Estimation Theory

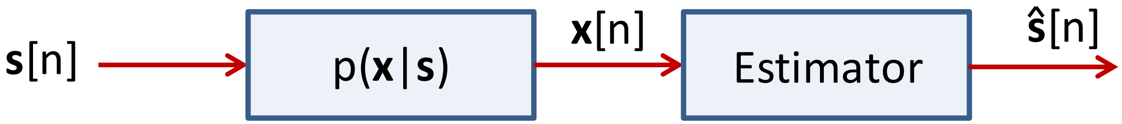
Fredrik Rusek

Chapter 13

Kalman Filters

Chapter 13 – Kalman Filters

Consider the estimation of a parameter –sequence s[n] from an observed sequence x[n]



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Chapter 13 – Kalman Filters

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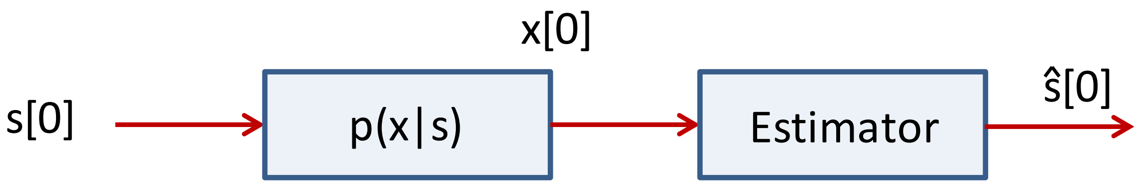

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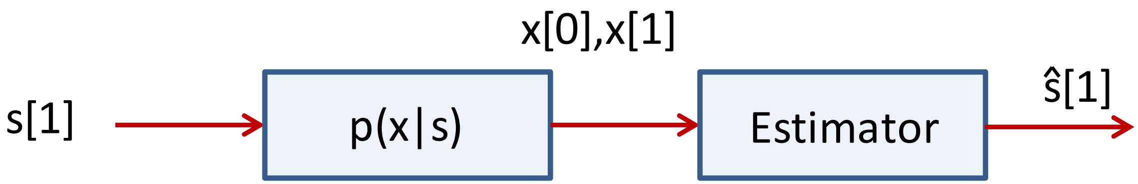

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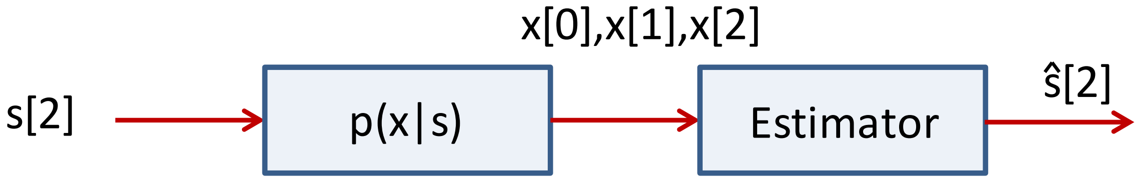

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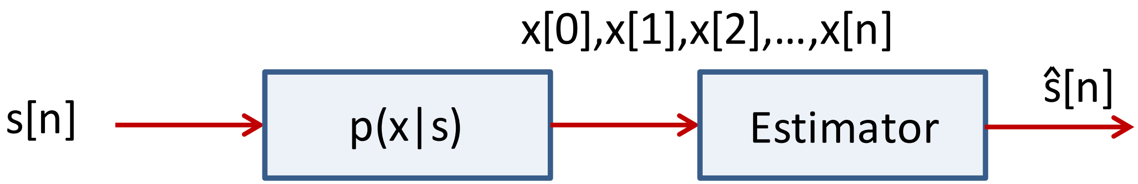

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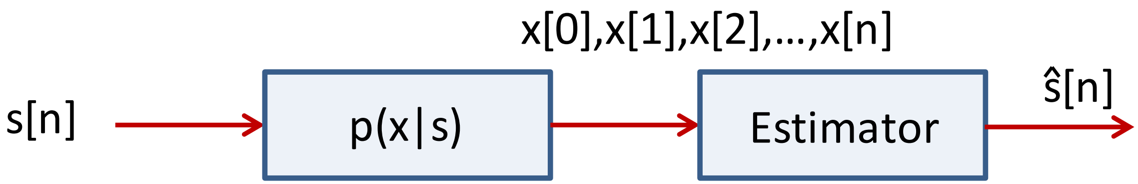

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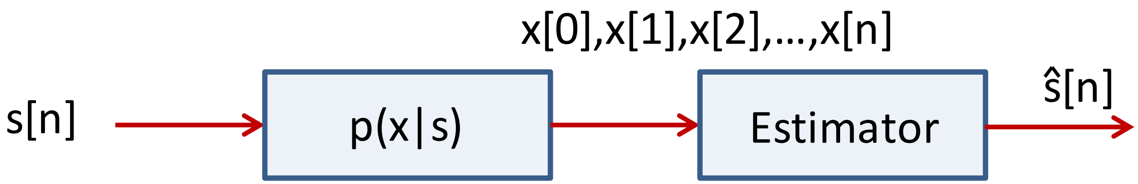

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*A linear combination of **x**[n]:* $s[n]=a_0x[n]+a_1x[n-1]+a_2x[n-2]+\ldots.$

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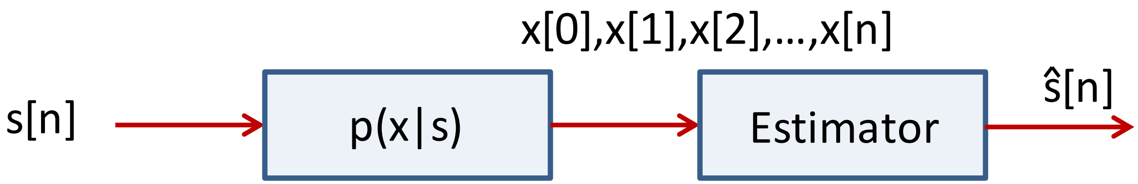

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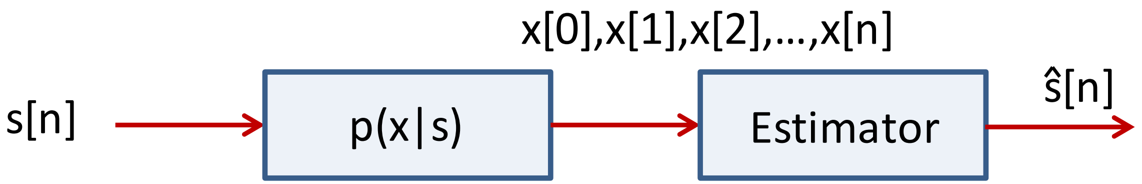

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$\mathbf{a}=R_{s[n]\mathbf{x}}R_{\mathbf{xx}}^{-1}$ where $\mathbf{x}=x[0],\ldots,x[n]$

Chapter 13 – Kalman Filters

Consider the estimation of a parameter –sequence s[n] from an observed sequence x[n]

If s[n] and x[n] are ***not*** jointly Gaussian but we keep a linear estimator

1. Is it optimal?
2. What it the form of the weightening coefficients?

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Chapter 13 – Kalman Filters

Consider the estimation of a parameter –sequence s[n] from an observed sequence x[n]

If s[n] and x[n] are ***not*** jointly Gaussian but we keep a linear estimator

1. Is it optimal? **NO**
2. What it the form of the weightening coefficients? $\mathbf{a}=R_{s[n]\mathbf{x}}R_{\mathbf{xx}}^{-1}$

If s[n] and x[n] are jointly Gaussian, what form does the estimator have at time n ?

