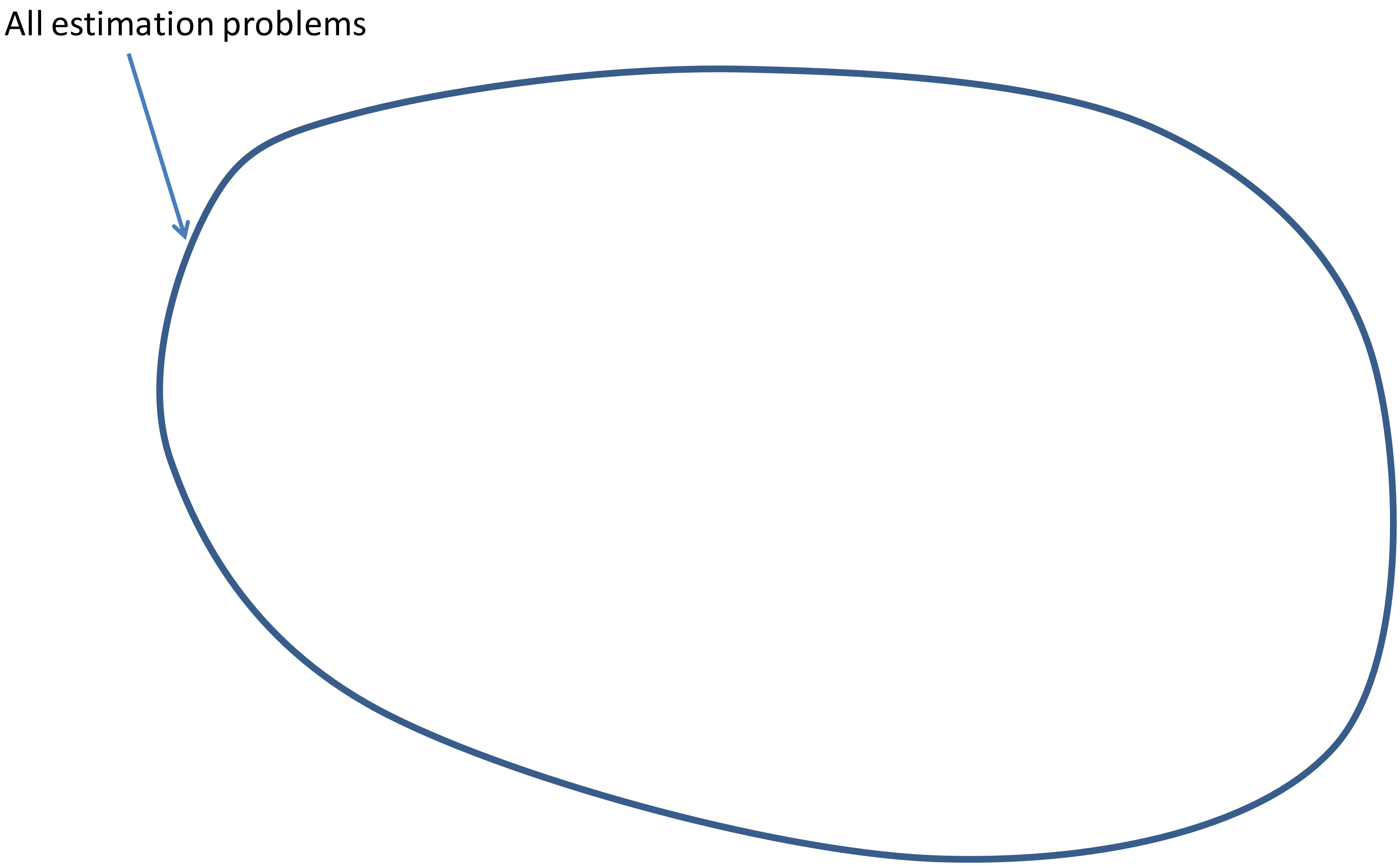
Estimation Theory
Fredrik Rusek

Chapters 6-7

Summary



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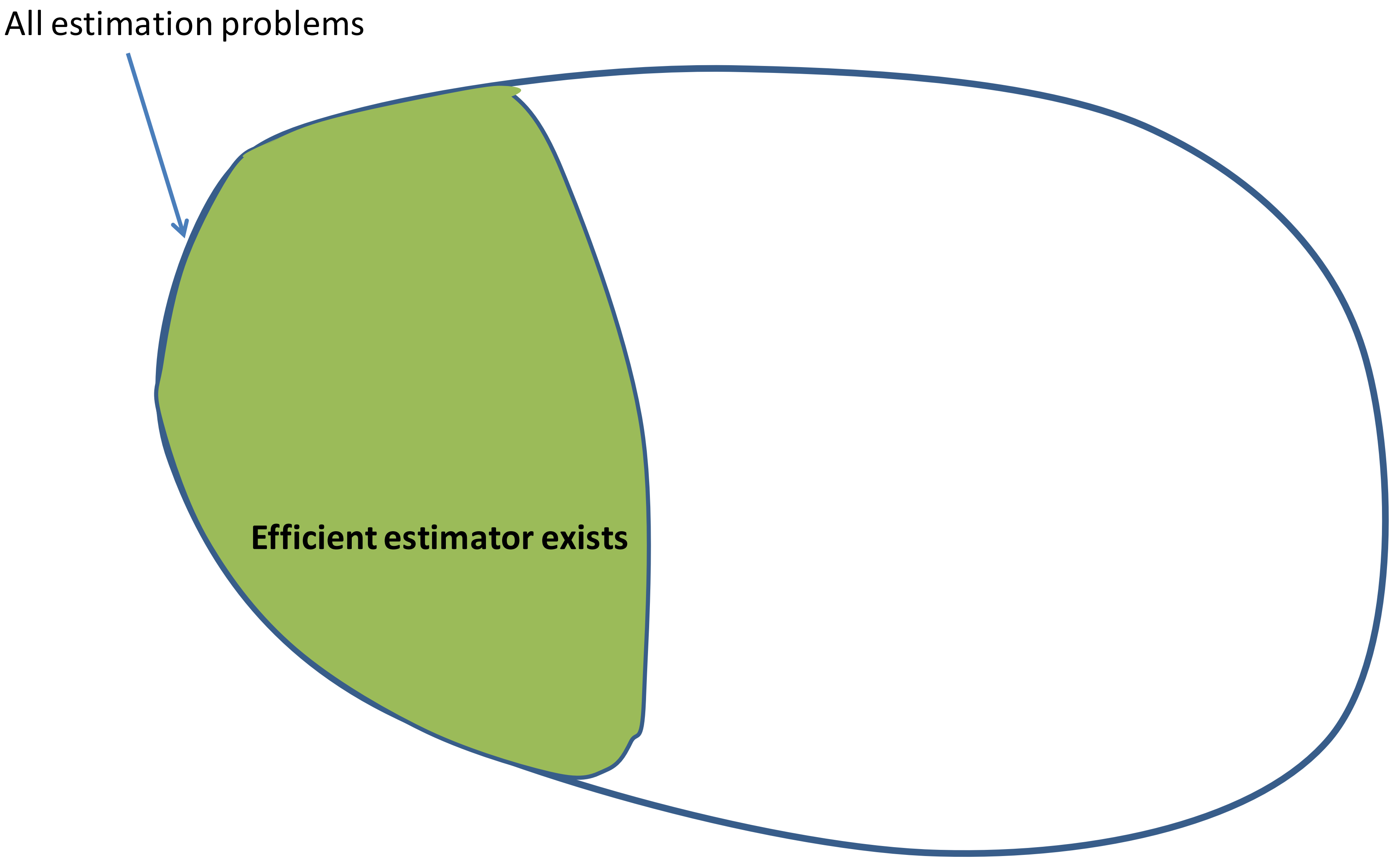

