Vector Waves and Probe Compensation

Lecture 9: Probe compensation

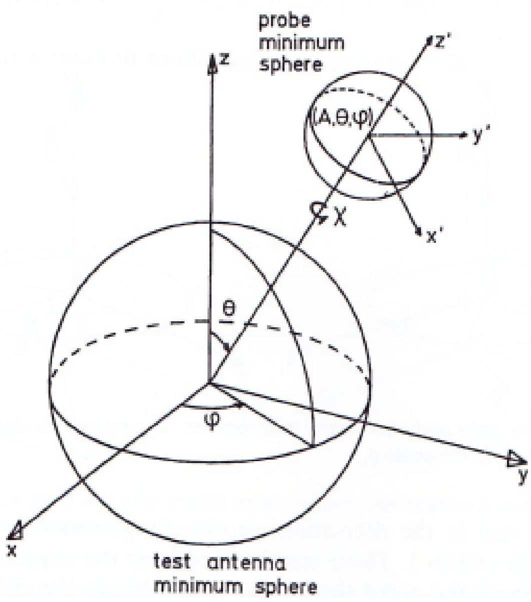
Gerhard Kristensson

Department of Electrical and Information Technology
Lund University

February 9, 2012

Experimental setup geometry I

- The two circumscribing (minimum) spheres do not overlap
- Assume there is no coupling between the two antenna systems (no multiple scattering — can be relaxed, see Hansen [2])



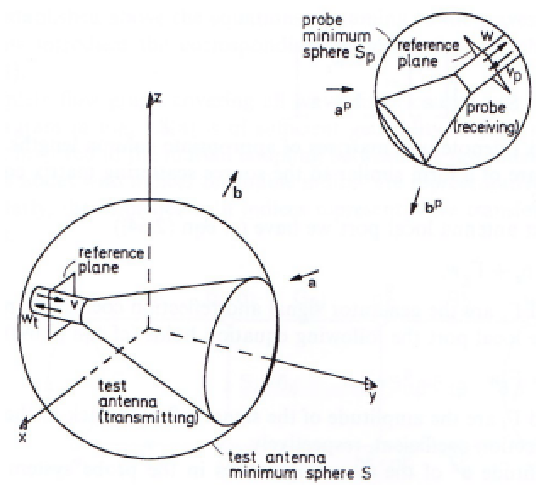
Experimental setup geometry II

Probe scattering description (receiving mode)

$$\begin{pmatrix} w^{\mathrm{p}} \\ \boldsymbol{b}^{\mathrm{p}} \end{pmatrix} = \begin{pmatrix} \Gamma^{\mathrm{p}} & \boldsymbol{R}^{\mathrm{p}} \\ \boldsymbol{T}^{\mathrm{p}} & \mathbf{S}^{\mathrm{p}} \end{pmatrix} \begin{pmatrix} v^{\mathrm{p}} \\ \boldsymbol{a}^{\mathrm{p}} \end{pmatrix}$$

Test antenna (AUT) scattering description (transmitting mode)

$$\begin{pmatrix} w^{\mathrm{t}} \\ \boldsymbol{b}^{\mathrm{t}} \end{pmatrix} = \begin{pmatrix} \Gamma^{\mathrm{t}} & \boldsymbol{R}^{\mathrm{t}} \\ \boldsymbol{T}^{\mathrm{t}} & \mathbf{S}^{\mathrm{t}} \end{pmatrix} \begin{pmatrix} v_{\mathrm{g}} \\ \boldsymbol{a}^{\mathrm{t}} \end{pmatrix}$$



Overview
Experimental setup geometry
Transmitting test antenna, receiving probe
Overview of methods
Without probe compensation
Naïve approach
Analytic method
With probe compensation
Naïve approach
Analytic method
Higher-order probe compensation
Not covered
Appendix
List of references
Assignment

Experimental setup geometry III

- The probe is matched to the load, $\Gamma_{\mathrm{l}} = 0$

$$w^{\mathrm{p}} = \frac{1}{1 - \Gamma_{\mathrm{l}}\Gamma^{\mathrm{p}}} \sum_n R_n^{\mathrm{p}} a_n^{\mathrm{p}} = \sum_n R_n^{\mathrm{p}} a_n^{\mathrm{p}}$$

- The AUT are matched to the generator, $\Gamma_{\mathrm{g}} = 0$, and no coupling $\boldsymbol{a}^{\mathrm{t}} = \mathbf{0}$ implies

$$b_n^{\mathrm{t}} = \frac{v_{\mathrm{g}}}{1 - \Gamma_{\mathrm{g}}\Gamma^{\mathrm{t}}} T_n^{\mathrm{t}} = v_{\mathrm{g}} T_n^{\mathrm{t}}$$

Overview
Experimental setup geometry
Transmitting test antenna, receiving probe
Overview of methods
Without probe compensation
Naïve approach
Analytic method
With probe compensation
Naïve approach
Analytic method
Higher-order probe compensation
Not covered
Appendix
List of references
Assignment