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Chapter 13 – Kalman Filters

Consider the estimation of a parameter –sequence s[n] from an observed sequence x[n]

Summary:

For linear estimators $s[n]=a_0x[n]+a_1x[n-1]+a_2x[n-2]+....$

the optimal taps are found according to

$$\mathbf{a}=R_{s[n]\mathbf{x}}R_{\mathbf{xx}}^{-1} \qquad \text{where } \mathbf{x}=x[0],\dots,x[n]$$

This is the Wiener filter discussed last lecture

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Kalman Filters can gain *<u>NOTHING</u>* over Wiener Filters in performance

Kalman filters is ***only*** clever "book-keeping" to reduce complexity and to avoid matrix inversion

Chapter 13 – Kalman Filters

Assume a linear dynamical state model for the signal

A linear dynamical model can includes, e.g., an **AR(p)** model

$$s[n] = -\sum_{k=1}^{p} a[k]s[n-k] + u[n]$$

Transition model
Not **a** from earlier
slides

noise a.k.a. innovations
Makes signal to be
estimated random

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In order to statistically characterize s[n], we need s[n-1],…,s[n-p]

These variables are termed the **state** of the process

$$\mathbf{s}[n-1] = \begin{bmatrix} s[n-p] \\ s[n-p+1] \\ \vdots \\ s[n-2] \\ s[n-1] \end{bmatrix}$$

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At the next time, the state becomes

$$\mathbf{s}[n] = \begin{bmatrix} s[n-p+1] \\ s[n-p+2] \\ \vdots \\ s[n-1] \\ s[n] \end{bmatrix}$$

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$$\begin{bmatrix} s[n-p+1] \\ s[n-p+2] \\ \vdots \\ s[n-1] \\ s[n] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a[p] & -a[p-1] & -a[p-2] & \cdots & -a[1] \end{bmatrix} \begin{bmatrix} s[n-p] \\ s[n-p+1] \\ \vdots \\ s[n-2] \\ s[n-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u[n]$$

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We can also have multiple innovations

Chapter 13 – Kalman Filters

Assume a linear dynamical state model for the signal

AR(p) models are one example, but in general we consider systems of the form

$$\boxed{\mathbf{s}[n] = \mathbf{A}\mathbf{s}[n-1] + \mathbf{B}\mathbf{u}[n]}$$

There can also be input signals that affect the state, but this is not used in the book.

Used in the Extended Kalman filter (treated in the book)

Chapter 13 – Kalman Filters

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The book makes a big deal out of the starting conditions:

- **An AR(p) model is assumed to start at -∞, but in the book it starts at n=0**
- **Then it is technically a Gauss-Markov process, and a model for s[-1] is required**
- **However, the initial condition fades out fairly quickly since the process is ergodic**

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We observe the process through noisy observations

$$\boxed{\mathbf{x}[n] = \mathbf{s}[n] + \mathbf{w}[n]}$$

And we assume that ***all*** means are zero

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The task is to estimate the state **s**[n] given **x**[n],**x**[n-1],….
with a linear estimator

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Chapter 13 – Kalman Filters

Since we assume a linear estimator, does it play a role if things are Gaussian or not ?

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Chapter 13 – Kalman Filters

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In this lecture we deal with the scalar case

$$s[n] = as[n-1] + u[n] \qquad n \geq 0 \qquad\qquad E(u^2[n]) = \sigma_u^2$$

$$x[n] = s[n] + w[n] \qquad\qquad E(w^2[n]) = \sigma_n^2$$

Chapter 13 – Kalman Filters

Let $\hat{s}[n|n]$ denote the estimate of s[n] computed based on the obsrevations at time n
$\hat{s}[n|n-1]$ be the estimate of s[n] computed at time n-1 (prediction) etc etc

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Chapter 13 – Kalman Filters

Define $\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$

so that
$$
\begin{aligned}
x[n] &= \tilde{x}[n] + \hat{x}[n|n-1] \\
&= \tilde{x}[n] + \sum_{k=0}^{n-1} a_k x[k]
\end{aligned}
$$

Note. Confusion in the book:
a_k are the estimation coefficients

But a is the parameter defining the AR(1) process

Chapter 13 – Kalman Filters

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Chapter 13 – Kalman Filters

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But if we are given $\mathbf{X}[n-1] = \{x[0], x[1], \ldots, x[n-1]\}$ and $\tilde{x}[n]$
we can compute $\mathbf{X}[n]$

So, $\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n])}$

Recall a slide from Lecture 7

Additive property

Independent observations $\mathbf{x}_1,\mathbf{x}_2$
Estimate $\boldsymbol{\theta}$
Assume that $\mathbf{x}_1,\mathbf{x}_2$, $\boldsymbol{\theta}$ are jointly Gaussian

$$\hat{\boldsymbol{\theta}} = E(\boldsymbol{\theta}|\mathbf{x}) = E(\boldsymbol{\theta}) + \mathbf{C}_{\theta x}\mathbf{C}_{xx}^{-1}(\mathbf{x} - E(\mathbf{x}))$$

$$\begin{aligned}
\hat{\boldsymbol{\theta}} &= E(\boldsymbol{\theta}) + \begin{bmatrix} \mathbf{C}_{\theta x_1} & \mathbf{C}_{\theta x_2} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{x_1 x_1}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{x_2 x_2}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 - E(\mathbf{x}_1) \\ \mathbf{x}_2 - E(\mathbf{x}_2) \end{bmatrix} \\
&= E(\boldsymbol{\theta}) + \boxed{\mathbf{C}_{\theta x_1}\mathbf{C}_{x_1 x_1}^{-1}(\mathbf{x}_1 - E(\mathbf{x}_1))} + \boxed{\mathbf{C}_{\theta x_2}\mathbf{C}_{x_2 x_2}^{-1}(\mathbf{x}_2 - E(\mathbf{x}_2))}.
\end{aligned}$$

MMSE estimate can be updated sequentially !!!

Chapter 13 – Kalman Filters

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$\tilde{x}[n]$ being the error of an MMSE estimate, is orthogonal to $\mathbf{X}[n-1]$ according to the ***orthogonality principle.*** Since it is Gaussian, it is also independent from $\mathbf{X}[n-1]$

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By definition $E(s[n]|\mathbf{X}[n-1]) = \hat{s}[n|n-1]$

Chapter 13 – Kalman Filters

Define $\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$

Let us now derive $\hat{s}[n|n-1] = E(s[n]|\mathbf{X}[n-1])$

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$$\begin{aligned}\hat{s}[n|n-1] &= E(s[n]|\mathbf{X}[n-1]) \\ &= E(as[n-1] + u[n]|\mathbf{X}[n-1])\end{aligned}$$

s[n] = as[n-1] + u[n]

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u[n] is uncorrelated with **X**[n-1]

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$$\begin{aligned} \hat{s}[n|n-1] &= E(s[n]|\mathbf{X}[n-1]) \\ &= E(as[n-1]+u[n]|\mathbf{X}[n-1]) \\ &= aE(s[n-1]|\mathbf{X}[n-1]) \\ &= a\hat{s}[n-1|n-1] \end{aligned}$$

By definition

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1],\tilde{x}[n]) = \hat{s}[n|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Define $\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$

Let us now derive

$$\begin{aligned}\hat{s}[n|n-1] &= E(s[n]|\mathbf{X}[n-1]) \\ &= E(as[n-1]+u[n]|\mathbf{X}[n-1]) \\ &= aE(s[n-1]|\mathbf{X}[n-1]) \\ &= a\hat{s}[n-1|n-1]\end{aligned}$$

Thus

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Summary so far: We seek to find $\hat{s}[n|n]$. This equals $a\hat{s}[n-1|n-1]$, which we have,

plus the term $E(s[n]|\tilde{x}[n])$. **Compute it and we are done!**

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore
$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \end{aligned}$$

Kalman Gain

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore

$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{E(s[n]\tilde{x}[n])}{E(\tilde{x}^2[n])}}$$

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore

$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{E(s[n]\tilde{x}[n])}{E(\tilde{x}^2[n])}}$$

However, $\hat{x}[n|n-1] = \hat{s}[n|n-1]$

x[n] = s[n] + w[n]

w[n] is independent from all other variables

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore

$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \\ &= K[n](x[n] - \hat{s}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{E(s[n]\tilde{x}[n])}{E(\tilde{x}^2[n])}}$$