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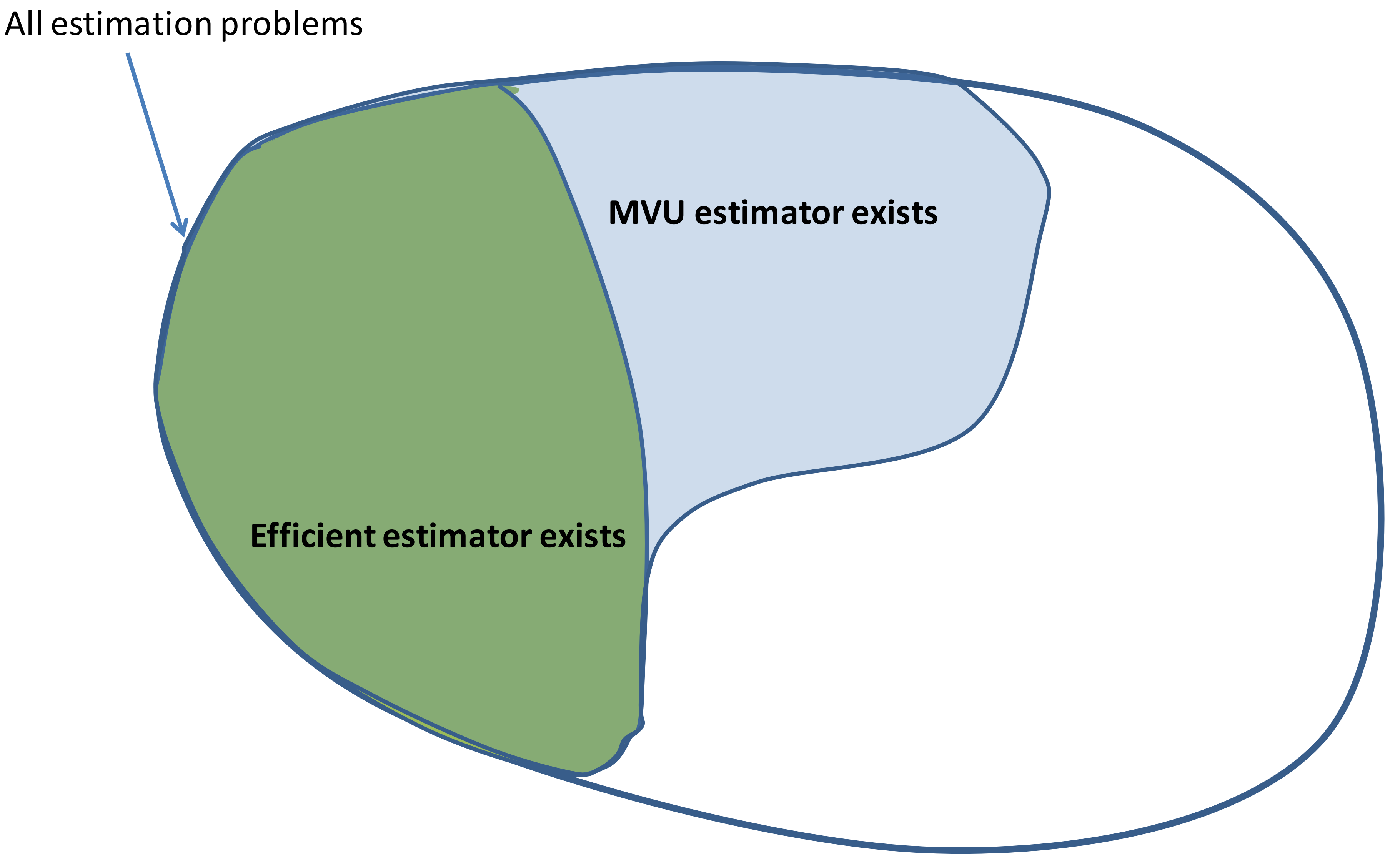
All estimation problems
MVU estimator exists
Efficient estimator exists