Transmitting AUT, receiving probe I

The radiated fields become

$$\begin{cases} \boldsymbol{E}^{\rm t}(\boldsymbol{r}) = k\sqrt{\eta_0\eta}v_{\rm g}\sum_n T_n^{\rm t}\boldsymbol{u}_n(k\boldsymbol{r}) \\ \boldsymbol{H}^{\rm t}(\boldsymbol{r}) = \dfrac{k}{{\rm i}\sqrt{\eta_0\eta}}v_{\rm g}\sum_n T_n^{\rm t}\boldsymbol{u}_{\bar{n}}(k\boldsymbol{r}) \end{cases} \qquad r > r_0$$

in the unprimed coordinate system (r, θ, ϕ) (only sum over $\tau = 1, 2$)

The truncation size is

$$J = 2L(L+2) \text{ where } l_{\text{max}} = L = [kr_0] + n_1 \text{ (empirical)}$$

where r_0 is the radius of the circumscribing sphere, and $[x]$ is the integer part of x

Transmitting AUT, receiving probe II

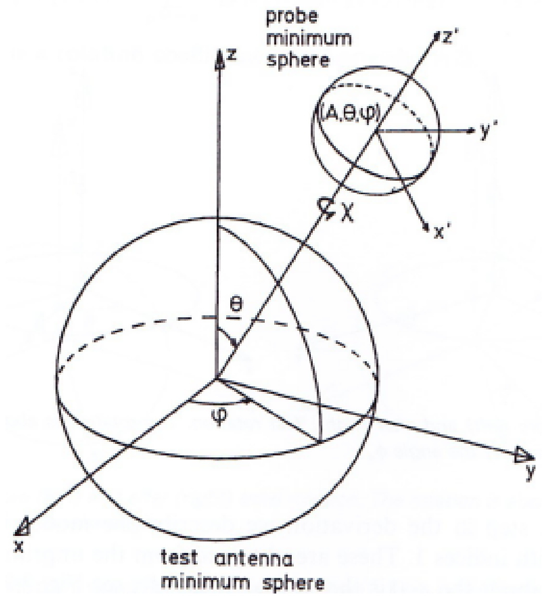
Goal: To find the transmission coefficients $T_n^{\rm t}$ of the test antenna

Aim: Transfer this description to the coordinate system of the probe!

The transformation consists of three parts, **1) rotation, 2) translation along z axis**, and a final **3) rotation**

Transmitting AUT, receiving probe III

1. Rotations ϕ and θ
2. Translation A along the new z axis
3. Final rotation χ around the z axis



Transmitting AUT, receiving probe IV

1. **Rotation:** From Lectures 5 & 6 we have

$$\begin{Bmatrix} \boldsymbol{v}_n(k\boldsymbol{r}) \\ \boldsymbol{u}_n(k\boldsymbol{r}) \end{Bmatrix} = \sum_{n'} \mathcal{D}_{n'n}(\alpha,\beta,\gamma) \begin{Bmatrix} \boldsymbol{v}'_{n'}(k\boldsymbol{r}') \\ \boldsymbol{u}'_{n'}(k\boldsymbol{r}') \end{Bmatrix}$$

$$\mathcal{D}_{n'n}(\alpha,\beta,\gamma) = \delta_{\tau\tau'}\delta_{ll'}\mathrm{e}^{\mathrm{i}m'\gamma} d^{(l)}_{m'm}(\beta)\mathrm{e}^{\mathrm{i}m\alpha}$$

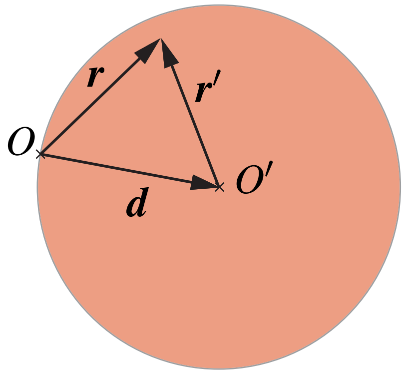
$$d^{(l)}_{m'm}(\beta) = \sqrt{\frac{(l+m')!\,(l-m')!}{(l+m)!\,(l-m)!}} \sum_j (-1)^{l-m'-j}$$

$$\binom{l+m}{l-m'-j}\binom{l-m}{j}\left(\cos\frac{\beta}{2}\right)^{2j+m+m'}\left(\sin\frac{\beta}{2}\right)^{2l-2j-m-m'}$$

Overview
Experimental setup geometry
Transmitting test antenna, receiving probe
Overview of methods
Without probe compensation
Naïve approach
Analytic method
With probe compensation
Naïve approach
Analytic method
Higher-order probe compensation
Not covered
Appendix
List of references
Assignment