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore
$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \\ &= K[n](x[n] - \hat{s}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{E(s[n]\tilde{x}[n])}{E(\tilde{x}^2[n])}}$$

We must now compute the **Kalman gain** K[n]. Start by the nominator

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore

$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \\ &= K[n](x[n] - \hat{s}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{E(s[n]\tilde{x}[n])}{E(\tilde{x}^2[n])}}$$

We must now compute the **Kalman gain** K[n]. Start by the nominator

$$E(s[n]\tilde{x}[n]) = E[s[n](x[n] - \hat{x}[n|n-1])]$$

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore

$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n]-\hat{x}[n|n-1]) \\ &= K[n](x[n]-\hat{s}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{E(s[n]\tilde{x}[n])}{E(\tilde{x}^2[n])}}$$

We must now compute the **Kalman gain** K[n]. Start by the nominator

$$E(s[n]\tilde{x}[n]) = E[s[n]\underbrace{(x[n]-\hat{x}[n|n-1])}_{\text{Error of MMSE estimate of } x[n] \text{ given } \mathbf{X}[n-1]}]$$

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore

$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n]-\hat{x}[n|n-1]) \\ &= K[n](x[n]-\hat{s}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{E(s[n]\tilde{x}[n])}{E(\tilde{x}^2[n])}}$$

We must now compute the **Kalman gain** K[n]. Start by the nominator

$$E(s[n]\tilde{x}[n]) = E[s[n]\underbrace{(x[n]-\hat{x}[n|n-1])}_{}]$$

Error of MMSE estimate of x[n] given **X**[n-1]
*Orthogonal to **X**[n-1]*

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore

$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \\ &= K[n](x[n] - \hat{s}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{E(s[n]\tilde{x}[n])}{E(\tilde{x}^2[n])}}$$

We must now compute the **Kalman gain** K[n]. Start by the nominator

$$E(s[n]\tilde{x}[n]) = E[s[n]\underbrace{(x[n] - \hat{x}[n|n-1])}]$$

Error of MMSE estimate of x[n] given **X**[n-1]

*Orthogonal to **X**[n-1]*

Therefore orthogonal to $\hat{s}[n|n-1]$ *which is a linear combination of **X**[n-1]*

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore

$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \\ &= K[n](x[n] - \hat{s}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{E(s[n]\tilde{x}[n])}{E(\tilde{x}^2[n])}}$$

We must now compute the **Kalman gain** K[n]. Start by the nominator

$$E(s[n]\tilde{x}[n]) = E[s[n](x[n] - \hat{x}[n|n-1])] = E[(s[n] - \hat{s}[n|n-1])(x[n] - \hat{s}[n|n-1])]$$

We can add this term freely

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore
$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \\ &= K[n](x[n] - \hat{s}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{E(s[n]\tilde{x}[n])}{E(\tilde{x}^2[n])}}$$

We must now compute the **Kalman gain** K[n]. Start by the nominator

$$\begin{aligned} E(s[n]\tilde{x}[n]) &= E[s[n](x[n] - \hat{x}[n|n-1])] = E[(s[n] - \hat{s}[n|n-1])(x[n] - \hat{s}[n|n-1])] \\ &= E[(s[n] - \hat{s}[n|n-1])(s[n] - \hat{s}[n|n-1])] \end{aligned}$$

w[n] independent from all other variables

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore
$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n]-\hat{x}[n|n-1]) \\ &= K[n](x[n]-\hat{s}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{E(s[n]\tilde{x}[n])}{E(\tilde{x}^2[n])}}$$

We must now compute the **Kalman gain** K[n]. Start by the nominator

$$E(s[n]\tilde{x}[n]) = E[s[n](x[n]-\hat{x}[n|n-1])] = E[(s[n]-\hat{s}[n|n-1])(x[n]-\hat{s}[n|n-1])]$$

$$= E[(s[n]-\hat{s}[n|n-1])(s[n]-\hat{s}[n|n-1])] = M[n|n-1]$$

Definition: 1-step prediction error

$$\tilde{x}[n] = x[n]-\hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1],\tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

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Computation of $E(s[n]|\tilde{x}[n])$

Eventually, we must also compute M

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore
$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \\ &= K[n](x[n] - \hat{s}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{M[n|n-1]}{E(\tilde{x}^2[n])}}$$

We must now compute the **Kalman gain** K[n]. Start by the nominator

$$E(s[n]\tilde{x}[n]) = E[s[n](x[n] - \hat{x}[n|n-1])] = E[(s[n] - \hat{s}[n|n-1])(x[n] - \hat{s}[n|n-1])]$$

$$= E[(s[n] - \hat{s}[n|n-1])(s[n] - \hat{s}[n|n-1])] = M[n|n-1]$$

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore

$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \\ &= K[n](x[n] - \hat{s}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{M[n|n-1]}{E(\tilde{x}^2[n])}}$$

We must now compute the **Kalman gain** K[n]. Now take the denominator

$$E(\tilde{x}^2[n]) = E[(x[n] - \hat{s}[n|n-1])^2]$$

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore

$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \\ &= K[n](x[n] - \hat{s}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{M[n|n-1]}{E(\tilde{x}^2[n])}}$$

We must now compute the **Kalman gain** K[n]. Now take the denominator

$$E(\tilde{x}^2[n]) = E[(x[n] - \hat{s}[n|n-1])^2] = E[(s[n] - \hat{s}[n|n-1] + w[n])^2]$$

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore
$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \\ &= K[n](x[n] - \hat{s}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{M[n|n-1]}{E(\tilde{x}^2[n])}}$$

We must now compute the **Kalman gain** K[n]. Now take the denominator

$$E(\tilde{x}^2[n]) = E[(x[n] - \hat{s}[n|n-1])^2] = E[(s[n] - \hat{s}[n|n-1] + w[n])^2]$$

$$= \sigma_n^2 + E[(s[n] - \hat{s}[n|n-1])^2]$$

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore

$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \\ &= K[n](x[n] - \hat{s}[n|n-1]) \end{aligned}$$

$$\boxed{K[n] = \frac{M[n|n-1]}{E(\tilde{x}^2[n])}}$$

We must now compute the **Kalman gain** K[n]. Now take the denominator

$$E(\tilde{x}^2[n]) = E[(x[n] - \hat{s}[n|n-1])^2] = E[(s[n] - \hat{s}[n|n-1] + w[n])^2]$$

$$= \sigma_n^2 + E[(s[n] - \hat{s}[n|n-1])^2] = \sigma_n^2 + M[n|n-1]$$

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

Chapter 13 – Kalman Filters

Computation of $E(s[n]|\tilde{x}[n])$

$E(s[n]|\tilde{x}[n])$ is an MMSE estimate of s[n] given $\tilde{x}[n]$

Therefore
$$\begin{aligned} E(s[n]|\tilde{x}[n]) &= K[n]\tilde{x}[n] \\ &= K[n](x[n] - \hat{x}[n|n-1]) \\ &= K[n](x[n] - \hat{s}[n|n-1]) \end{aligned}$$

We must now compute the **Kalman gain** K[n]. Now take the denominator

$$E(\tilde{x}^2[n]) = E[(x[n] - \hat{s}[n|n-1])^2] = E[(s[n] - \hat{s}[n|n-1] + w[n])^2]$$

$$= \sigma_n^2 + E[(s[n] - \hat{s}[n|n-1])^2] = \sigma_n^2 + M[n|n-1]$$

$$\boxed{K[n] = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}}$$

$$\tilde{x}[n] = x[n] - \hat{x}[n|n-1]$$

$$\boxed{\hat{s}[n|n] = E(s[n]|\mathbf{X}[n-1], \tilde{x}[n]) = a\hat{s}[n-1|n-1] + E(s[n]|\tilde{x}[n])}$$

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Summary so far

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Chapter 13 – Kalman Filters

Summary so far

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

This is the estimate at the previous time slot

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Summary so far

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

This parameter is from the model and must be known

Chapter 13 – Kalman Filters

Summary so far

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = \underbrace{a\hat{s}[n-1|n-1]} + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

