Problem 1

- Clearly $H(X) = H(Y) = \log 6 = 1 + \log 3 \approx 2.5850$. The probabilities for Z is shown in the table

z	2	3	4	5	6	7	8	9	10	11	12
$P(Z=z)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Hence, $H(Z) = -\sum_z P(Z=z)\log P(Z=z) \approx 3.2744$.

- Since X and Y are independent $H(Y|X) = H(Y) = \log 6$. The entropy of Z when $X = x$ is known is $H(Z|X=x) = \log 6$. Hence $H(Z|X) = \log 6$. The entropy $H(X|Z=z)$ goes back to the number of possible X-values for that z,

z	2	3	4	5	6	7	8	9	10	11	12
$H(X\|Z=z)$	0	$\log 2$	$\log 3$	$\log 4$	$\log 5$	$\log 6$	$\log 5$	$\log 4$	$\log 3$	$\log 2$	0

$H(X|Z) = \sum_z H(X|Z=z)P(Z=z) \approx 1.8955$ (which is less than $H(X)$).

- $H(X,Z) = H(X) + H(Z|X) = H(Z) + H(X|Z) \approx 5.1699$. Since $Z = X+Y$ and X and Y independent

$$H(X,Y,Z) = H(X) + \underbrace{H(Y|X)}_{=H(Y)} + \underbrace{H(Z|X,Y)}_{=0} = H(X) + H(Y) = 2\log 6 \approx 5.1699$$

We can also see that $H(X,Y,Z) = H(X) + H(Z|X) + \underbrace{H(Y|X,Z)}_{=0} = H(X,Z)$.

- $I(Z;X) = H(X) - H(X|Z) = H(Z) - H(Z|X) \approx 0.6894$.

Problem 2

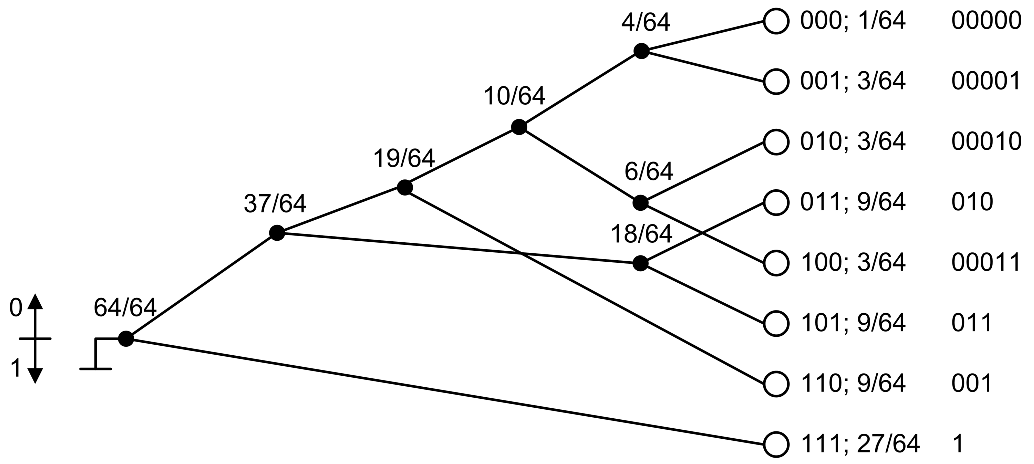
Let k be the number 1s in the source block. Then we get the following table for the probabilities

k	$\#\boldsymbol{x}$	$P(\boldsymbol{x})$
0	1	$\left(\frac{3}{4}\right)^3 = \frac{27}{64}$
1	3	$\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) = \frac{9}{64}$
2	3	$\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 = \frac{3}{64}$
3	1	$\left(\frac{1}{4}\right)^3 = \frac{1}{64}$

The entropy for the source blocks is

$$H(\boldsymbol{X}) = 3H(X) = 3h(\tfrac{1}{4}) \approx 2.4338$$

(a) The Huffman code is defined in the tree where the leaves are labelled with the source symbol, its probability and the code symbol.