Transmitting AUT, receiving probe V

2 Translation:



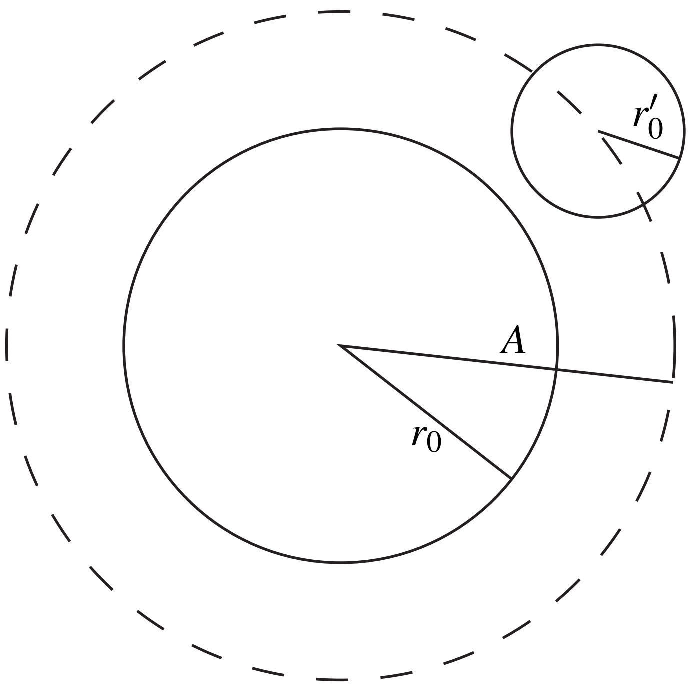
$$\boldsymbol{u}_n(k\boldsymbol{r}) = \sum_{n'} \mathcal{P}_{nn'}(k\boldsymbol{d})\boldsymbol{v}_{n'}(k\boldsymbol{r}'), \quad r' < d$$

The matrix $\mathcal{P}_{nn'}(k\boldsymbol{d})$ is given by

$$\mathcal{P}_{\tau n \tau' n'}(k\boldsymbol{d}) = \begin{array}{c|ccc} \tau\backslash\tau' & 1 & 2 & 3 \\ \hline 1 & A_{nn'}(k\boldsymbol{d}; h_\lambda^{(1)}) & B_{nn'}(k\boldsymbol{d}; h_\lambda^{(1)}) & 0 \\ 2 & B_{nn'}(k\boldsymbol{d}; h_\lambda^{(1)}) & A_{nn'}(k\boldsymbol{d}; h_\lambda^{(1)}) & 0 \\ 3 & 0 & 0 & C_{nn'}(k\boldsymbol{d}; h_\lambda^{(1)}) \end{array}$$

$A_{nn'}(k\boldsymbol{d}; h_\lambda^{(1)})$, $B_{nn'}(k\boldsymbol{d}; h_\lambda^{(1)})$, and $C_{nn'}(k\boldsymbol{d}; h_\lambda^{(1)})$ are explicitly given in Lecture 6

Transmitting AUT, receiving probe VI



Transmitting AUT, receiving probe VII

The connection between the two coordinate systems (r, θ, ϕ) and (r', θ', ϕ') are written in a compact notation

$$\boldsymbol{u}_n(k\boldsymbol{r}) = \sum_{n'} \mathcal{C}_{nn'}(k\boldsymbol{d})\boldsymbol{v}'_{n'}(k\boldsymbol{r}'), \quad r' < (A - r_0)$$

where $\mathcal{C}_{nn'}(k\boldsymbol{d})$ is a combination of matrices that represent the appropriate rotations and translation

$$\begin{aligned}\mathcal{C}_{nn'}(k\boldsymbol{d}) &= \sum_{n''n'''} \mathcal{D}_{n''n}(\phi, \theta, 0)\mathcal{P}_{n''n'''}(kA\hat{\boldsymbol{z}})\mathcal{D}_{n'n'''}(0, 0, \chi)\\ &= \mathrm{e}^{\mathrm{i}m\phi} d^{(l)}_{m'm}(\theta)\mathcal{P}_{\tau lm'\tau'l'm'}(kA\hat{\boldsymbol{z}})\mathrm{e}^{\mathrm{i}m'\chi}\end{aligned}$$

Remember $\mathcal{D}_{n'n'''}(0, 0, \chi)$ and $\mathcal{D}_{n''n}(\phi, \theta, 0)$ diagonal in τ and l indices, and $\mathcal{P}_{n''n'''}(kA\hat{\boldsymbol{z}})$ diagonal in the m index

Transmitting AUT, receiving probe VIII

$$\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r}) = k\sqrt{\eta_0\eta}v_{\mathrm{g}} \sum_n T^{\mathrm{t}}_n \boldsymbol{u}_n(k\boldsymbol{r}) = k\sqrt{\eta_0\eta}v_{\mathrm{g}} \sum_{nn'} T^{\mathrm{t}}_n \mathcal{C}_{nn'}(k\boldsymbol{d})\boldsymbol{v}'_{n'}(k\boldsymbol{r}')$$

This is the description of the electric field in the primed system (r', θ', ϕ')

From $2\boldsymbol{v}'_n(k\boldsymbol{r}') = \boldsymbol{u}'_n(k\boldsymbol{r}') + \boldsymbol{w}'_n(k\boldsymbol{r}')$, we get ($r' < (A - r_0)$)

$$\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r}) = \frac{k\sqrt{\eta_0\eta}v_{\mathrm{g}}}{2} \sum_{nn'} T^{\mathrm{t}}_n \mathcal{C}_{nn'}(k\boldsymbol{d}) \left(\boldsymbol{u}_{n'}(k\boldsymbol{r}') + \boldsymbol{w}_{n'}(k\boldsymbol{r}')\right)$$