This is the predicted estimate at time n-1, $\hat{s}[n|n-1]$

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Summary so far

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Observation at time n

Chapter 13 – Kalman Filters

Summary so far

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Noise density

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Summary so far

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Error made when predicting s[n] from **X**[n-1]

Remains to be computed

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Summary so far

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Error made when predicting s[n] from **X**[n-1]

Remains to be computed

$M[n|n]$ **is also of interest and should be computed as well**

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of $M[n|n-1]$

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of $M[n|n-1] = E[(s[n] - \hat{s}[n|n-1])^2]$ By definition

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of $M[n|n-1] = E[(s[n] - \hat{s}[n|n-1])^2]$

s[n] = s[n-1] + u[n] $\quad = E[(as[n-1] + u[n] - \hat{s}[n|n-1])^2]$

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of

$$\begin{aligned} M[n|n-1] &= E[(s[n] - \hat{s}[n|n-1])^2] \\ &= E[(as[n-1] + u[n] - \hat{s}[n|n-1])^2] \\ &= E[(a(s[n-1] - \hat{s}[n-1|n-1]) + u[n])^2] \end{aligned}$$

$$\boxed{\hat{s}[n|n-1] = a\hat{s}[n-1|n-1]}$$

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of $M[n|n-1] = E[(s[n] - \hat{s}[n|n-1])^2]$

$$= E[(as[n-1] + u[n] - \hat{s}[n|n-1])^2]$$

$$= E[(a(s[n-1] - \hat{s}[n-1|n-1]) + u[n])^2]$$

u[n] is uncorrelated with all other variables $= E[(a(s[n-1] - \hat{s}[n-1|n-1]))^2] + \sigma_u^2$

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of
$$\begin{aligned} M[n|n-1] &= E[(s[n] - \hat{s}[n|n-1])^2] \\ &= E[(as[n-1] + u[n] - \hat{s}[n|n-1])^2] \\ &= E[(a(s[n-1] - \hat{s}[n-1|n-1]) + u[n])^2] \\ &= E[(a(s[n-1] - \hat{s}[n-1|n-1]))^2] + \sigma_u^2 \\ &= a^2 M[n-1|n-1] + \sigma_u^2. \end{aligned}$$

By definition

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of
$$\begin{aligned}
M[n|n-1] &= E[(s[n] - \hat{s}[n|n-1])^2] \\
&= E[(as[n-1] + u[n] - \hat{s}[n|n-1])^2] \\
&= E[(a(s[n-1] - \hat{s}[n-1|n-1]) + u[n])^2] \\
&= E[(a(s[n-1] - \hat{s}[n-1|n-1]))^2] + \sigma_u^2 \\
&= a^2 M[n-1|n-1] + \sigma_u^2.
\end{aligned}$$

We can now get M[n|n-1] from M[n-1|n-1]

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of

$$\begin{aligned} M[n|n-1] &= E[(s[n] - \hat{s}[n|n-1])^2] \\ &= E[(as[n-1] + u[n] - \hat{s}[n|n-1])^2] \\ &= E[(a(s[n-1] - \hat{s}[n-1|n-1]) + u[n])^2] \\ &= E[(a(s[n-1] - \hat{s}[n-1|n-1]))^2] + \sigma_u^2 \\ &= a^2 M[n-1|n-1] + \sigma_u^2. \end{aligned}$$

We can now get M[n|n-1] from M[n-1|n-1]

In next time step, we need M[n+1|n] from M[n|n]

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of
$$\begin{aligned} M[n|n-1] &= E[(s[n] - \hat{s}[n|n-1])^2] \\ &= E[(as[n-1] + u[n] - \hat{s}[n|n-1])^2] \\ &= E[(a(s[n-1] - \hat{s}[n-1|n-1]) + u[n])^2] \\ &= E[(a(s[n-1] - \hat{s}[n-1|n-1]))^2] + \sigma_u^2 \\ &= a^2 M[n-1|n-1] + \sigma_u^2. \end{aligned}$$

We can now get M[n|n-1] from M[n-1|n-1]

In next time step, we need M[n+1|n] from M[n|n]

Thus, we need M[n|n] from M[n|n-1]

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of $M[n|n]$

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of $M[n|n] = E[(s[n] - \hat{s}[n|n])^2]$ By definition

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of

$$\begin{aligned} M[n|n] &= E[(s[n] - \hat{s}[n|n])^2] \\ &= E[(s[n] - \hat{s}[n|n-1] - K[n](x[n] - \hat{s}[n|n-1]))^2] \end{aligned}$$

Since the boxed equation above can be written as

$$\hat{s}[n|n] = \hat{s}[n|n-1] + K[n](x[n] - \hat{s}[n|n-1])$$

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of
$$\begin{aligned} M[n|n] &= E[(s[n] - \hat{s}[n|n])^2] \\ &= E[(s[n] - \hat{s}[n|n-1] - K[n](x[n] - \hat{s}[n|n-1]))^2] \\ &= E[(s[n] - \hat{s}[n|n-1])^2] - 2K[n]E[(s[n] - \hat{s}[n|n-1])(x[n] - \hat{s}[n|n-1])] \\ &\quad + K^2[n]E[(x[n] - \hat{s}[n|n-1])^2]. \end{aligned}$$

Expand the power of 2

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of

$$\begin{aligned} M[n|n] &= E[(s[n]-\hat{s}[n|n])^2] \\ &= E[(s[n]-\hat{s}[n|n-1]-K[n](x[n]-\hat{s}[n|n-1]))^2] \\ &= \boxed{E[(s[n]-\hat{s}[n|n-1])^2]} - 2K[n]E[(s[n]-\hat{s}[n|n-1])(x[n]-\hat{s}[n|n-1])] \\ &\quad + K^2[n]E[(x[n]-\hat{s}[n|n-1])^2]. \end{aligned}$$

By definition

$$= M[n|n-1] -$$

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of
$$\begin{aligned} M[n|n] &= E[(s[n] - \hat{s}[n|n])^2] \\ &= E[(s[n] - \hat{s}[n|n-1] - K[n](x[n] - \hat{s}[n|n-1]))^2] \\ &= E[(s[n] - \hat{s}[n|n-1])^2] - \boxed{2K[n]E[(s[n] - \hat{s}[n|n-1])(x[n] - \hat{s}[n|n-1])]} \\ &\quad + K^2[n]E[(x[n] - \hat{s}[n|n-1])^2]. \\ &= M[n|n-1] - 2K^2[n](M[n|n-1] + \sigma_n^2) + \end{aligned}$$

since

$$K[n] = \frac{E[(s[n] - \hat{s}[n|n-1])(x[n] - \hat{s}[n|n-1])]}{E[(s[n] - \hat{s}[n|n-1] + w[n])^2]} = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of

$$\begin{aligned} M[n|n] &= E[(s[n] - \hat{s}[n|n])^2] \\ &= E[(s[n] - \hat{s}[n|n-1] - K[n](x[n] - \hat{s}[n|n-1]))^2] \\ &= E[(s[n] - \hat{s}[n|n-1])^2] - 2K[n]E[(s[n] - \hat{s}[n|n-1])(x[n] - \hat{s}[n|n-1])] \\ &\quad \boxed{+ K^2[n]E[(x[n] - \hat{s}[n|n-1])^2]}. \\ &= M[n|n-1] - 2K^2[n](M[n|n-1] + \sigma_n^2) + K[n]M[n|n-1] \end{aligned}$$

