m}(\theta, \phi), \quad \theta \in [0, \pi], \phi \in [0, 2\pi]$$

where the Fourier-coefficients $a_{\tau l m}$ are determined by the integrals

$$a_{\tau l m} = \iint_{\Omega} F(\hat{r}) \cdot A_{\tau l m}^\dagger(\hat{r}) \, d\Omega$$

Note that the expansion coefficients in this expansion are scalars
The index set \( \tau = 1, 2, 3, \)  
\( l = 0, 1, 2, \ldots, \)  and  
\( m = -l, \ldots, -1, 0, 1, \ldots, l \) is most conveniently mapped to a single index (countable set) by the formula

\[
j = 3 \{l(l + 1) + m\} + \tau - 2
\]

\[
\sum_{\tau=1}^{3} \sum_{l=0}^{L} \sum_{m=-l}^{l} = \sum_{j=1}^{J}
\]

where \( J = 3(L + 1)^2 - 2 \)
List of references


Assignment

Derive the divergence and the curl relations

\[
\begin{align*}
\nabla \cdot \mathbf{A}_{1l} \hat{\mathbf{r}} & = 0 \\
\nabla \cdot \mathbf{A}_{2l} \hat{\mathbf{r}} & = -\sqrt{l(l+1)} \frac{Y_{lm} \hat{\mathbf{r}}}{r} \\
\nabla \cdot \mathbf{A}_{3l} \hat{\mathbf{r}} & = \frac{2Y_{lm} \hat{\mathbf{r}}}{r}
\end{align*}
\]

and

\[
\begin{align*}
\nabla \times \mathbf{A}_{1l} \hat{\mathbf{r}} & = \frac{\mathbf{A}_{2l} \hat{\mathbf{r}} + \sqrt{l(l+1)} \mathbf{A}_{3l} \hat{\mathbf{r}}}{r} \\
\nabla \times \mathbf{A}_{2l} \hat{\mathbf{r}} & = -\frac{\mathbf{A}_{1l} \hat{\mathbf{r}}}{r} \\
\nabla \times \mathbf{A}_{3l} \hat{\mathbf{r}} & = \frac{\sqrt{l(l+1)} \mathbf{A}_{1l} \hat{\mathbf{r}}}{r}
\end{align*}
\]