The average code length is

$$E[W] = \frac{64+37+19+10+18+4+6}{64} = \frac{158}{64} \approx 2.4688$$

(b) The source sequence is encoded as

$$\boldsymbol{x} = \underline{111}\ \underline{011}\ \underline{101}\ \underline{110}\ \underline{111}\ \underline{100}\ \underline{111}\dots$$
$$\boldsymbol{y} = \underline{1}\ \underline{010}\ \underline{011}\ \underline{001}\ \underline{1}\ \underline{00011}\ \underline{1}\dots$$

The average codeword length in the encoded sequence is

$$\frac{L(\boldsymbol{y})}{7} = \frac{17}{7} \approx 2.4286$$

Problem 3

(a) The channel can be partitioned into two BSC, one from $0,1$ to $0,1$ with error probability $1/4$ and capacity $C_1 = 1-h(1/4)$, and one from $0,1$ to Δ_0,Δ_1 with error probability $1/2$ and capacity $C_2 = 1-h(1/2)=0$. They are equally likely to be used and we can derive the capacity for the complete channel

$$C = \frac{1}{2}\Big(1-h(\tfrac{1}{4})\Big) = \tfrac{1}{2} - \tfrac{1}{2}h(\tfrac{1}{4}) \approx 0.0944$$

Alternatively, it can be derived by maximising $I(X;Y) = H(Y)-H(Y|X)$ with respect to $P = P(X=0)$. We notice that $H(Y|X=x)$ is independent of P (the channel is dispersive). Therefore, $H(Y|X) = H(\frac{3}{8},\frac{1}{4},\frac{1}{4},\frac{1}{8}) = -\frac{3}{8}\log 3 + \frac{5}{2}$. The probabilities for Y are

$$P(Y=0) = \tfrac{1}{8} + \tfrac{1}{4}P$$
$$P(Y=\Delta_0) = P(Y=\Delta_1) = \tfrac{1}{4}$$
$$P(Y=1) = \tfrac{3}{8} - \tfrac{1}{4}P$$

Hence, $H(Y)$ is maximised for $P=\frac{1}{2}$ yielding $H(Y) = \log 4 = 2$. That gives the capacity

$$C = \max_P I(X;Y) = 2 - \big(-\tfrac{3}{8}\log 3 + \tfrac{5}{2}\big) = -\tfrac{1}{2} + \tfrac{\log}{3} = \tfrac{1}{2} - \tfrac{1}{2}\log 4 + \tfrac{3}{8}\log 3$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\Big(\tfrac{1}{4}\log\tfrac{1}{4} + \tfrac{3}{4}\log\tfrac{3}{4}\Big) = \tfrac{1}{2} - \tfrac{1}{2}h(\tfrac{1}{4})$$

(b) We see that the probability to receive Δ_i is independent of both the transmitted x and the index i. Therefore, we need only to consider one of them and treat the other the same way. The ML criteria, maximise $P(\boldsymbol{y}|\boldsymbol{x})$, means now that count the number of 0s and 1s and chose the one in majority. The decoding table becomes (a $\times$ means that we might chose that codeword to get an ML decoder, two $\times$ means that we chose randomly)

$\boldsymbol{y}$	$\boldsymbol{x}$ = 000	111
000	$\times$	
00Δ	$\times$	
001	$\times$	
0Δ0	$\times$	
0ΔΔ	$\times$	
0Δ1	$\times$	$\times$
010	$\times$	
01Δ	$\times$	$\times$
011		$\times$

$\boldsymbol{y}$	$\boldsymbol{x}$ = 000	111
Δ00	$\times$	
Δ0Δ	$\times$	
Δ01	$\times$	$\times$
ΔΔ0	$\times$	
ΔΔΔ	$\times$	$\times$
ΔΔ1		$\times$
Δ10	$\times$	$\times$
Δ1Δ		$\times$
Δ11		$\times$