Identify the input, a^{p}_n, in the primed system

$$\boxed{a^{\mathrm{p}}_{n'} = \frac{v_{\mathrm{g}}}{2} \sum_n T^{\mathrm{t}}_n \mathcal{C}_{nn'}(k\boldsymbol{d})}$$

Overview
Experimental setup geometry
Transmitting test antenna, receiving probe
Overview of methods
Without probe compensation
Naïve approach
Analytic method
With probe compensation
Naïve approach
Analytic method
Higher-order probe compensation
Not covered
Appendix
List of references
Assignment

Transmitting AUT, receiving probe IX

In the primed system, the antenna scattering description is (matched conditions $\Gamma^{\mathrm{p}} = 0$)

$$w^{\mathrm{p}}(A, \chi, \theta, \phi) = \sum_{n'} R^{\mathrm{p}}_{n'} a^{\mathrm{p}}_{n'} = \frac{v_{\mathrm{g}}}{2} \sum_{nn'} T^{\mathrm{t}}_n \mathcal{C}_{nn'}(k\boldsymbol{d}) R^{\mathrm{p}}_{n'}$$

where R^{p}_n are the receiving coefficients of the probe, which are assumed known from a previous calibration process

Problem in a nutshell: From the knowledge of the received wave $w^{\mathrm{p}}(A, \chi, \theta, \phi)$, determine the unknown transmitting coefficients T^{t}_n

Overview of methods

- Without probe compensation
  - Naïve methods
  - Analytical methods
    - Full field
    - Radial field component [2]
    - Wood's method [2]
- With probe compensation
  - Naïve methods
  - Analytical methods

Without probe compensation

These methods rely on the assumption that the electric and the magnetic fields are accurately measured with *e.g.*, short electric and magnetic dipoles

$$\begin{cases} \boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r}) = k\sqrt{\eta_0\eta}\sum_n b_n^{\mathrm{t}}\boldsymbol{u}_n(k\boldsymbol{r}) \\ \boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r}) = \dfrac{k}{\mathrm{i}\sqrt{\eta_0\eta}}\sum_n b_n^{\mathrm{t}}\boldsymbol{u}_{\bar{n}}(k\boldsymbol{r}) \end{cases} \qquad r > r_0$$

The probe should have **low** directivity and located at not-to-close distance

Goal: Find the expansion coefficients b_n^{t}

Naïve approach

Measure the fields, $\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r}_i)$, at a set of points $\boldsymbol{r}_i$, $i = 1, 2, \ldots, N$ ($\hat{\boldsymbol{e}}_i$ given unit vector, may vary with the points $\boldsymbol{r}_i$)

$$\hat{\boldsymbol{e}}_i\cdot\boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r}_i) = k\sqrt{\eta_0\eta}\sum_n b_n^{\mathrm{t}}\hat{\boldsymbol{e}}_i\cdot\boldsymbol{u}_n(k\boldsymbol{r}_i) = \sum_n P_{in}b_n^{\mathrm{t}}, \quad i = 1, \ldots, N$$

where the known matrix P_{in} is

$$P_{in} = k\sqrt{\eta_0\eta}\hat{\boldsymbol{e}}_i\cdot\boldsymbol{u}_n(k\boldsymbol{r}_i)$$

Procedure: Measure a set where $N \geq J$ (J is the truncation level of spherical vector waves) and solve the overdetermined system with SVD

Hansen [2] is sceptical to this method — depends critically on the sampling

Overview
Experimental setup geometry
Transmitting test antenna, receiving probe
Overview of methods
Without probe compensation
Naïve approach
Analytic method
With probe compensation
Naïve approach
Analytic method
Higher-order probe compensation
Not covered
Appendix
List of references
Assignment

Analytic method I

Assume accurate measurements of the electric field on a spherical surface $r =$ constant $> r_0$ are available

$$\begin{cases} \boldsymbol{E}^{\mathrm{t}}(\boldsymbol{r}) = k\sqrt{\eta_0\eta}\sum_n b_n^{\mathrm{t}}\boldsymbol{u}_n(k\boldsymbol{r}) \\ \boldsymbol{H}^{\mathrm{t}}(\boldsymbol{r}) = \dfrac{k}{\mathrm{i}\sqrt{\eta_0\eta}}\sum_n b_n^{\mathrm{t}}\boldsymbol{u}_{\bar{n}}(k\boldsymbol{r}) \end{cases} \quad r > r_0$$

Analytic method II

Scalar multiplication with $\boldsymbol{A}^{\dagger}_{\tau n}(\hat{\boldsymbol{r}})$ and integrate over the unit sphere Ω (in practice numerical integration)

$$\begin{cases} \displaystyle\iint_{\Omega} \boldsymbol{A}^{\dagger}_{1n}(\hat{\boldsymbol{r}}) \cdot \boldsymbol{E}^{\mathrm{t}}(r\hat{\boldsymbol{r}})\,\mathrm{d}\Omega = k\sqrt{\eta_0\eta}b^{\mathrm{t}}_{1n}h_l^{(1)}(kr) \\ \displaystyle\iint_{\Omega} \boldsymbol{A}^{\dagger}_{2n}(\hat{\boldsymbol{r}}) \cdot \boldsymbol{E}^{\mathrm{t}}(r\hat{\boldsymbol{r}})\,\mathrm{d}\Omega = k\sqrt{\eta_0\eta}b^{\mathrm{t}}_{2n}\frac{(krh_l^{(1)}(kr))'}{kr} \end{cases} \quad r > r_0$$

