Problem 1

(a) From the probability table we have

Y	0	1	2
$P(Y)$	3/8	1/4	3/8

X	0	1
$P(X)$	1/4	3/4

Hence,

$$H(X) = h(\tfrac{1}{4}) = 2 - \tfrac{3}{4}\log 3 \approx 0.8113$$
$$H(Y) = H(\tfrac{3}{8}, \tfrac{2}{8}, \tfrac{3}{8}) = \tfrac{11}{4} - \tfrac{6}{8}\log 3 \approx 1.5613$$

(b) Using $P(A|B) = \frac{P(A,B)}{P(B)}$ we get

$P(Y\|X=i)$		Y: 0	1	2
X	0	1/2	0	1/2
	1	1/3	1/3	1/3

$P(Y\|X=i)$		Y: 0	1	2
X	0	1/3	0	1/3
	1	2/3	1	2/3

and

$$H(X|Y) = \tfrac{6}{8}h(\tfrac{1}{3}) = \tfrac{6}{8}\log 3 - \tfrac{1}{2} \approx 0.6887$$
$$H(Y|X) = \tfrac{1}{4}h(\tfrac{1}{2}) + \tfrac{3}{4}H(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}) = \tfrac{1}{4} + \tfrac{3}{4}\log 3 \approx 1.4387$$

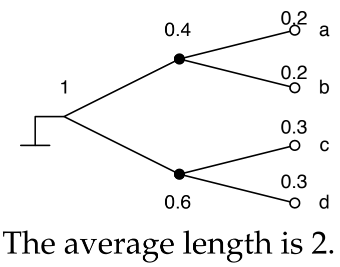
(c) Then,

$$H(X,Y) = H(X) + H(Y|X) = \tfrac{9}{4} = 2.25$$
$$I(X;Y) = H(X) - H(X|Y) = \tfrac{10}{4} - \tfrac{3}{2}\log 3 \approx 0.1226$$

Problem 2

$x \in X$	$P(x)$
a	0.2
b	0.2
c	0.3
d	0.3

(a) An optimal code can be construceted as a Huffman code, tree and table below



X	a	b	c	d
$\ell(X)$	00	01	10	11

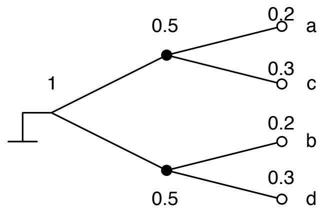
The average length is 2.

(b) Since all codewords have equal length, all codes with that property are optimal. That is

$$\text{Num Codewords} = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

As a check we should also try other lengths to find all optimal codes. If we shorten the codeword for one symbol we should choose one with high probability (0.3). Then the two least probable codewords will have length 3. That will result in a code with average length 2.1, which is more than the optimal. This concludes that we have found all possible optimal codes

(c) We can modify the tree in (a) as below



X	a	b	c	d
$\ell(X)$	00	10	01	11

This is still an optimal code, but the tree does not follow the rules for a Huffman code.

Problem 3

The channel can be visualized as a figure with 64 elements to the right (X) and 64 values to the left (Y). From the problem it is clear that there are four arrows leaving each X-value, with equal probablity. Since the channel is error is calculated modulu 64, each Y-element will have one arrow leading to it, corresponding to each of the numbers $\{1, 5, 17, 26\}$, with equal probability. This means that the channel is symetric and

$$C = \log 64 - H(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}) = 4$$

Problem 4

The behaviour at the playground can be modelled as a Markov source with three states corresponding to the swing, rocking horse and climbing net. The probability transition matrix becomes

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

The stationary distribution then can be derived as

$$w = (2/9 \quad 4/9 \quad 1/3)$$

(a) From the Markov model we see that the probability that Bob will stay at an attraction, independent of which, is $1/2$. Therefore, when he has started to play at an attraction the avererage time he will stay there is

$$T = \sum_{i=0}^{\infty} i(\tfrac{1}{2})^i = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2$$

Note: The sum above can be viewed as a standard sum. It can be found from

$$\sum_{i=0}^{\infty} p^i = \frac{1}{1-p}, \qquad \text{if } |p| < 1$$

Taking the derivative on both sides we get

$$\sum_{i=1}^{\infty} ip^{i-1} = \frac{1}{(1-p)^2}, \qquad \text{if } |p| < 1$$

and we conclude that

$$\sum_{i=0}^{\infty} ip^i = \frac{p}{(1-p)^2}, \qquad \text{if } |p| < 1$$

(b) From the stationary distribution we get $P(\text{swing}) = \frac{2}{9}$.

(c) The bound for compression is the entropy rate,

$$\begin{aligned} H_\infty(X) &= \sum_i w_i H(X_2|X_1 = i) \\ &= \tfrac{2}{9}H(\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{4}) + \tfrac{4}{9}H(\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{4}) + \tfrac{1}{3}H(\tfrac{1}{2},\tfrac{1}{2}) = \tfrac{4}{3} \approx 1.33 \end{aligned}$$

This can be compared to the compression rate when only considering the steady state distribution $H(\frac{2}{9},\frac{4}{9},\frac{1}{3}) \approx 1.53$, or when not taking the model in consideration $H(\frac{1}{3},\frac{1}{3},\frac{1}{3}) \approx 1.58$.

Problem 5

From $D(p||q) \geqslant 0$ we get that $\sum_i p_i \log p_i \geqslant \sum_i p_i \log q_i$. This can be used in the following derivations

$$\begin{aligned} h\Big(\tfrac{\sum_i p_i}{n}\Big) &= -\tfrac{\sum_i p_i}{n}\log\tfrac{\sum_i p_i}{n} - \Big(1-\tfrac{\sum_i p_i}{n}\Big)\log\Big(1-\tfrac{\sum_i p_i}{n}\Big) \\ &= -\tfrac{\sum_i p_i}{n}\log\tfrac{\sum_i p_i}{n} - \tfrac{\sum_i 1-p_i}{n}\log\Big(1-\tfrac{\sum_i p_i}{n}\Big) \\ &= -\frac{1}{n}\sum_i \underbrace{\Big(p_i\log\tfrac{\sum_i p_i}{n} + (1-p_i)\log\Big(1-\tfrac{\sum_i p_i}{n}\Big)\Big)}_{\leqslant p_i\log p_i + (1-p_i)\log(1-p_i)} \\ &\geqslant -\frac{1}{n}\sum_i p_i\log p_i + (1-p_i)\log(1-p_i) \\ &= \frac{\sum_i h(p_i)}{n} \end{aligned}$$