$\boldsymbol{y}$	$\boldsymbol{x}$ = 000	111
100	$\times$	
10Δ	$\times$	$\times$
101		$\times$
1Δ0	$\times$	$\times$
1ΔΔ		$\times$
1Δ1		$\times$
110		$\times$
11Δ		$\times$
111		$\times$

The Bhattacharyya bound gives

$$
\begin{aligned}
P_B &\leqslant \prod_{i=1}^{n}\sum_{y}\sqrt{p(y|0)p(y|1)}\\
&= \Big(\sqrt{p(0|0)p(0|1)}+\sqrt{p(\Delta_0|0)p(\Delta_0|1)}+\sqrt{p(\Delta_1|0)p(\Delta_1|1)}+\sqrt{p(1|0)p(1|1)}\Big)^3\\
&= \Big(\sqrt{\tfrac{3}{8}\tfrac{1}{8}}+\sqrt{\tfrac{1}{4}\tfrac{1}{4}}+\sqrt{\tfrac{1}{4}\tfrac{1}{4}}+\sqrt{\tfrac{1}{8}\tfrac{3}{8}}\Big)^3 = \Big(\tfrac{\sqrt{3}+2}{4}\Big)^3 \approx 0.2030
\end{aligned}
$$

Problem 4

(a) The travel route follows a Markov chain according to the probability matrix

$$
\Pi = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0\\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix}
$$

Let $\boldsymbol{P} = \begin{pmatrix} P_0 & P_1 & P_2 & P_3\end{pmatrix}$ be the stationary distribution. Then, the equation system $\boldsymbol{P}\Pi = \boldsymbol{P}$ together with the condition $\sum_i P_i = 1$ gives the solution

$$
\boldsymbol{P} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{2}{9} & \frac{1}{9}\end{pmatrix}
$$

which is the distribution of the islands.

(b) The minimum number of bits per code symbol is entropy rate,

$$
H_\infty = \tfrac{1}{3}\log 3 + \tfrac{1}{3}\log 3 + \tfrac{2}{9}\log 2 + \tfrac{1}{9}\log 1 = \tfrac{2}{9} + \tfrac{2}{3}\log 3
$$

Problem 5

Consider a sequence of n cuts and let $\boldsymbol{x} = x_1x_2\dots x_n$ be the the outcome where x_i is the part saved in cut i. If in k of the cuts we save the long part and in $n-k$ the short part, the length becomes $L_k = (\frac{2}{3})^k(\frac{1}{3})^{(n-k)} = \frac{2^k}{3^n}$. The probability for such a sequence is $P(\boldsymbol{x}) = (\frac{3}{4})^k(\frac{1}{4})^{(n-k)} = \frac{3^k}{4^n}$. On the other hand we know that the most probable sequences are the typical, represented by the set $A_\varepsilon(\boldsymbol{X})$. Hence, if we consider a typical sequence we know that the probability is bounded by

$$
2^{-n(H(X)+\varepsilon)} \leqslant P(\boldsymbol{x}) \leqslant 2^{-n(H(X)-\varepsilon)}
$$

To the first order of the exponent (assume ε very small), this gives that $P(\boldsymbol{x}) = 2^{-nH(X)}$, where $H(X) = h(\frac{1}{4})$. Combining the two expressions for the probability gives

$$
3^k = 2^{2n}\cdot 2^{-nh(\frac{1}{4})} = 2^{n(2-h(\frac{1}{4}))}
$$

or, equivalently,

$$
k = n\frac{2-h(\frac{1}{4})}{\log 3} = n\frac{2+\frac{1}{4}\log\frac{1}{4}+\frac{3}{4}\log\frac{1}{4}+\frac{3}{4}\log 3}{\log 3} = n\frac{3}{4}
$$

Going back to the remaining length we get

$$
L_k = \frac{2^{n\frac{3}{4}}}{3^n} = \Big(\frac{2^{\frac{3}{4}}}{3}\Big)^n
$$

and we conclude that, in average, we keep $\frac{2^{3/4}}{3}$ of the length at each cut.