Solve for b_n ($h_l^{(1)}(z)$ and $(krh_l^{(1)}(kr))'$ have no real zeros, see Kristensson [4, Lemma B.1 & B.2])

Analytic method III

Hansen [2] multiplies with $\boldsymbol{u}_n^\dagger(k\boldsymbol{r})$ and integrates over the unit sphere Ω

The result is

$$\begin{cases} \displaystyle\iint_\Omega \boldsymbol{u}_{1n}^\dagger(kr\hat{\boldsymbol{r}})\cdot\boldsymbol{E}^{\mathrm{t}}(r\hat{\boldsymbol{r}})\,\mathrm{d}\Omega = k\sqrt{\eta_0\eta}b_{1n}^{\mathrm{t}}\left(h_l^{(1)}(kr)\right)^2 \\ \displaystyle\iint_\Omega \boldsymbol{u}_{2n}^\dagger(kr\hat{\boldsymbol{r}})\cdot\boldsymbol{E}^{\mathrm{t}}(r\hat{\boldsymbol{r}})\,\mathrm{d}\Omega = k\sqrt{\eta_0\eta}b_{2n}^{\mathrm{t}}\left\{\left(\frac{(krh_l^{(1)}(kr))'}{kr}\right)^2 \right. & r > r_0 \\ \displaystyle\qquad\qquad\left. + l(l+1)\left(\frac{h_l^{(1)}(kr)}{kr}\right)^2\right\} \end{cases}$$

These two approaches are very similar

With probe compensation

From above

$$w^{\mathrm{p}}(A,\chi,\theta,\phi) = \frac{v_{\mathrm{g}}}{2}\sum_{nn'} T_n^{\mathrm{t}} \mathcal{C}_{nn'}(k\boldsymbol{d}) R_{n'}^{\mathrm{p}}$$

where

$$\mathcal{C}_{nn'}(k\boldsymbol{d}) = \mathrm{e}^{\mathrm{i}m\phi} d_{m'm}^{(l)}(\theta)\mathcal{P}_{\tau l m' \tau' l' m'}(kA\hat{\boldsymbol{z}})\mathrm{e}^{\mathrm{i}m'\chi}$$

Notice that the sensitivity to different polarization of the probe signal is included in the receiving coefficients R_n^{p}

As above, the indices are:

$$\tau = 1,2, \quad l = 1,2,\dots,L, \quad m = -l,\dots,0,\dots,l$$
$$\tau' = 1,2, \quad l' = 1,2,\dots,L', \quad m' = -l',\dots,0,\dots,l'$$

Rule of thumb Hansen [2]: $L = [kr_0] + 10$ and $L' = [kr_0'] + 10$

Naïve approach

Measure the response, $w^{\mathrm{p}}(A_i, \chi_i, \theta_i, \phi_i)$, at a set of points $\boldsymbol{r}_i$ or $(A_i, \chi_i, \theta_i, \phi_i)$ (the method allows for different A_i), $i = 1, 2, \ldots, N$

$$w^{\mathrm{p}}(A_i, \chi_i, \theta_i, \phi_i) = \frac{v_{\mathrm{g}}}{2} \sum_{nn'} T_n^{\mathrm{t}} \mathcal{C}_{nn'}(k\boldsymbol{r}_i) R_{n'}^{\mathrm{p}} = \sum_n P_{in} T_n^{\mathrm{t}}$$

where the known matrix P_{in} is

$$P_{in} = \frac{v_{\mathrm{g}}}{2} \sum_{n'} \mathcal{C}_{nn'}(k\boldsymbol{r}_i) R_{n'}^{\mathrm{p}}$$

Procedure: Measure a set where $N \geq J$ (J is the truncation level of spherical vector waves) and solve the overdetermined system with SVD

Hansen [2] claims this method is ill-conditioned — depends critically on the sampling

Analytic method I

Rewrite as

$$\begin{aligned} w^{\mathrm{p}}(A, \chi, \theta, \phi) &= \frac{v_{\mathrm{g}}}{2} \sum_{nn'} T_n^{\mathrm{t}} \mathcal{C}_{nn'}(k\boldsymbol{d}) R_{n'}^{\mathrm{p}} \\ &= \frac{v_{\mathrm{g}}}{2} \sum_{nn'} T_n^{\mathrm{t}} \mathrm{e}^{\mathrm{i}m\phi} d_{m'm}^{(l)}(\theta) \mathcal{P}_{\tau l m' \tau' l' m'}(kA\hat{\boldsymbol{z}}) \mathrm{e}^{\mathrm{i}m'\chi} R_{n'}^{\mathrm{p}} \\ &= v_{\mathrm{g}} \sum_{\tau l m m'} T_{\tau l m}^{\mathrm{t}} \mathrm{e}^{\mathrm{i}m\phi} d_{m'm}^{(l)}(\theta) \mathrm{e}^{\mathrm{i}m'\chi} P_{\tau l m'}(kA) \end{aligned}$$

where the **probe response constants** are

$$\boxed{P_{\tau l m'}(kA) = \frac{1}{2} \sum_{\tau' l'} \mathcal{P}_{\tau l m' \tau' l' m'}(kA\hat{\boldsymbol{z}}) R_{\tau' l' m'}^{\mathrm{p}}, \quad m' = -L', \ldots, L'}$$