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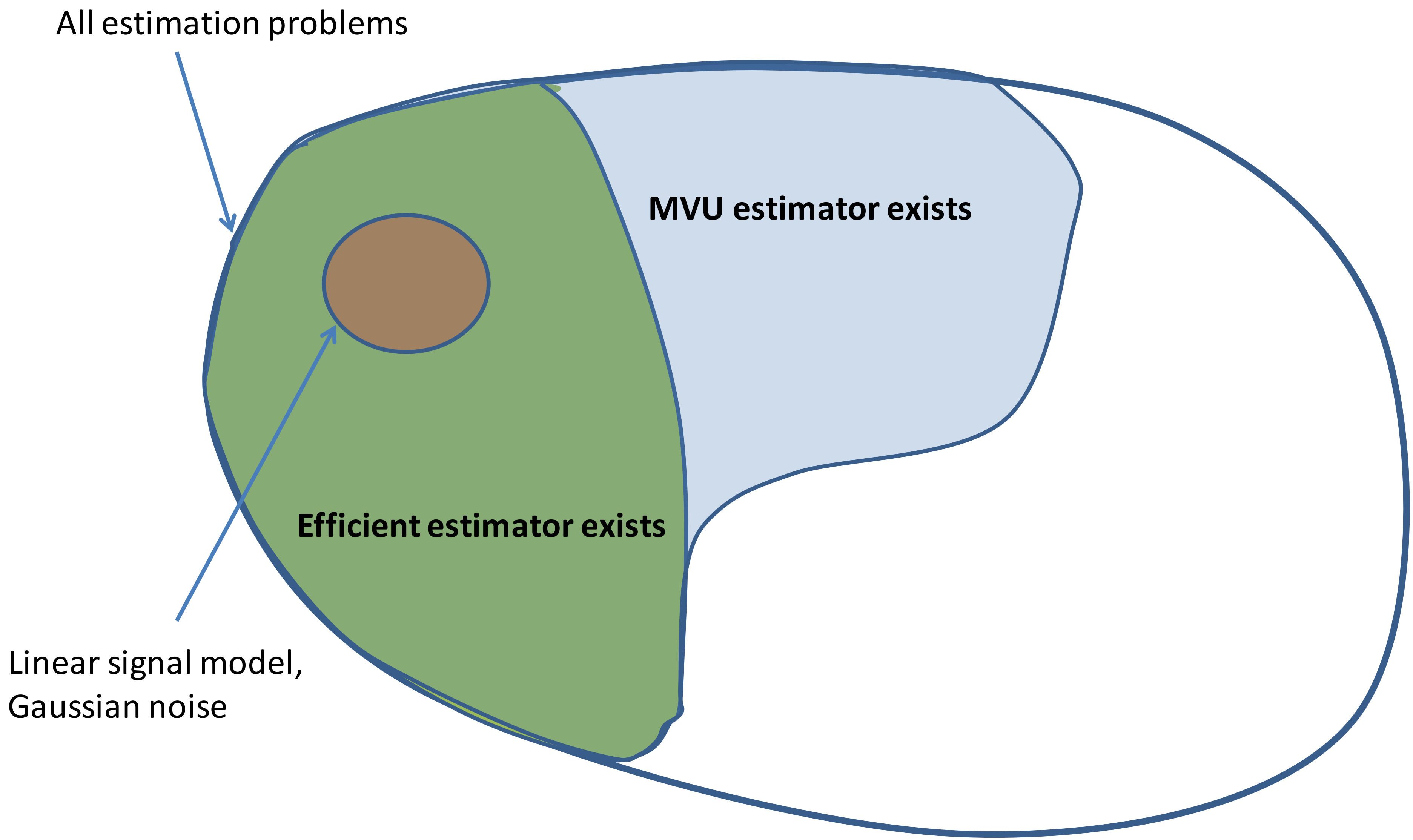
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MVU estimator exists
Efficient estimator exists
Linear signal model,
Gaussian noise