since

$$K[n] = \frac{E[(s[n] - \hat{s}[n|n-1])(x[n] - \hat{s}[n|n-1])]}{E[(s[n] - \hat{s}[n|n-1] + w[n])^2]} = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of
$$\begin{aligned} M[n|n] &= E[(s[n] - \hat{s}[n|n])^2] \\ &= E[(s[n] - \hat{s}[n|n-1] - K[n](x[n] - \hat{s}[n|n-1]))^2] \\ &= E[(s[n] - \hat{s}[n|n-1])^2] - 2K[n]E[(s[n] - \hat{s}[n|n-1])(x[n] - \hat{s}[n|n-1])] \\ &\quad + K^2[n]E[(x[n] - \hat{s}[n|n-1])^2]. \\ &= M[n|n-1] - 2K^2[n](M[n|n-1] + \sigma_n^2) + K[n]M[n|n-1] \\ &= M[n|n-1] - \boxed{2K[n]M[n|n-1]} + K[n]M[n|n-1] \end{aligned}$$

since

$$\sigma_n^2 + M[n|n-1] = \frac{M[n|n-1]}{K[n]}$$

Chapter 13 – Kalman Filters

The estimate of s[n] computed for all data x[0],…,x[n] is

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Computation of

$$\begin{aligned}
M[n|n] &= E[(s[n] - \hat{s}[n|n])^2] \\
&= E[(s[n] - \hat{s}[n|n-1] - K[n](x[n] - \hat{s}[n|n-1]))^2] \\
&= E[(s[n] - \hat{s}[n|n-1])^2] - 2K[n]E[(s[n] - \hat{s}[n|n-1])(x[n] - \hat{s}[n|n-1])] \\
&\quad + K^2[n]E[(x[n] - \hat{s}[n|n-1])^2]. \\
&= M[n|n-1] - 2K^2[n](M[n|n-1] + \sigma_n^2) + K[n]M[n|n-1] \\
&= M[n|n-1] - 2K[n]M[n|n-1] + K[n]M[n|n-1] \\
&= (1 - K[n])M[n|n-1].
\end{aligned}$$

Chapter 13 – Kalman Filters

$$\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])$$

Summary

At time n-1, we computed

- $\hat{s}[n-1|n-1]$
- $M[n-1|n-1]$

At time n, we receive x[n]. We need

- $\hat{s}[n|n]$
- $M[n|n]$

Chapter 13 – Kalman Filters

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Summary

At time n-1, we computed

- $\hat{s}[n-1|n-1]$
- $M[n-1|n-1]$

At time n, we receive x[n]. We need

- $\hat{s}[n|n]$
- $M[n|n]$

We do

Prediction:

$$\hat{s}[n|n-1] = a\hat{s}[n-1|n-1]$$

Minimum Prediction MSE:

$$M[n|n-1] = a^2 M[n-1|n-1] + \sigma_u^2$$

Chapter 13 – Kalman Filters

$$\boxed{\hat{s}[n|n] = a\hat{s}[n-1|n-1] + \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}(x[n] - \hat{s}[n|n-1])}$$

Summary

At time n-1, we computed

- $\hat{s}[n-1|n-1]$
- $M[n-1|n-1]$

At time n, we receive x[n]. We need

- $\hat{s}[n|n]$
- $M[n|n]$

We do

Prediction:

$$\hat{s}[n|n-1] = a\hat{s}[n-1|n-1]$$

Minimum Prediction MSE:

$$M[n|n-1] = a^2 M[n-1|n-1] + \sigma_u^2$$

Kalman Gain: $K[n] = \dfrac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$

Correction:

$$\boxed{\hat{s}[n|n] = \hat{s}[n|n-1] + K[n](x[n] - \hat{s}[n|n-1])}$$

Minimum MSE:

$$M[n|n] = (1 - K[n])M[n|n-1]$$

Chapter 13 – Kalman Filters

Last lecture we discussed the Wiener filter and discussed that as time n grows, the **Wiener filter** converged to a stationary solution

However, this solution required us to solve the **Wiener-Hopf equations** which are rather tough (requires spectral factorization)

To do **Wiener prediction**, we need to solve the **Yule-Walker equations**. Rather tough as well

With the Kalman model we can obtain the solutions much easier.

Iterating the Kalman equation will force the Kalman gain to a stationary value (for WSS signals)

$$K[n] \to K[\infty]$$

Chapter 13 – Kalman Filters

The Kalman estimator

$$\hat{s}[n|n] = \hat{s}[n|n-1] + K[n](x[n] - \hat{s}[n|n-1])$$

becomes asymptotically

$$\hat{s}[n|n] = \hat{s}[n|n-1] + K[\infty](x[n] - \hat{s}[n|n-1])$$

Chapter 13 – Kalman Filters

The Kalman estimator

$$\hat{s}[n|n] = \hat{s}[n|n-1] + K[n](x[n] - \hat{s}[n|n-1])$$

becomes asymptotically

$$\begin{aligned}\hat{s}[n|n] &= \hat{s}[n|n-1] + K[\infty](x[n] - \hat{s}[n|n-1]) \\ &= a\hat{s}[n-1|n-1] + K[\infty](x[n] - a\hat{s}[n-1|n-1])\end{aligned}$$

Chapter 13 – Kalman Filters

The Kalman estimator

$$\hat{s}[n|n] = \hat{s}[n|n-1] + K[n](x[n] - \hat{s}[n|n-1])$$

becomes asymptotically

$$\begin{aligned}
\hat{s}[n|n] &= \hat{s}[n|n-1] + K[\infty](x[n] - \hat{s}[n|n-1]) \\
&= a\hat{s}[n-1|n-1] + K[\infty](x[n] - a\hat{s}[n-1|n-1]) \\
&= a(1 - K[\infty])\hat{s}[n-1|n-1] + K[\infty]x[n]
\end{aligned}$$

Chapter 13 – Kalman Filters

The Kalman estimator

$$\hat{s}[n|n] = \hat{s}[n|n-1] + K[n](x[n] - \hat{s}[n|n-1])$$

becomes asymptotically

$$\begin{aligned}\hat{s}[n|n] &= \hat{s}[n|n-1] + K[\infty](x[n] - \hat{s}[n|n-1]) \\ &= a\hat{s}[n-1|n-1] + K[\infty](x[n] - a\hat{s}[n-1|n-1]) \\ &= a(1-K[\infty])\hat{s}[n-1|n-1] + K[\infty]x[n]\end{aligned}$$

Thus,

$$\hat{s}[z] - a(1-K[\infty])\,\hat{s}[z]\; z^{-1} \;=\; K[\infty]x[z]$$

Delay operator

Chapter 13 – Kalman Filters

The Kalman estimator

$$\hat{s}[n|n] = \hat{s}[n|n-1] + K[n](x[n] - \hat{s}[n|n-1])$$

becomes asymptotically

$$\begin{aligned}\hat{s}[n|n] &= \hat{s}[n|n-1] + K[\infty](x[n] - \hat{s}[n|n-1]) \\ &= a\hat{s}[n-1|n-1] + K[\infty](x[n] - a\hat{s}[n-1|n-1]) \\ &= a(1-K[\infty])\hat{s}[n-1|n-1] + K[\infty]x[n]\end{aligned}$$

Thus,

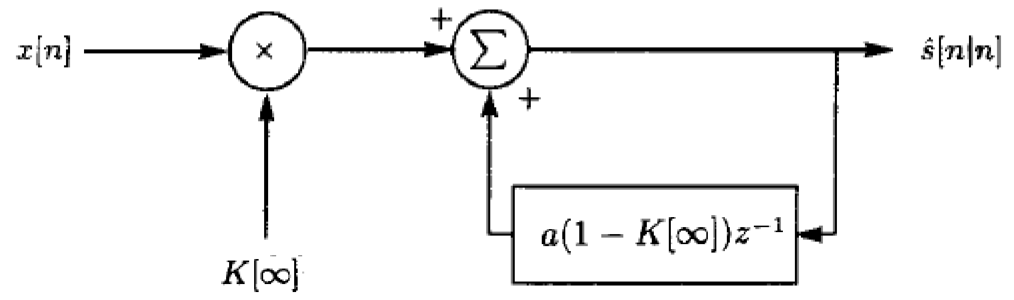
$$\hat{s}[z] - a(1-K[\infty])\,\hat{s}[z]\,z^{-1} = K[\infty]x[z]$$