These constants contain only known quantities — specific for each range and probe

Overview
Experimental setup geometry
Transmitting test antenna, receiving probe
Overview of methods
Without probe compensation
Naïve approach
Analytic method
With probe compensation
Naïve approach
Analytic method
Higher-order probe compensation
Not covered
Appendix
List of references
Assignment

Analytic method II

We assume measurement are made in a near-field range on a spherical surface $A =$ constant

Three types of integrations retrieve the transmitting coefficients $T^{\mathrm{t}}_{\tau lm}$

1. Integrate over χ over $[0, 2\pi)$ (polarization)
2. Integrate over the azimuth angle ϕ over $[0, 2\pi)$
3. Integrate over the polar angle θ over $[0, \pi]$

These three integrations are done numerical by proper sampling, see Hansen [2]

Analytic method III

1. First step (the polarization χ)

With the use of orthogonality of $\mathrm{e}^{\mathrm{i}m'\chi}$ over $[0, 2\pi)$

$$w^{\mathrm{p}}_{m'}(A, \theta, \phi) = v_{\mathrm{g}} \sum_{\tau lm} T^{\mathrm{t}}_{\tau lm} \mathrm{e}^{\mathrm{i}m\phi} d^{(l)}_{m'm}(\theta) P_{\tau lm'}(kA)$$

where

$$w^{\mathrm{p}}_{m'}(A, \theta, \phi) = \frac{1}{2\pi} \int_0^{2\pi} w^{\mathrm{p}}(A, \chi, \theta, \phi) \mathrm{e}^{-\mathrm{i}m'\chi} \, \mathrm{d}\chi$$

and (approximation)

$$w^{\mathrm{p}}(A, \chi, \theta, \phi) = \sum_{m'=-L'}^{L'} w^{\mathrm{p}}_{m'}(A, \theta, \phi) \mathrm{e}^{\mathrm{i}m'\chi}$$

Overview
Experimental setup geometry
Transmitting test antenna, receiving probe
Overview of methods
Without probe compensation
Naïve approach
Analytic method
With probe compensation
Naïve approach
Analytic method
Higher-order probe compensation
Not covered
Appendix
List of references
Assignment

Analytic method IV

2. Second step (the azimuth angle ϕ)

With the use of orthogonality of $\mathrm{e}^{\mathrm{i}m\phi}$ over $[0, 2\pi)$

$$w^{\mathrm{p}}_{mm'}(A,\theta) = v_{\mathrm{g}} \sum_{\tau l} T^{\mathrm{t}}_{\tau lm} d^{(l)}_{m'm}(\theta) P_{\tau l m'}(kA)$$

where

$$w^{\mathrm{p}}_{mm'}(A,\theta) = \frac{1}{2\pi}\int_0^{2\pi} w^{\mathrm{p}}_{m'}(A,\theta,\phi)\mathrm{e}^{-\mathrm{i}m\phi}\,\mathrm{d}\phi$$

and (approximation)

$$w^{\mathrm{p}}(A,\chi,\theta,\phi) = \sum_{m=-L}^{L}\sum_{m'=-L'}^{L'} w^{\mathrm{p}}_{mm'}(A,\theta)\mathrm{e}^{\mathrm{i}m'\chi}\mathrm{e}^{\mathrm{i}m\phi}$$

Analytic method V

3. Third step (the polar angle θ)

With the use of orthogonality of $d^{(l)}_{m'm}(\theta)$ over $[0, \pi]$

$$w^{\mathrm{p}}_{lmm'}(A) = v_{\mathrm{g}} \sum_{\tau} T^{\mathrm{t}}_{\tau lm} P_{\tau l m'}(kA) \qquad (\star)$$

where

$$w^{\mathrm{p}}_{lmm'}(A) = \frac{2l+1}{2}\int_0^{\pi} w^{\mathrm{p}}_{mm'}(A,\theta) d^{(l)}_{m'm}(\theta)\,\sin\theta\,\mathrm{d}\theta$$

We have here used the orthogonality of Wigner's function $d^{(l)}_{m'm}(\theta)$, see Appendix and Edmonds [1]

$$\int_0^{\pi} d^{(l')}_{m'm}(\theta) d^{(l)}_{m'm}(\theta)\,\sin\theta\,\mathrm{d}\theta = \delta_{ll'}\frac{2}{2l+1}$$

Overview
Experimental setup geometry
Transmitting test antenna, receiving probe
Overview of methods
Without probe compensation
Naïve approach
Analytic method
With probe compensation
Naïve approach
Analytic method
Higher-order probe compensation
Not covered
Appendix
List of references
Assignment