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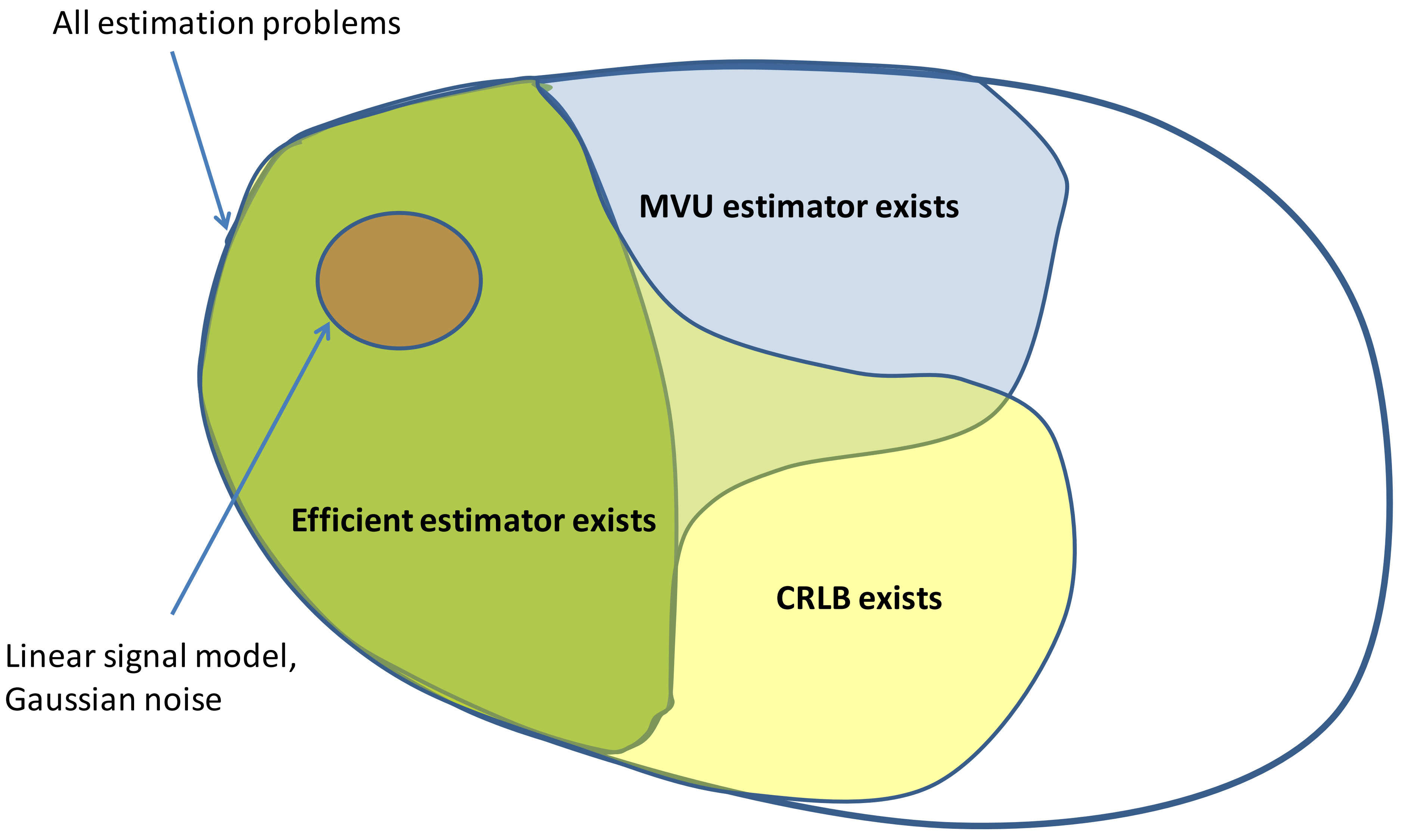
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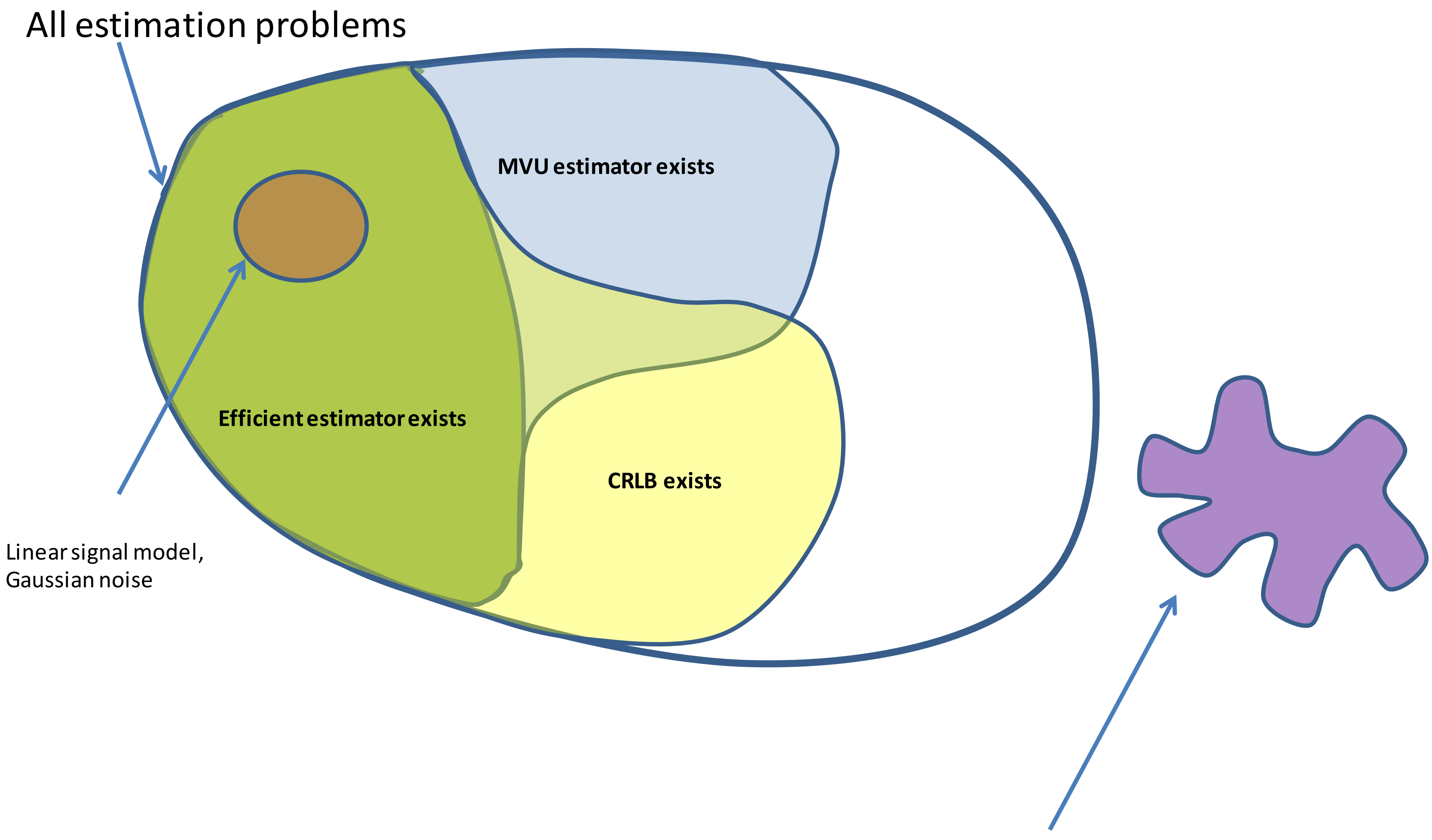
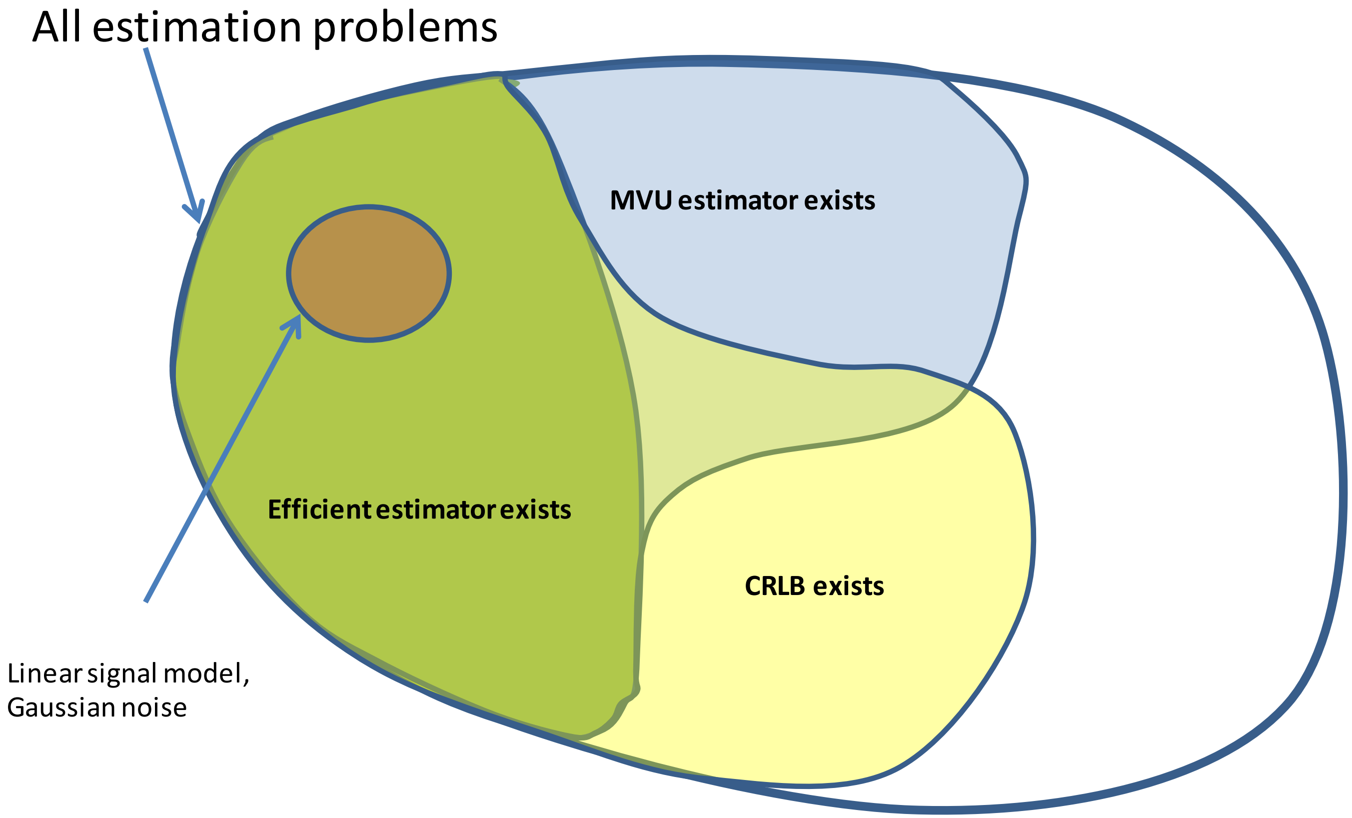
MVU estimator exists
Efficient estimator exists
CRLB exists
Linear signal model,
Gaussian noise