Transfer function of recursive filter becomes

$$\boxed{\mathcal{H}_\infty(z) = \frac{K[\infty]}{1 - a(1-K[\infty])z^{-1}}}$$

Chapter 13 – Kalman Filters

IIR filter, one-pole for an AR(1) process



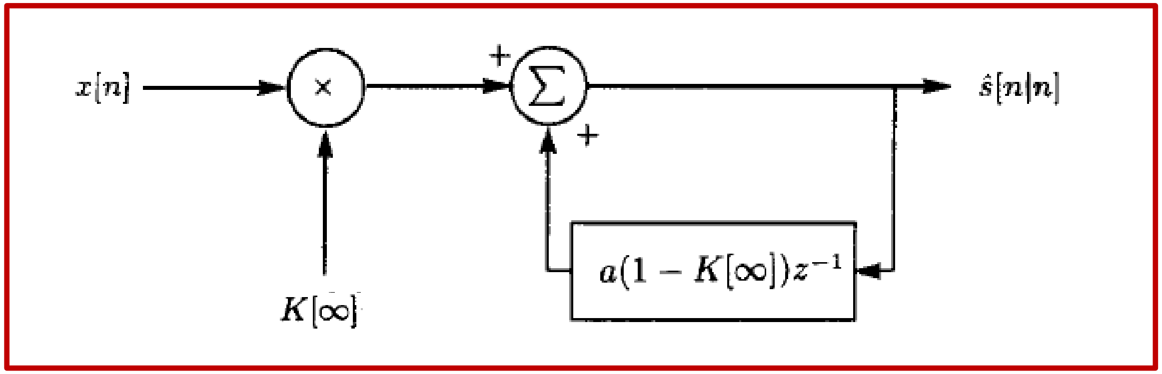
$$\mathcal{H}_\infty(z) = \frac{K[\infty]}{1 - a(1 - K[\infty])z^{-1}}$$

Chapter 13 – Kalman Filters

Solve for $K[\infty]$ $\qquad K[n] = \dfrac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$

Solve for $M_p[\infty]$ $\qquad M[n|n-1] \to M_p[\infty]$ definition

IIR filter, one-pole for an AR(1) process



$$\boxed{\mathcal{H}_\infty(z) = \frac{K[\infty]}{1 - a(1-K[\infty])z^{-1}}}$$

Chapter 13 – Kalman Filters

Solve for $K[\infty]$

$$K[n] = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$

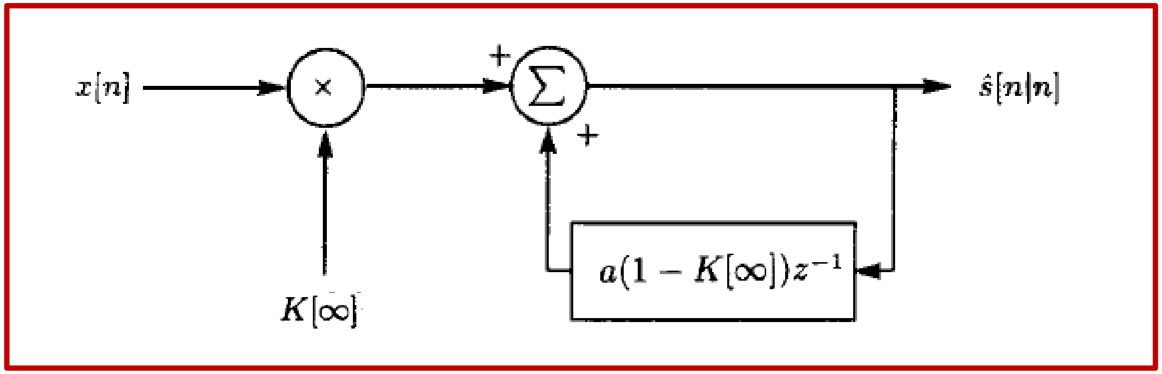
Solve for $M_p[\infty]$

$$M[n|n-1] \rightarrow M_p[\infty]$$

$$M[n|n-1] = a^2 M[n-1|n-1] + \sigma_u^2 \quad \rightarrow \quad a^2 M[\infty] + \sigma_u^2$$

Update formula from before

IIR filter, one-pole for an AR(1) process



$$\boxed{\mathcal{H}_\infty(z) = \frac{K[\infty]}{1 - a(1-K[\infty])z^{-1}}}$$

Chapter 13 – Kalman Filters

Solve for $K[\infty]$

$$K[n] = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$

Solve for $M_p[\infty]$

$$M[n|n-1] \to M_p[\infty]$$

$$M[n|n-1] = a^2 M[n-1|n-1] + \sigma_u^2 \quad \to \quad a^2 M[\infty] + \sigma_u^2$$

So:

$$\boxed{K[\infty] = \frac{a^2 M[\infty] + \sigma_u^2}{\sigma_n^2 + a^2 M[\infty] + \sigma_u^2}}$$

Chapter 13 – Kalman Filters

Solve for $K[\infty]$ $$K[n] = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$

Solve for $M_p[\infty]$ $$M[n|n-1] \to M_p[\infty]$$

$$M[n|n-1] = a^2 M[n-1|n-1] + \sigma_u^2 \quad \to \quad a^2 M[\infty] + \sigma_u^2$$

So:

$$\boxed{K[\infty] \quad = \quad \frac{a^2 M[\infty] + \sigma_u^2}{\sigma_n^2 + \ a^2 M[\infty] + \sigma_u^2}}$$

$$M[n|n] = (1 - K[n])M[n|n-1]$$

Chapter 13 – Kalman Filters

Solve for $K[\infty]$ $$K[n] = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$

Solve for $M_p[\infty]$ $$M[n|n-1] \to M_p[\infty]$$

$$M[n|n-1] = a^2 M[n-1|n-1] + \sigma_u^2 \quad \to \quad a^2 M[\infty] + \sigma_u^2$$

So:

$$\boxed{K[\infty] \quad = \quad \frac{a^2 M[\infty] + \sigma_u^2}{\sigma_n^2 + \ a^2 M[\infty] + \sigma_u^2}}$$

$$M[n|n] = (1 - K[n]) M[n|n-1]$$

$$\begin{aligned} M[\infty] &= \left(1 - \frac{M_p[\infty]}{\sigma^2 + M_p[\infty]}\right) M_p[\infty] \\ &= \frac{\sigma^2 M_p[\infty]}{M_p[\infty] + \sigma^2} \\ &= \frac{\sigma^2 (a^2 M[\infty] + \sigma_u^2)}{a^2 M[\infty] + \sigma_u^2 + \sigma^2} \end{aligned}$$

Chapter 13 – Kalman Filters

Solve for $K[\infty]$ $$K[n] = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$

Solve for $M_p[\infty]$ $$M[n|n-1] \rightarrow M_p[\infty]$$

$$M[n|n-1] = a^2 M[n-1|n-1] + \sigma_u^2 \quad \rightarrow \quad a^2 M[\infty] + \sigma_u^2$$

So:

$$\boxed{K[\infty] \quad = \quad \frac{a^2 M[\infty] + \sigma_u^2}{\sigma_n^2 + \ a^2 M[\infty] + \sigma_u^2}}$$

$$\boxed{M[\infty] \quad = \quad \frac{\sigma^2 (a^2 M[\infty] + \sigma_u^2)}{a^2 M[\infty] + \sigma_u^2 + \sigma^2}}$$

Fixed-point equation, 2nd order. Easy to solve.

Chapter 13 – Kalman Filters

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The same solution would also result if we solved

$\lim_{n \to \infty} \ 1 - R_{s[n]\mathbf{x}} R_{\mathbf{xx}}^{-1} R_{\mathbf{x}s[n]}$

This is the Schur complement, but is much Harder to evalaute

Chapter 13 – Kalman Filters

Solve for $K[\infty]$

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Solve for $M_p[\infty]$

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Which equation is the most important ?

