Problem 1

From the probability function in the problem we get

$P(X)$

$X=0$	$X=1$
3/8	5/8

$P(Y)$

$Y=0$	$Y=1$
1/4	3/4

$P(X|Y)$

$X \backslash Y$	0	1
0	1/2	1/3
1	1/2	2/3

$P(Y|X)$

$X \backslash Y$	0	1
0	1/3	2/3
1	1/5	4/5

From this we can derive

(a) Entropies

$$H(X) = h(\tfrac{3}{8}) = 3 - \tfrac{3}{8}\log 3 - \tfrac{5}{8}\log 5 \approx 0.9544$$
$$H(Y) = h(\tfrac{1}{4}) = 2 - \tfrac{3}{4}\log 3 \approx 0.8113$$

(b) Conditional entropies

$$H(X|Y) = \tfrac{1}{4}h(\tfrac{1}{2}) + \tfrac{3}{4}h(\tfrac{1}{3}) = \tfrac{3}{4}\log 3 - \tfrac{1}{4} \approx 0.9387$$
$$H(Y|X) = \tfrac{3}{8}h(\tfrac{1}{3}) + \tfrac{5}{8}h(\tfrac{1}{5}) = \tfrac{3}{8}\log 3 + \tfrac{5}{8}\log 5 - \tfrac{5}{4} \approx 0.7956$$

(c) Joint entropy and mutual information

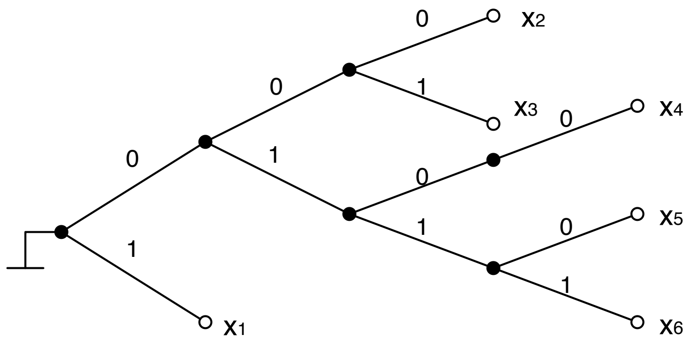
$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y) = H(\tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{4}, \tfrac{1}{2}) \approx 1.75$$
$$I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X) = H(X) + H(Y) - H(X,Y) \approx 0.0157$$

(d) Relative entropies

$$D(p(x)||p(y)) = P(X=0)\log\tfrac{P(X=0)}{P(Y=0)} + P(X=1)\log\tfrac{P(X=1)}{P(Y=1)} = \tfrac{5}{8}\log 5 - \tfrac{1}{4}\log 3 - 1 \approx 0.0550$$
$$D(p(y)||p(x)) = P(Y=0)\log\tfrac{P(Y=0)}{P(X=0)} + P(Y=1)\log\tfrac{P(Y=1)}{P(X=1)} = 1 + \tfrac{1}{2}\log 3 - \tfrac{3}{4}\log 5 \approx 0.0510$$

Problem 2

(a) A code is prefix-free if and only if we can draw it in a tree, where the codewords are represented by leaves. For this code we get



From the tree we can actually see that the code is sub-optiomal, since the codeword for x_4 can be shortened to 010.

(b) The entropy is $H(0.5, 0.15, 0.1, 0.1, 0.1, 0.05) \approx 2.1232$ and the length $E[\ell] \approx 2.25$.

(c) In the table a Huffman code is given, which is an optimal code.

X	C_H
x_1	0
x_2	110
x_3	100

X	C_H
x_4	101
x_5	1110
x_6	1111

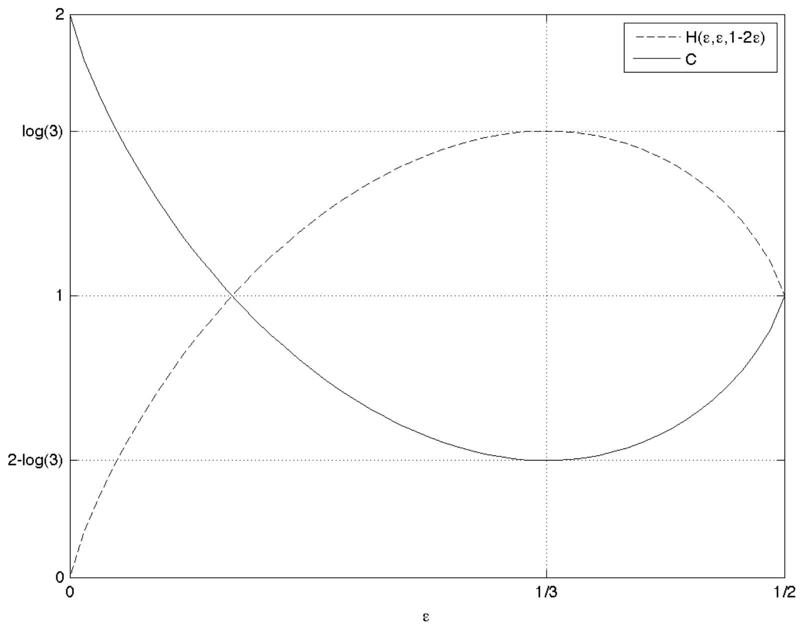
The optimal codeword length is $E[\ell_H] = 2.15$. Hence the code in the problem is not optimal.