Analytic method VI

Final, approximate expression

$$w^{\mathrm{p}}(A,\chi,\theta,\phi)=\sum_{l=0}^{L}\sum_{m=-L}^{L}\sum_{m'=-L'}^{L'} w^{\mathrm{p}}_{lmm'}(A)\mathrm{e}^{\mathrm{i}m'\chi}\mathrm{e}^{\mathrm{i}m\phi}d^{(l)}_{m'm}(\theta)$$

Analytic method VII

4. Fourth step (the transmitting coefficients T^{t}_n)

Let $m' = \pm 1$ in ($\star$), Wacker [6], (first-order probe compensation) *i.e.*,

$$\begin{cases} v_{\mathrm{g}}T^{\mathrm{t}}_{1lm}P_{1l-1}(kA)+v_{\mathrm{g}}T^{\mathrm{t}}_{2lm}P_{2l-1}(kA)=w^{\mathrm{p}}_{lm-1}(A) \\ v_{\mathrm{g}}T^{\mathrm{t}}_{1lm}P_{1l1}(kA)+v_{\mathrm{g}}T^{\mathrm{t}}_{2lm}P_{2l1}(kA)=w^{\mathrm{p}}_{lm1}(A) \end{cases}$$

Solve this 2×2 system for each l, m values for the two unknowns T^{t}_{1lm} and T^{t}_{2lm}, v_{g} known

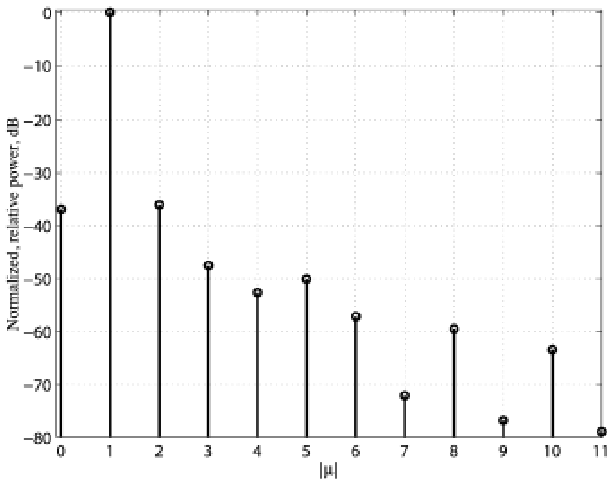
This can be done provided

$$\begin{vmatrix} P_{1l-1}(kA) & P_{2l-1}(kA) \\ P_{1l1}(kA) & P_{2l1}(kA) \end{vmatrix} \neq 0$$

Overview
Experimental setup geometry
Transmitting test antenna, receiving probe
Overview of methods
Without probe compensation
Naïve approach
Analytic method
With probe compensation
Naïve approach
Analytic method
Higher-order probe compensation
Not covered
Appendix
List of references
Assignment

Higher-order probe compensation I

In practice, real probes are never ideal first-order probes but possess, due to, *e.g.*, the manufacturing errors, also a minor level of high-order modes [5]



Overview
Experimental setup geometry
Transmitting test antenna, receiving probe
Overview of methods
Without probe compensation
Naïve approach
Analytic method
With probe compensation
Naïve approach
Analytic method
Higher-order probe compensation
Not covered
Appendix
List of references
Assignment

Higher-order probe compensation II

This requires higher-order probe compensation (hopefully more at DTU)

We follow [5] and write (superscripts $^{\rm p}$ and $^{\rm t}$ are omitted)

$$w(A,\chi,\theta,\phi) = v_{\rm g} \sum_{\tau l m m'} T_{\tau l m} {\rm e}^{{\rm i}m\phi} d^{(l)}_{m'm}(\theta) {\rm e}^{{\rm i}m'\chi} P_{\tau l m'}(kA)$$

as $w(A,\chi,\theta,\phi) = w_{\rm d}(A,\chi,\theta,\phi) + w_{\rm e}(A,\chi,\theta,\phi)$

$$w_{\rm d}(A,\chi,\theta,\phi) = v_{\rm g} \sum_{\tau l m} \sum_{m'=\pm 1} T_{\tau l m} {\rm e}^{{\rm i}m\phi} d^{(l)}_{m'm}(\theta) {\rm e}^{{\rm i}m'\chi} P_{\tau l m'}(kA) = \Phi_{\rm d}\{T\}$$

and

$$w_{\rm e}(A,\chi,\theta,\phi) = v_{\rm g} \sum_{\tau l m} \sum_{m'\neq\pm 1} T_{\tau l m} {\rm e}^{{\rm i}m\phi} d^{(l)}_{m'm}(\theta) {\rm e}^{{\rm i}m'\chi} P_{\tau l m'}(kA) = \Phi_{\rm e}\{T\}$$