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Chapter 13 – Kalman Filters

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With the first equation, one can do or implement something
With the second, one can understand something (the asymptotic Bmse)

Chapter 13 – Kalman Filters

Extended Kalman Filters: used for non-linear models

- **Not optimal**
- **Not always very accurate**
- **Hard to analyze asymptotic performance – not suitable for analytical work.**
- **Good solutions if you dont know what to do, but can model the system as a non-linear dynamical system**

- **Do not believe that extended Kalman filters have any optimality properties. They are just low-complex solutions to hard problems!**

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Mathematical model:

$$\begin{aligned} \mathbf{s}[n] &= \mathbf{a}(\mathbf{s}[n-1]) + \mathbf{B}\mathbf{u}[n] \\ \mathbf{x}[n] &= \mathbf{h}(\mathbf{s}[n]) + \mathbf{w}[n] \end{aligned}$$

a( ) and h( ) are non-linear (possibly vector valued) functions

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Prediction:

$$\hat{s}[n|n-1] = a\hat{s}[n-1|n-1]$$

Correction:

$$\hat{s}[n|n] = \hat{s}[n|n-1] + K[n](x[n] - \hat{s}[n|n-1])$$

Recall the KF for a linear model

We cannot do the prediction step since our case is non-linear

Chapter 13 – Kalman Filters

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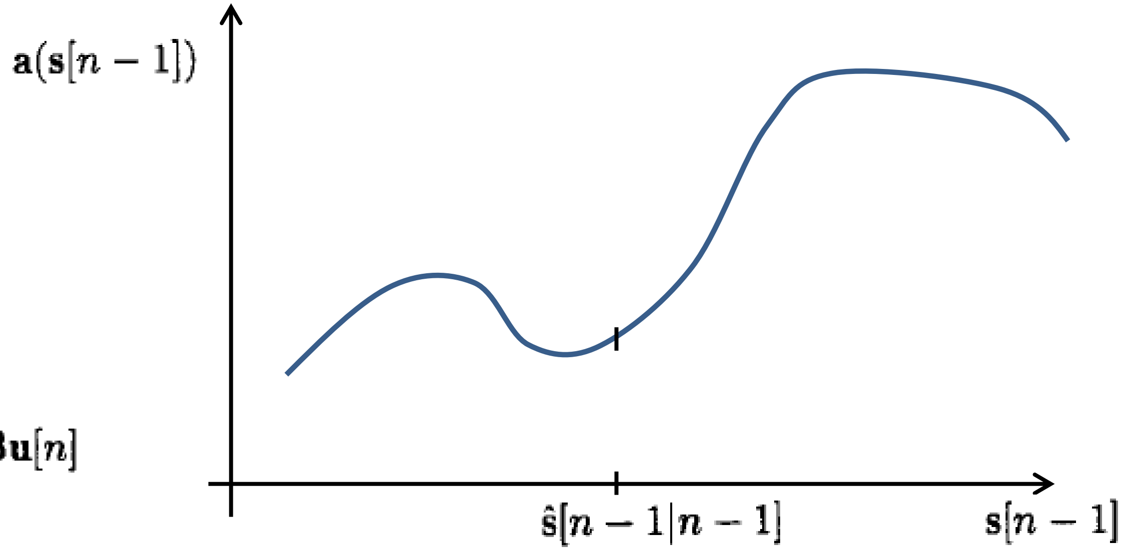
Linearize the model!
That is, linearize a( ) and h( )

Chapter 13 – Kalman Filters

Extended Kalman Filters: used for non-linear models



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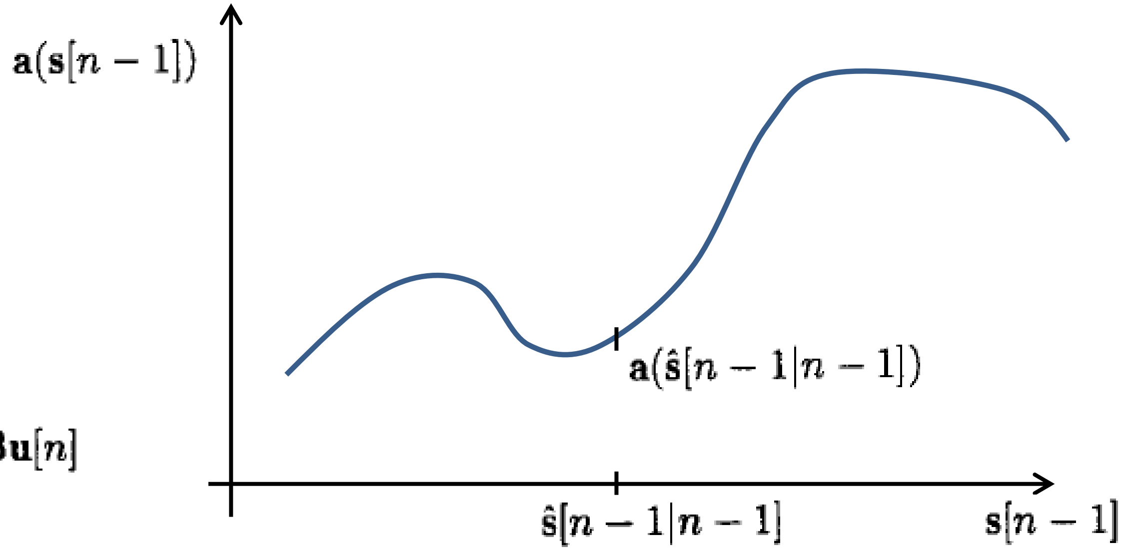
\end{aligned}
$$

First order Taylor series expansion around $\hat{\mathbf{s}}[n-1|n-1]$ **which is available at time n-1**

$$\mathbf{a}(\mathbf{s}[n-1]) \approx$$

Chapter 13 – Kalman Filters

Extended Kalman Filters: used for non-linear models



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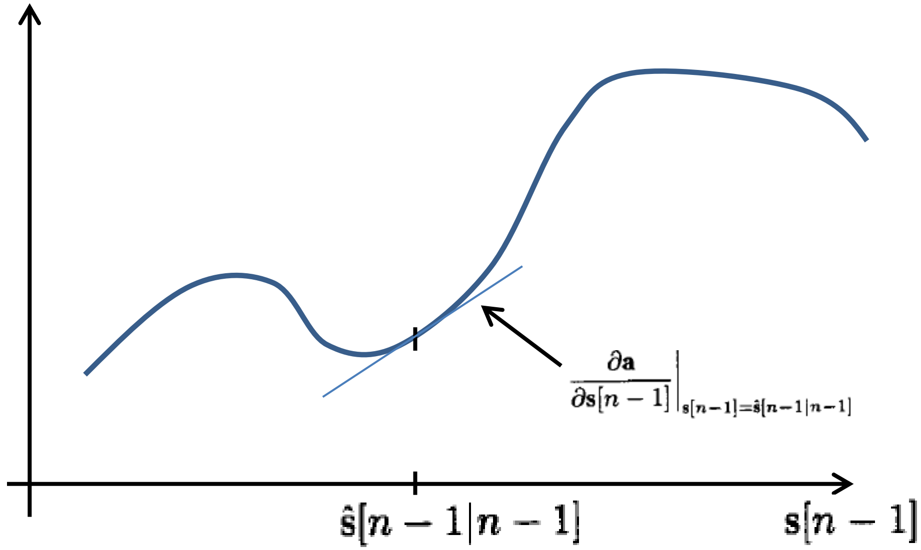
Chapter 13 – Kalman Filters