Problem 3

(a) Since it is a symmetric channel we get directly

$$C = \log 4 - H(\varepsilon, \varepsilon, 1 - 2\varepsilon) = 2 + 2\varepsilon \log \varepsilon + (1 - 2\varepsilon) \log(1 - 2\varepsilon)$$

(b) The entropy function is bounded by $0 \leqslant H(\varepsilon, \varepsilon, 1 - 2\varepsilon) \leqslant \log 3$ with equality to the left when $\varepsilon = 0$ and equality to right when $\varepsilon = \frac{1}{3}$. The maximum ε is $\frac{1}{2}$, for which the entropy becomes $H(\frac{1}{2}, \frac{1}{2}, 0) = 1$. Hence, the capacity lies in the interval $2 - \log 3 \leqslant C \leqslant 2$ with equality to the left for $\varepsilon = \frac{1}{3}$ and to the left for $\varepsilon = 0$. The second endpoint is $\varepsilon = \frac{1}{2}$ that gives the capacity $C = 1$. A plot of the entropy and the capacity for all possible ε between 0 and $\frac{1}{2}$ is given below.



Problem 4

This problem can give different solutions depending on the interpreteation of the codeword when the repetiotion cannot be found in the buffer. Here it is $(0, 0, x)$. The encoded bits and the match in the bufer are underlined. Since both N and B are 7, we need three bits each for them, and also one bit for the next symbol.

(a) Encoding

buffer|sequence	Codeword	Binary
0000000|1011 ⋯	$(0,0,1)$	000 000 1
0000001|01111 ⋯	$(2,2,1)$	010 010 1
0001011|1100 ⋯	$(1,2,0)$	001 010 0
1011110|01011 ⋯	$(6,2,0)$	110 010 0
1110010|1100010 ⋯	$(6,4,0)$	110 100 0
1011000|101100 ⋯	$(7,?,?)$	111 ??? ?

Notice that the last codeword cannot be decided since it depends on the next bit, and, if there is a match, also on the bit after that.

(b) Decoding is the same table but starting with the codewords.

Problem 5

(a) With Lagrangian optimization of the capacity with the restriction on the total power, we have the function

$$J = \sum_i C_i + \lambda\Big(\sum_i P_i - P\Big) = \sum_i \Delta \log\Big(1 + \frac{P_i H_i^2}{N_{0,i}\Delta}\Big) + \lambda\Big(\sum_i P_i - P\Big)$$

Set the derivative to zero

$$\frac{\partial}{\partial P_i} J = \frac{\Delta}{\ln 2}\frac{H_i^2}{N_{0,i}\Delta}\frac{1}{1 + \frac{P_i H_i^2}{N_{0,i}\Delta}} + \lambda = 0$$

and rewrite as

$$\frac{1}{\frac{N_{0,i}}{H_i^2} + \frac{P_i}{\Delta}} = -\lambda \ln 2 \qquad \Rightarrow \qquad P_i = \Delta\Big(-\tfrac{1}{\lambda \ln 2} - \frac{N_{0,i}}{H_i^2}\Big)$$

Assigning $B = -\frac{1}{\lambda \ln 2}$ gives the result in the problem.

(b) With $H_i^2 = (0.5, 0.4, 0.3, 0.2, 0.1)$ and $N_{0,i} = \big(\frac{0.1}{1000}, \frac{0.2}{1000}, \frac{0.3}{1000}, \frac{0.4}{1000}, \frac{0.5}{1000},\big)$ we get

$$\frac{N_{0,i}}{H_i^2} = \big(\tfrac{1/5}{1000}, \tfrac{1/2}{1000}, \tfrac{1}{1000}, \tfrac{2}{1000}, \tfrac{5}{1000}\big)$$

Set P_i according to (a)

$$\sum_i P_i = 1000\Big(5B - \frac{1/5 + 1/2 + 1 + 2 + 5}{1000}\Big) = 40$$

Let $\tilde{B} = 1000B$. Then

$$5\tilde{B} - \big(\tfrac{1}{5} + \tfrac{1}{2} + 1 + 2 + 5\big) = 40 \qquad \Rightarrow \qquad \tilde{B} = \tfrac{487}{50} \approx 9.74$$

Then the power per sub-chennel is

$$P_i = \Delta\Big(B - \frac{N_{0,i}}{H_i^2}\Big) = \tilde{B} - \Delta\frac{N_{0,i}}{H_i^2} = (9.54, 9.24, 8.74, 7.74, 4.74)$$

The corresponding bit rate is

$$C = \sum_i \Delta \log\Big(1 + \frac{P_i H_i^2}{N_{0,i}\Delta}\Big) = \Delta \sum_i \log\Big(1 + \frac{P_i}{\Delta \frac{N_{0,i}}{H_i^2}}\Big)$$

$$\approx 5.6058 + 4.2839 + 3.2839 + 2.2839 + 0.9620 = 16.4196\text{kb/s}$$