Problem 1

To get the optimal codeword length construct a Huffman code. Compare the codeword length of the code in the problem with this.

U	$P(U)$	W	ℓ	W_H	ℓ_H
000	27/125	0	1	00	2
001	18/125	100	3	100	3
010	18/125	110	3	101	3
011	12/125	11100	5	010	3
100	18/125	101	3	110	3
101	12/125	11101	5	011	3
110	12/125	11110	5	1110	4
111	8/125	11111	5	1111	4

From this table (or the path length lemma of the corresponding trees) we see that

$$L = E[\ell] = 3.272$$
$$L_H = E[\ell_H] = 2.944$$
$$L - L_H = 0.328$$

and we see that the original code is not optimal. The entropy is

$$H(\tfrac{27}{125}, \dots, \tfrac{8}{125}) = 2.9129$$

and $L_H - H = 0.0311$.

Problem 2

(a) Since a BSC is considered the ML decoder is equivalent to a minimum Hamming distance decoder. There can be many such encoders. In the table below all possibla received codewords are listed together with all possible least weight codewords that are possible in an optimal decoder. When specifying a specific decoder one has to decide which estimated codeword should be used, one exmample is the underlined codewords in the table.

000	<u>001</u>,010,100
001	<u>001</u>
010	<u>010</u>
011	001,<u>010</u>,111
100	<u>100</u>
101	001,<u>100</u>,111
110	010,100,<u>111</u>
111	<u>111</u>

(b) $3^4 = 81$.

(c) —

(d) —

Problem 3

(This problem is similar to Problem 11 in the course 2010)
Let Y = Number heads and X = Identity, T = True, F = Counterfait. Then we can set up

the following tables.

		$P(Y\|X=T)$	$P(Y\|X=F)$
	0	1/4	1/16
Y	1	2/4	6/16
	2	1/4	9/16

		X	
	$P(X;Y)$	T	F
	0	1/8	1/32
Y	1	2/8	6/32
	2	1/8	9/32

$P(Y)$
5/32
14/32
13/32

Hence,

$$\begin{aligned} H(Y|X) &= H(Y|X=T)P(X=T) + H(Y|X=F)P(X=F) \\ &= \tfrac{1}{2}\big(H(\tfrac{1}{4},\tfrac{2}{4},\tfrac{1}{4}) + H(\tfrac{1}{16},\tfrac{6}{16},\tfrac{9}{16})\big) \approx 1.37 \\ H(Y) &= H(\tfrac{5}{32},\tfrac{14}{32},\tfrac{13}{32}) \approx 1.47 \\ I(X;Y) &= H(Y) - H(Y|X) \approx 0.094 \end{aligned}$$

Problem 4

The state transition matrix is

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

The steady state distrribution $\boldsymbol{w} = (w_0\ w_1\ w_2)$ can be found by solving the equation system $\boldsymbol{w}P = \boldsymbol{w}$ together with $\sum_i w_i = 1$. The first equation is equaivalent to $\boldsymbol{w}(P-I) = \boldsymbol{0}$, but here the matrix $P-I$ does not have full rank and is not invertible. Exchange one column (e.g. the last one) with the all-one vector to get the matrix

$$Z = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & 1 \\ \frac{1}{3} & -1 & 1 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$$

which is invertible. Then solve the equation system $\boldsymbol{w}Z = (0\ 0\ 1)$ with

$$\boldsymbol{w} = (0\ 0\ 1)Z^{-1} = (\tfrac{9}{22}\ \tfrac{3}{22}\ \tfrac{10}{22})$$

The entropy rate becomes

$$\begin{aligned} H_\infty(X) &= H(X|s_0)w_0 + H(X|s_1)w_1 + H(X|s_2)w_2 \\ &= H(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3})\tfrac{9}{22} + H(\tfrac{1}{3},\tfrac{2}{3})\tfrac{3}{22} + H(\tfrac{1}{2},\tfrac{1}{2})\tfrac{10}{22} \approx 1.2282 \end{aligned}$$

Problem 5

(a) Let $p = P(X=0)$. From the figure, the distribution for Y can be derived as

$$\begin{aligned} P(Y=0) &= p(1-\alpha) + (1-p)\beta \\ P(Y=1) &= p\alpha + (1-p)(1-\beta) \end{aligned}$$

Hence $H(Y) = h(P(Y=0)) = h(p(1-\alpha-\beta)+\beta)$. The conditioned entropy is

$$H(Y|X) = ph(\alpha) + (1-p)h(\beta)$$

Combining with $I(X;Y) = H(Y) - H(Y|X)$ we get the result.

(b) Setting the derivative wequal to zero yields

$$\frac{\partial}{\partial p} I(X;Y) = (1-\alpha-\beta)\log\frac{1-\big(p(1-\alpha-\beta)+\beta\big)}{p(1-\alpha-\beta)+\beta} - h(\alpha) + h(\beta) = 0$$
$$\frac{1-\big(p(1-\alpha-\beta)+\beta\big)}{p(1-\alpha-\beta)+\beta} = 2^A$$
$$p(1-\alpha-\beta)\big(1+2^A\big) = 1-\beta\big(1+2^A\big)$$
$$p = \frac{1}{1-\alpha-\beta}\Big(\frac{1}{1+2^A}-\beta\Big)$$

where $A = \frac{h(\alpha)-h(\beta)}{1-\alpha-\beta}$.

(c) Inserting δ in the place of α and β gives that $A = 0$ and

$$p = \frac{1}{2}$$
$$I(X;Y) = 1 - h(\delta)$$