LUND UNIVERSITY
Electrical and Information Technology

# Exam in
# Information Theory, EIT080

## June 1, 2010, kl 14-19

- Write your name on each paper.
- Start a new solution on a new sheet of paper. Use only one side of the paper.
- Solutions should clearly show the line of reasoning.
- Aid:
  - Course book.
  - Handed out material.
  - Lecture slides and articles from the home page.
  - Calculator.

## Good luck!

## Problem 1

The joint probability for the random variables $X$ and $Y$ is given by the following table

| $P(X,Y)$ | | $Y=0$ | $Y=1$ |
|---|---|---|---|
| $X$ | 0 | 1/8 | 1/4 |
| | 1 | 1/8 | 1/2 |

Calculate

(a) $H(X)$ and $H(Y)$

(b) $H(X|Y)$ and $H(Y|X)$

(c) $H(X,Y)$ and $I(X;Y)$

(d) $D(p(x)||p(y))$ and $D(p(y)||p(x))$

(2+3+2+3=10p)

## Problem 2

Consider the random variable $X$, taking values in $\{x_1, x_2, x_3, x_4, x_5, x_6\}$, and the binary code

| $X$ | $P(X)$ | $Y$ |
|---|---|---|
| $x_1$ | 0.5 | 1 |
| $x_2$ | 0.15 | 000 |
| $x_3$ | 0.1 | 001 |

| $X$ | $P(X)$ | $Y$ |
|---|---|---|
| $x_4$ | 0.1 | 0100 |
| $x_5$ | 0.1 | 0110 |
| $x_6$ | 0.05 | 0111 |

(a) Show that the code if prefix-free by drawing a tree represention.

(b) Derive the entropy of the source, $H(X)$, and the average codeword length of the code above.

(c) Is the code above optimal? If not, construct an optimal code and specify its average codeword length.

(2+3+5=10p)

## Problem 3

Consider the discrete channel in Figure 3.1.

(a) Derive an expression for the channel capacity as a function of $\varepsilon$.

(b) Find the maximum and minimum values for the capacity, and sketch a plot of the capacity as $\varepsilon$ varies, where you show minimum, maximum and end points.

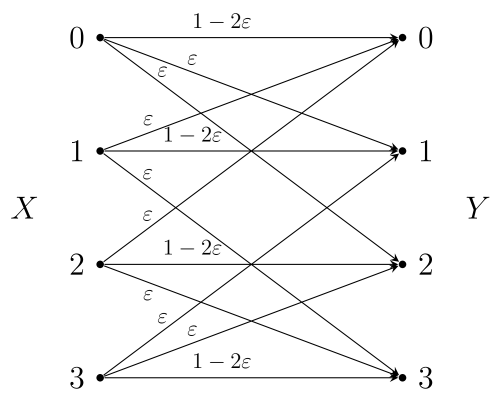


**Figure 3.1:** A discrete channel

**Remark:** The channel can be seen as a model when using 4-QAM modulation.

(5+5=10p)

## Problem 4

The binary sequence

$$\boldsymbol{x} = 0000000 \Big| 101111001011000101100\ldots$$

should be compressed with the LZ77 algorithm, using a search buffer of length $N = 7$ and a look ahead buffer of size $B = 7$. The first seven zeros of the sequence is the initialization of the buffer.

(a) Encode the sequence.

(b) Decode the sequence in part (a).

(5+5=10p)

## Problem 5

In an OFDM modulation system a wide-band channel is split into several, say $N$, independent narrow band sub-channels of frequency band-width $\Delta$. In general, both the noise spectra $R_Z(f)$ and the signal attenuation $|H(f)|^2$ varies over the wide-band channel, but it is often assumed that both parameters are constant within a narrow band sub-channel. With this assumption it is possible to derive the capacity for the channel according to the following. For sub-channel $i$ the transmitted power is $P_i$. Then the received power is $P_{i,\text{rec}} = P_i H_i^2$ where $H_i^2$ is the attenuation for the sub-channel. The corresponding noise has the power spectral density

$$R_{Z,i}(f) = \begin{cases} \frac{N_{0,i}}{2}, & f_i \leqslant f < f_i + \Delta \\ 0, & \text{otherwise} \end{cases}$$

Hence, for sub-channel $i$ the channel capacity, in bits/seconds, is

$$C_i = \Delta \log\Big(1 + \frac{P_{i,\text{rec}}}{N_{0,i}\Delta}\Big) = \Delta \log\Big(1 + \frac{P_i H_i^2}{N_{0,i}\Delta}\Big)$$

And for the whole wide-band channel it becomes

$$C = \sum_{i=1}^{N} C_i$$

(a) Show that under the power constraint $\sum_i P_i \leqslant P$, where $P_i > 0$ for all $i$, the optimal choice of sub-channel use is through the equation system

$$P_i = \Delta\Big(B - \frac{N_{0,i}}{H_i^2}\Big)$$

where $B$ is chosen such that

$$\sum_i P_i = P$$

**Hint:** $D(\log_2 x) = D(x)\dfrac{1}{x \ln 2}$.

(b) Consider an OFDM system with $N = 5$ sub-channels and frequency spacing $\Delta = 1\text{kHz}$. The channel parameters are

$$H_i^2 = (0.5; 0.4; 0.3; 0.2; 0.1)$$
$$N_{0,i} = (0.0001; 0.0002; 0.0003; 0.0004; 0.0005)$$

How many bits per second is it possible to transmit over the channel if $P = 40W$?

(5+5=10p)