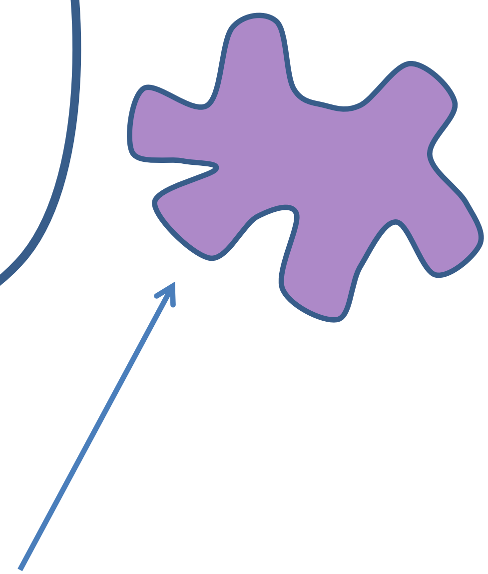
Summary



Today we will place the following piece:
Linear signal in non-Gaussian noise

Summary





Chapter 6 deals with non-Gaussian noise
This is not very well pointed out

Today we will place the following piece:
Linear signal in non-Gaussian noise

Chapter 6 – Best linear unbiased estimators

Definition of the BLUE

Linear in received data

$$\boxed{\hat{\theta} = \sum_{n=0}^{N-1} a_n x[n]}$$

Many possible estimators possible.

The BLUE is the one that is:

- **Unbiased**
- **Smallest possible variance over all $\{a_n\}$**

Chapter 6 – Best linear unbiased estimators

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Optimal when

- **MVU is linear in the data**
- **Suboptimal otherwise**

Chapter 6 – Best linear unbiased estimators

Definition of the BLUE

Linear in received data

$$\boxed{\hat{\theta} = \sum_{n=0}^{N-1} a_n x[n]}$$

Examples:

<u>DC level in white noise</u>

MVU: $\hat{\theta} = \bar{x} = \sum_{n=0}^{N-1} \frac{1}{N} x[n]$ = BLUE

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Chapter 6 – Best linear unbiased estimators

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<u>German tank problem</u>

MVU: $\hat{\theta} = \frac{N+1}{2N} \max x[n]$

BLUE: (can be shown) $\hat{\theta} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$

BLUE ≠ MVU

Chapter 6 – Best linear unbiased estimators

Definition of the BLUE

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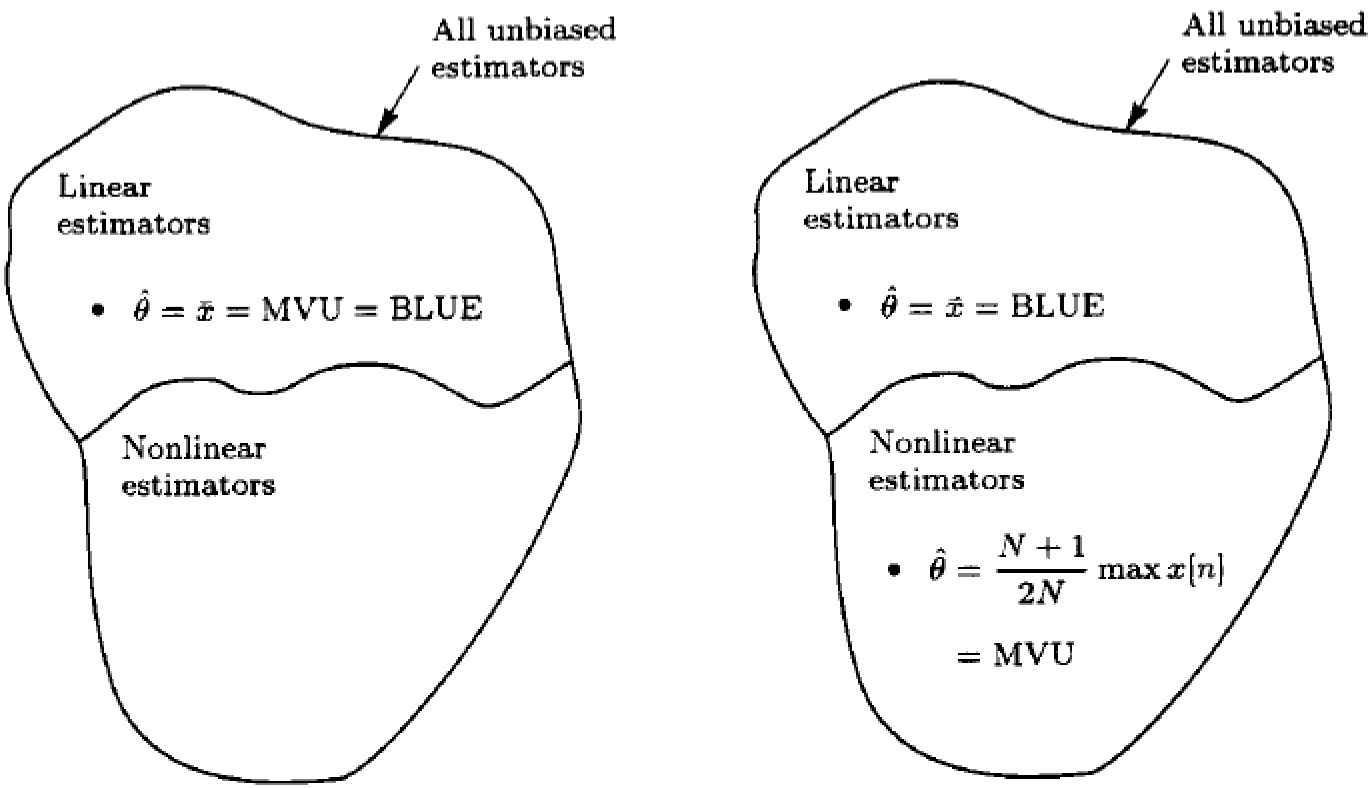
BLUE ≠ MVU