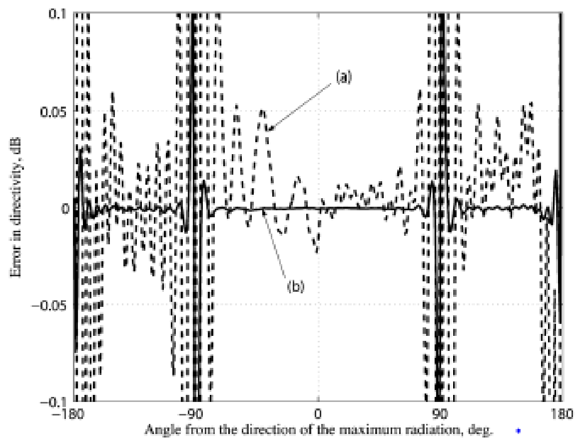
Overview
Experimental setup geometry
Transmitting test antenna, receiving probe
Overview of methods
Without probe compensation
Naïve approach
Analytic method
With probe compensation
Naïve approach
Analytic method
Higher-order probe compensation
Not covered
Appendix
List of references
Assignment

Higher-order probe compensation III

Formally, create an iteration scheme

$$\begin{cases} T^{(i)} = \Phi_{\mathrm{d}}^{-1}\{w - w_{\mathrm{e}}^{(i)}\} & \text{(first-order probe compensation)} \\ w_{\mathrm{e}}^{(i+1)} = \Phi_{\mathrm{e}}\{T^{(i)}\} \end{cases}$$

The iteration index $i = 0, 1, \ldots, I$, and the iteration scheme is initialized by $w_{\mathrm{e}}^{(0)} = 0$



What has not been covered, Hansen [2]

- Friis' transmission formula
- Far field aspects and transformations
- Measurement techniques
- Experimental setups
- Discretization and sampling
- Multiple coupling
- More on higher-order probe compensation, maybe solved at DTU
- ...
- ...
- ...

Appendix, Wigner's function $d^{(l)}_{mm'}(\theta)$

Rewrite in terms of Jacobi polynomials $P_m^{(\alpha,\beta)}(\cos\theta)$

$$d^{(l)}_{m'm}(\theta) = \sqrt{\frac{(l+m')!}{(l+m)!}\frac{(l-m')!}{(l-m)!}}\left(\cos\frac{\theta}{2}\right)^{m'+m}\left(\sin\frac{\theta}{2}\right)^{m'-m} P^{(m'-m,m'+m)}_{l-m'}(\cos\theta)$$

and use the orthogonality relation of these polynomials [3, Lemma 5.8]

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)\,\mathrm{d}x = \frac{\delta_{n,m}2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}$$

$$\int_0^{\pi} d^{(l)}_{m'm}(\theta)d^{(l')}_{m'm}(\theta)\,\sin\theta\,\mathrm{d}\theta = \sqrt{\frac{(l+m')!}{(l+m)!}\frac{(l-m')!}{(l-m)!}}\sqrt{\frac{(l'+m')!}{(l'+m)!}\frac{(l'-m')!}{(l'-m)!}}$$

$$\times\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{m'-m}\left(\frac{1+x}{2}\right)^{m'+m}P^{(m'-m,m'+m)}_{l-m'}(x)P^{(m'-m,m'+m)}_{l'-m'}(x)\,\mathrm{d}x$$

$$=\delta_{ll'}2\frac{(l+m')!}{(l+m)!}\frac{(l-m')!}{(l-m)!}\frac{(l-m)!(l+m)!}{(l-m')!(2l+1)(l+m')!} = \delta_{ll'}\frac{2}{2l+1}$$

List of references

[1] A. R. Edmonds.
Angular Momentum in Quantum Mechanics.
Princeton University Press, Princeton, New Jersey, second edition, 1960.

[2] J. E. Hansen, editor.
Spherical Near-Field Antenna Measurements.
Number 26 in IEE electromagnetic waves series. Peter Peregrinus Ltd., Stevenage, UK, 1988.
ISBN: 0-86341-110-X.

[3] G. Kristensson.
Second order differential equations — Special functions and their classification.
Springer-Verlag, London, 2010.

[4] G. Kristensson.
Spherical vector waves, 2011.
This booklet is an excerpt of the textbook Gerhard Kristensson, *Scattering of Electromagnetic Waves*.

[5] T. Laitinen, S. Pivnenko, and O. Breinbjerg.
Iterative probe correction technique for spherical near-field antenna measurements.
Antennas and Wireless Propagation Letters, IEEE, **4**, 221 – 223, 2005.

[6] P. Wacker.
Non-planar near field measurements: Spherical scanning.
Technical report, 1975.

Assignment 9

Make a probe compensation on a electric dipole system — both the AUT and the probe are electric dipoles

Perform the following steps:

1. Determine the transmitting and receiving coefficients of an electric dipole oriented along the three Cartesian axes (use the result of a vertical dipole and the result of the Assignment 5 to derive the two remaining orientations, *cf.*, with the results in Hansen [2])
2. Determine the received signal, $w^{\mathrm{p}}(A, \chi, \theta, \phi)$, from a vertical electrical dipole in spherical near field range if the probe is an electrical dipole (do not expand the translation coefficients more than necessary to perform the next step)
3. From the result in item 2, make a probe compensation to retrieve the original AUT coefficients