Extended Kalman Filters: used for non-linear models



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\mathbf{a}(\mathbf{s}[n-1]) \approx\ & \mathbf{a}(\hat{\mathbf{s}}[n-1|n-1]) \\
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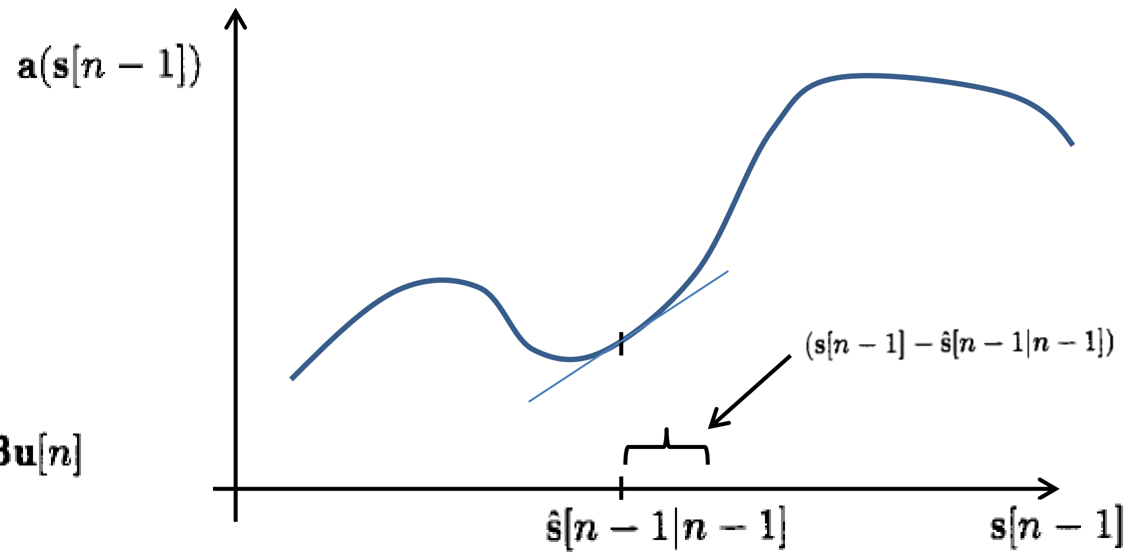
\end{aligned}
$$

Chapter 13 – Kalman Filters

Extended Kalman Filters: used for non-linear models



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Chapter 13 – Kalman Filters

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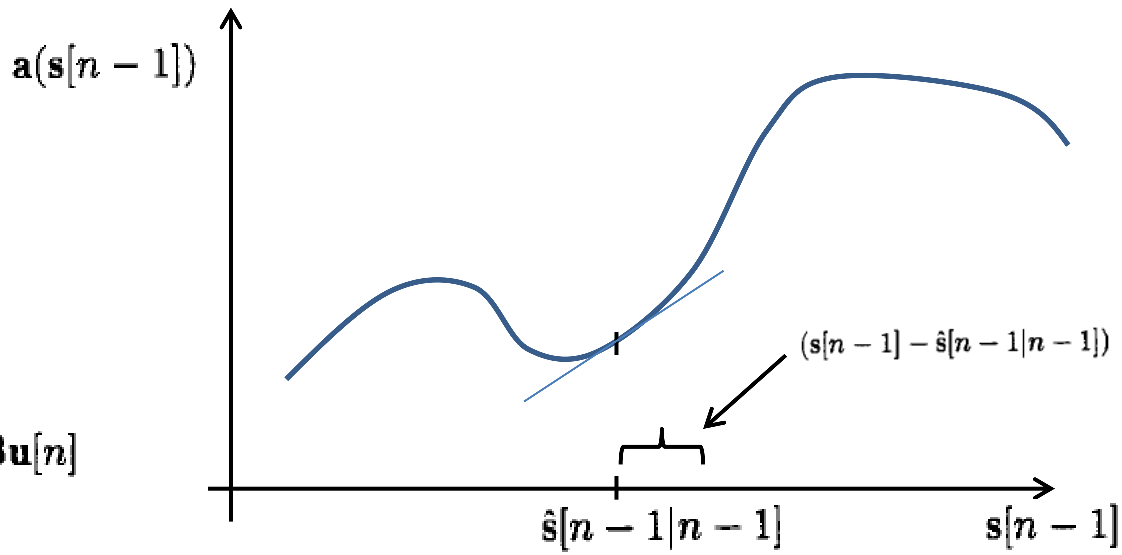

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$$

$$
\mathbf{h}(\mathbf{s}[n]) \approx \mathbf{h}(\hat{\mathbf{s}}[n|n-1]) + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{s}[n]}\right|_{\mathbf{s}[n]=\hat{\mathbf{s}}[n|n-1]} (\mathbf{s}[n] - \hat{\mathbf{s}}[n|n-1])
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Extended Kalman Filters: used for non-linear models

Mathematical model:

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First order Taylor series expansion around $\hat{\mathbf{s}}[n-1|n-1]$ **which is available at time n-1**

$$\begin{aligned}\mathbf{a}(\mathbf{s}[n-1]) &\approx \mathbf{a}(\hat{\mathbf{s}}[n-1|n-1]) \\ &+ \mathbf{A}[n-1]\;(\mathbf{s}[n-1] - \hat{\mathbf{s}}[n-1|n-1])\end{aligned}$$

$$\mathbf{h}(\mathbf{s}[n]) \approx \mathbf{h}(\hat{\mathbf{s}}[n|n-1]) + \mathbf{H}[n]\;(\mathbf{s}[n] - \hat{\mathbf{s}}[n|n-1])$$

Chapter 13 – Kalman Filters

Extended Kalman Filters: used for non-linear models

Mathematical model:

$$\begin{aligned} \mathbf{s}[n] &= \mathbf{a}(\hat{\mathbf{s}}[n-1|n-1]) + \mathbf{A}[n-1]\ (\mathbf{s}[n-1] - \hat{\mathbf{s}}[n-1|n-1]) + \mathbf{B}\mathbf{u}[n] \\ \mathbf{x}[n] &= \mathbf{h}(\hat{\mathbf{s}}[n|n-1]) + \mathbf{H}[n]\ (\mathbf{s}[n] - \hat{\mathbf{s}}[n|n-1]) + \mathbf{w}[n] \end{aligned}$$

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Chapter 13 – Kalman Filters

Extended Kalman Filters: used for non-linear models

These are known input signals

Mathematical model:

$$\begin{aligned} \mathbf{s}[n] &= \overbrace{\mathbf{a}(\hat{\mathbf{s}}[n-1|n-1])} + \mathbf{A}[n-1]\ (\mathbf{s}[n-1] - \overbrace{\hat{\mathbf{s}}[n-1|n-1]}) + \mathbf{B}\mathbf{u}[n] \\ \mathbf{x}[n] &= \underbrace{\mathbf{h}(\hat{\mathbf{s}}[n|n-1])} + \mathbf{H}[n]\ (\mathbf{s}[n] - \underbrace{\hat{\mathbf{s}}[n|n-1]}) + \mathbf{w}[n] \end{aligned}$$

First order Taylor series expansion around $\hat{\mathbf{s}}[n-1|n-1]$ which is available at time n-1

$$\begin{aligned} \mathbf{a}(\mathbf{s}[n-1]) &\approx \mathbf{a}(\hat{\mathbf{s}}[n-1|n-1]) \\ &\quad + \mathbf{A}[n-1]\ (\mathbf{s}[n-1] - \hat{\mathbf{s}}[n-1|n-1]) \end{aligned}$$

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Chapter 13 – Kalman Filters

Extended Kalman Filters: used for non-linear models

These are known input signals
The extended Kalman filter is identical to the normal Kalman filter with known input signals

Mathematical model:

$$\begin{aligned} \mathbf{s}[n] &= \mathbf{a}(\hat{\mathbf{s}}[n-1|n-1]) + \mathbf{A}[n-1]\ (\mathbf{s}[n-1] - \hat{\mathbf{s}}[n-1|n-1]) + \mathbf{B}\mathbf{u}[n] \\ \mathbf{x}[n] &= \mathbf{h}(\hat{\mathbf{s}}[n|n-1]) + \mathbf{H}[n]\ (\mathbf{s}[n] - \hat{\mathbf{s}}[n|n-1]) + \mathbf{w}[n] \end{aligned}$$

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$$\mathbf{a}(\mathbf{s}[n-1]) \approx \mathbf{a}(\hat{\mathbf{s}}[n-1|n-1]) + \mathbf{A}[n-1]\ (\mathbf{s}[n-1] - \hat{\mathbf{s}}[n-1|n-1])$$

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