The MVU is often not linear, hence the name ”best linear”

Chapter 6 – Best linear unbiased estimators



(a) DC level in WGN; BLUE is optimal

(b) Mean of uniform noise; BLUE is suboptimal

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Chapter 6 – Best linear unbiased estimators

Definition of the BLUE

Sometimes LE is very bad

<u>Noise power estimation</u> (x[n]=w[n])

MVU $\hat{\sigma^2} = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$

BLUE $\hat{\sigma^2} = \sum_{n=0}^{N-1} a_n x[n]$

$$E(\hat{\sigma^2}) = \sum_{n=0}^{N-1} a_n E(x[n]) = 0$$

Chapter 6 – Best linear unbiased estimators

Definition of the BLUE

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BLUE + ”cleverness”

- Transform data as $y[n] = x^2[n]$
- Apply BLUE to y[n]

Chapter 6 – Best linear unbiased estimators

Finding the BLUE

Constraint

$$E(\hat{\theta}) = \sum_{n=0}^{N-1} a_n E(x[n]) = \theta$$

Chapter 6 – Best linear unbiased estimators

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$$E(\hat{\theta}) = \sum_{n=0}^{N-1} a_n E(x[n]) = \theta$$

Variance

$$\text{var}(\hat{\theta}) = E\left[\hat{\theta} - E(\hat{\theta})\right]$$

Chapter 6 – Best linear unbiased estimators

Finding the BLUE

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$$E(\hat{\theta}) = \sum_{n=0}^{N-1} a_n E(x[n]) = \theta$$

Variance

$$\begin{aligned}\operatorname{var}(\hat{\theta}) &= E\left[\hat{\theta} - E(\hat{\theta})\right] \\ &= E\left[\left(\sum_{n=0}^{N-1} a_n x[n] - E\left(\sum_{n=0}^{N-1} a_n x[n]\right)\right)^2\right]\end{aligned}$$

Chapter 6 – Best linear unbiased estimators

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Use vector notation

$$\boxed{\sum_{n=0}^{N-1} a_n x[n] = \mathbf{a}^T \mathbf{x}}$$

Chapter 6 – Best linear unbiased estimators

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$$\begin{aligned}
\mathrm{var}(\hat{\theta}) &= E\left[\hat{\theta} - E(\hat{\theta})\right] \\
&= E\left[\left(\sum_{n=0}^{N-1} a_n x[n] - E\left(\sum_{n=0}^{N-1} a_n x[n]\right)\right)^2\right] \\
&= E\left[\left(\mathbf{a}^T\mathbf{x} - \mathbf{a}^T E(\mathbf{x})\right)^2\right]
\end{aligned}$$

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Chapter 6 – Best linear unbiased estimators

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&= E\left[\left(\mathbf{a}^T(\mathbf{x} - E(\mathbf{x}))\right)^2\right]
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Chapter 6 – Best linear unbiased estimators

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&= E\left[\left(\mathbf{a}^T(\mathbf{x} - E(\mathbf{x}))\right)^2\right] \\
&= E\left[\mathbf{a}^T(\mathbf{x} - E(\mathbf{x}))(\mathbf{x} - E(\mathbf{x}))^T\mathbf{a}\right]
\end{aligned}$$

Chapter 6 – Best linear unbiased estimators

Finding the BLUE

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&= \mathbf{a}^T\mathbf{C}\mathbf{a}.
\end{aligned}$$

Chapter 6 – Best linear unbiased estimators

Finding the BLUE

Constraint

$$E(\hat{\theta}) = \sum_{n=0}^{N-1} a_n E(x[n]) = \theta$$

Optimization problem

$$\min_{\mathbf{a}} \quad \mathbf{a}^T \mathbf{C} \mathbf{a}$$

$$E\left[\mathbf{a}^T \mathbf{x}\right] = \theta$$

Variance

$$\begin{aligned}
\operatorname{var}(\hat{\theta}) &= E\left[\hat{\theta} - E(\hat{\theta})\right] \\
&= E\left[\left(\sum_{n=0}^{N-1} a_n x[n] - E\left(\sum_{n=0}^{N-1} a_n x[n]\right)\right)^2\right] \\
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Chapter 6 – Best linear unbiased estimators

Finding the BLUE

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$$E\left[\mathbf{a}^T \mathbf{x}\right] = \theta$$

To satisfy $E\left[\mathbf{a}^T \mathbf{x}\right] = \theta$ we must have $E(x[n]) = s[n]\theta$

Chapter 6 – Best linear unbiased estimators

Finding the BLUE

Constraint $$E(\hat{\theta}) = \sum_{n=0}^{N-1} a_n E(x[n]) = \theta$$

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Now write $$x[n] = E(x[n]) + [x[n] - E(x[n])]$$

Chapter 6 – Best linear unbiased estimators

Finding the BLUE

Constraint $$E(\hat{\theta}) = \sum_{n=0}^{N-1} a_n E(x[n]) = \theta$$

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Define $$w[n] = [x[n] - E(x[n])]$$

Chapter 6 – Best linear unbiased estimators

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Define $$w[n] = [x[n] - E(x[n])]$$

Which yields $$\boxed{x[n] = \theta s[n] + w[n]}$$

w[n] not Gaussian, but zero-mean

This is the linear model, but with non-Gaussian noise

Chapter 6 – Best linear unbiased estimators

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We have $E\left[\mathbf{a}^T \mathbf{x}\right] = \mathbf{a}^T \mathbf{s}\, \theta$

Define $$w[n] = [x[n] - E(x[n])]$$

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Chapter 6 – Best linear unbiased estimators

Finding the BLUE

Constraint $$E(\hat{\theta}) = \sum_{n=0}^{N-1} a_n E(x[n]) = \theta$$

Optimization problem

$$\min_{\mathbf{a}} \ \mathbf{a}^T \mathbf{C} \mathbf{a}$$

$$\mathbf{a}^T \mathbf{s} = 1$$

To satisfy $E\left[\mathbf{a}^T \mathbf{x}\right] = \theta$ we must have $E(x[n]) = s[n]\theta$

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Define $$w[n] = [x[n] - E(x[n])]$$

Which yields $$\boxed{x[n] = \theta s[n] + w[n]}$$

Chapter 6 – Best linear unbiased estimators

Finding the BLUE

Lagrangian $J = \mathbf{a}^T \mathbf{C} \mathbf{a} + \lambda(\mathbf{a}^T \mathbf{s} - 1)$

Optimization problem

$$\min_{\mathbf{a}} \ \mathbf{a}^T \mathbf{C} \mathbf{a}$$

$$\mathbf{a}^T \mathbf{s} = 1$$

Chapter 6 – Best linear unbiased estimators

Finding the BLUE

Lagrangian $J = \mathbf{a}^T\mathbf{C}\mathbf{a} + \lambda(\mathbf{a}^T\mathbf{s} - 1)$

$$\frac{\partial J}{\partial \mathbf{a}} = 2\mathbf{C}\mathbf{a} + \lambda\mathbf{s} \qquad \mathbf{a} = -\frac{\lambda}{2}\mathbf{C}^{-1}\mathbf{s}.$$

Optimization problem

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Chapter 6 – Best linear unbiased estimators

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To find λ, use the constraint function $\mathbf{a}^T\mathbf{s} = 1$ **with** $\mathbf{a} = -\frac{\lambda}{2}\mathbf{C}^{-1}\mathbf{s}$

Chapter 6 – Best linear unbiased estimators

Finding the BLUE

Lagrangian $J = \mathbf{a}^T\mathbf{C}\mathbf{a} + \lambda(\mathbf{a}^T\mathbf{s} - 1)$

Optimization problem

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$$\mathbf{a}^T\mathbf{s} = -\frac{\lambda}{2}\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s} = 1$$

$$-\frac{\lambda}{2} = \frac{1}{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}$$

Chapter 6 – Best linear unbiased estimators

Finding the BLUE

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Plug back: $\mathbf{a}_{\mathrm{opt}} = \frac{\mathbf{C}^{-1}\mathbf{s}}{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}$

Chapter 6 – Best linear unbiased estimators

Finding the BLUE

Lagrangian $J = \mathbf{a}^T\mathbf{C}\mathbf{a} + \lambda(\mathbf{a}^T\mathbf{s} - 1)$

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Plug back: $\mathbf{a}_{\text{opt}} = \frac{\mathbf{C}^{-1}\mathbf{s}}{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}$

Variance
$$\begin{aligned}\operatorname{var}(\hat{\theta}) &= \mathbf{a}_{\text{opt}}^T\mathbf{C}\mathbf{a}_{\text{opt}} \\ &= \frac{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{C}\mathbf{C}^{-1}\mathbf{s}}{(\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s})^2} \\ &= \frac{1}{\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}}.\end{aligned}$$

Chapter 6 – Best linear unbiased estimators

To compute the BLUE, we need

- Know that $E(\mathbf{x}) = \mathbf{s}\theta$
- Know $\mathbf{s}$
- Know $\mathbf{C}$

Connections to the MVU estimator for the linear model will soon be made

Chapter 6 – Best linear unbiased estimators

BLUE for vector parameter

$$\hat{\theta}_i = \sum_{n=0}^{N-1} a_{in} x[n] \qquad i = 1, 2, \ldots, p$$

$$\hat{\boldsymbol{\theta}} = \mathbf{A}\mathbf{x}$$

Chapter 6 – Best linear unbiased estimators

BLUE for vector parameter

$$\hat{\theta}_i = \sum_{n=0}^{N-1} a_{in} x[n] \qquad i = 1, 2, \ldots, p$$

$$\hat{\boldsymbol{\theta}} = \mathbf{A}\mathbf{x}$$

Constraint

$$E(\hat{\boldsymbol{\theta}}) = \mathbf{A}E(\mathbf{x}) = \boldsymbol{\theta}$$

$$E(\mathbf{x}) = \mathbf{H}\boldsymbol{\theta}$$

Chapter 6 – Best linear unbiased estimators

BLUE for vector parameter

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Constraint

$$E(\hat{\boldsymbol{\theta}}) = \mathbf{A}E(\mathbf{x}) = \boldsymbol{\theta}$$

$$E(\mathbf{x}) = \mathbf{H}\boldsymbol{\theta}$$

Which becomes

$$\boxed{\mathbf{A}\mathbf{H} = \mathbf{I}}$$

Chapter 6 – Best linear unbiased estimators

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$$\hat{\theta}_i = \sum_{n=0}^{N-1} a_{in} x[n] \qquad i = 1, 2, \ldots, p$$

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Constraint

$$E(\hat{\boldsymbol{\theta}}) = \mathbf{A}E(\mathbf{x}) = \boldsymbol{\theta}$$

$$E(\mathbf{x}) = \mathbf{H}\boldsymbol{\theta}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_p^T \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \ldots & \mathbf{h}_p \end{bmatrix}$$

Which becomes

$$\boxed{\mathbf{A}\mathbf{H} = \mathbf{I}}$$

Chapter 6 – Best linear unbiased estimators

BLUE for vector parameter

$$\hat{\theta}_i = \sum_{n=0}^{N-1} a_{in} x[n] \qquad i = 1, 2, \ldots, p$$

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$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 & \ldots & \mathbf{h}_p \end{bmatrix}$$

Which becomes

$$\mathbf{a}_i^T \mathbf{h}_j = \delta_{ij} \qquad i = 1, 2, \ldots, p; j = 1, 2, \ldots, p$$

Chapter 6 – Best linear unbiased estimators

BLUE for vector parameter

$$\hat{\theta}_i = \sum_{n=0}^{N-1} a_{in} x[n] \qquad i = 1, 2, \ldots, p$$

$$\hat{\boldsymbol{\theta}} = \mathbf{A}\mathbf{x}$$

Constraint

$$E(\hat{\boldsymbol{\theta}}) = \mathbf{A}E(\mathbf{x}) = \boldsymbol{\theta}$$

$$E(\mathbf{x}) = \mathbf{H}\boldsymbol{\theta}$$

Cost function to optimize

$$\mathrm{var}(\hat{\theta}_i) = \mathbf{a}_i^T \mathbf{C} \mathbf{a}_i$$

Which becomes

$$\mathbf{a}_i^T \mathbf{h}_j = \delta_{ij} \qquad i = 1, 2, \ldots, p; j = 1, 2, \ldots, p$$

Chapter 6 – Best linear unbiased estimators

BLUE for vector parameter

Optimal solution

$$\hat{\boldsymbol{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{C}^{-1}\mathbf{x}$$

$$\mathbf{C}_{\hat{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}.$$

Chapter 6 – Best linear unbiased estimators

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Summary

- For a model **y=Hθ+n,** a linear estimator has the same form no matter the distribution of the noise (for the same covariance **C** of it)

Chapter 6 – Best linear unbiased estimators

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Chapter 6 – Best linear unbiased estimators

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Chapter 6 – Best linear unbiased estimators

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- For a model **y=Hθ+n,** a linear estimator has the same form no matter the distribution of the noise (for the same covariance **C** of it)
- If **n** is Gaussian, then a linear estimator is MVU
- If **n** is not Gaussian, the BLUE is the best linear estimator, but not MVU
- Performance of the BLUE for non-Gaussian noise is identical to the performance with Gaussian noise

Chapter 6 – Best linear unbiased estimators

Est. Problem with Gaussian noise | **Est. Problem with Non-Gaussian noise**

Variance of linear estimator

Chapter 6 – Best linear unbiased estimators

	Est. Problem with Gaussian noise	Est. Problem with Non-Gaussian noise
Variance of linear estimator	$\mathbf{C}_{\hat{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}.$	$\mathbf{C}_{\hat{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}.$

Chapter 6 – Best linear unbiased estimators

	Est. Problem with Gaussian noise	Est. Problem with Non-Gaussian noise
Variance of linear estimator	$\mathbf{C}_{\hat{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}.$	$\mathbf{C}_{\hat{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}.$
	Estimator is Optimal	Estimator is not optimal

Chapter 6 – Best linear unbiased estimators

	Est. Problem with Gaussian noise	Est. Problem with Non-Gaussian noise
Variance of linear estimator	$\mathbf{C}_{\hat{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}.$ Estimator is Optimal	$\mathbf{C}_{\hat{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}.$ Estimator is not optimal
Variance of non-linear estimator	The same	?? But smaller than $\mathbf{C}_{\hat{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}.$

Therefore, Gaussian noise is the worst noise possible

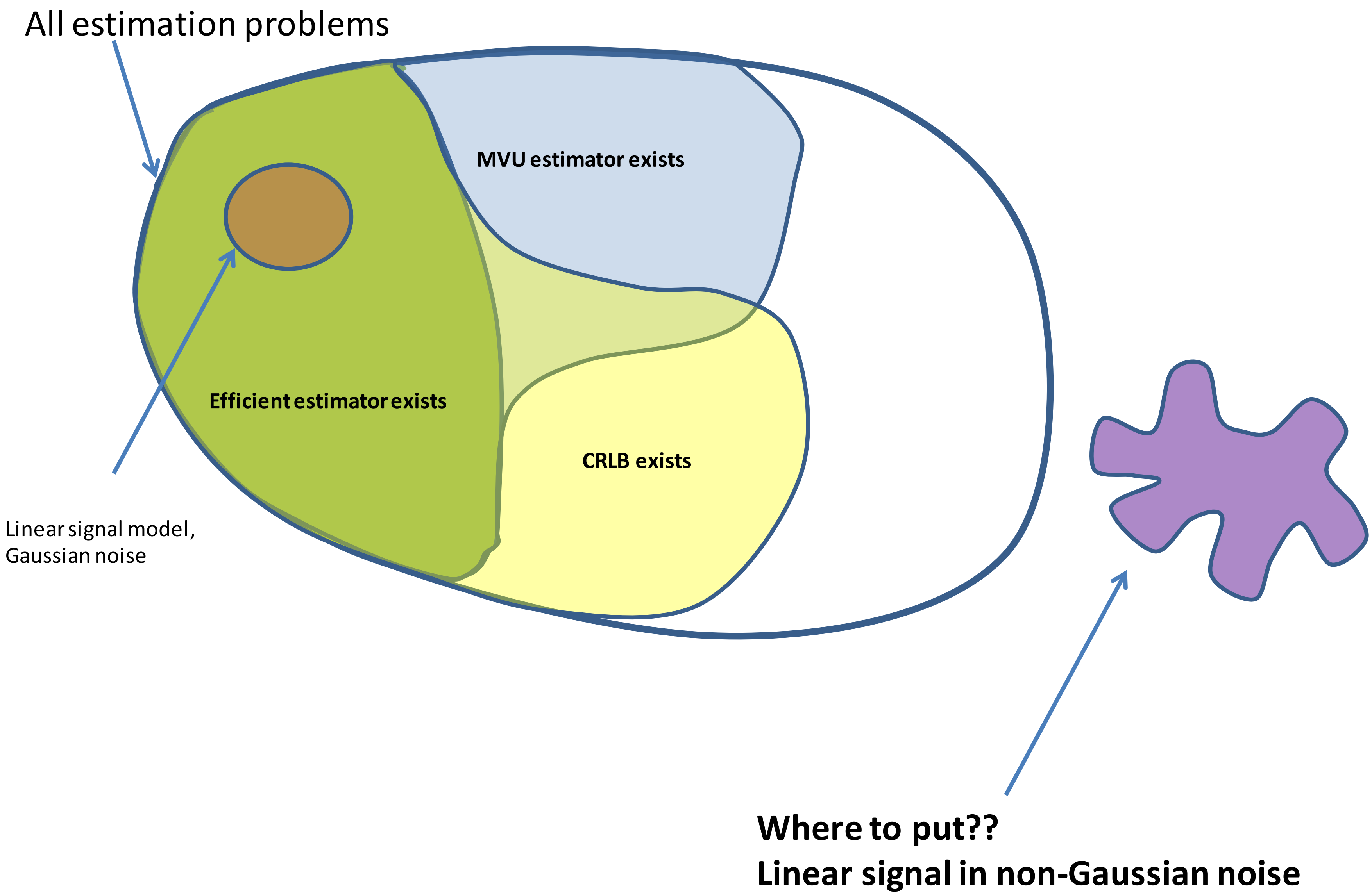
Chapter 6 – Best linear unbiased estimators

	Est. Problem with Gaussian noise	Est. Problem with Non-Gaussian noise
Variance of linear estimator	$\mathbf{C}_{\hat{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}.$ Estimator is Optimal	$\mathbf{C}_{\hat{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}.$ Estimator is not optimal
Variance of non-linear estimator	The same	?? But smaller than $\mathbf{C}_{\hat{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}.$

Alternative proofs

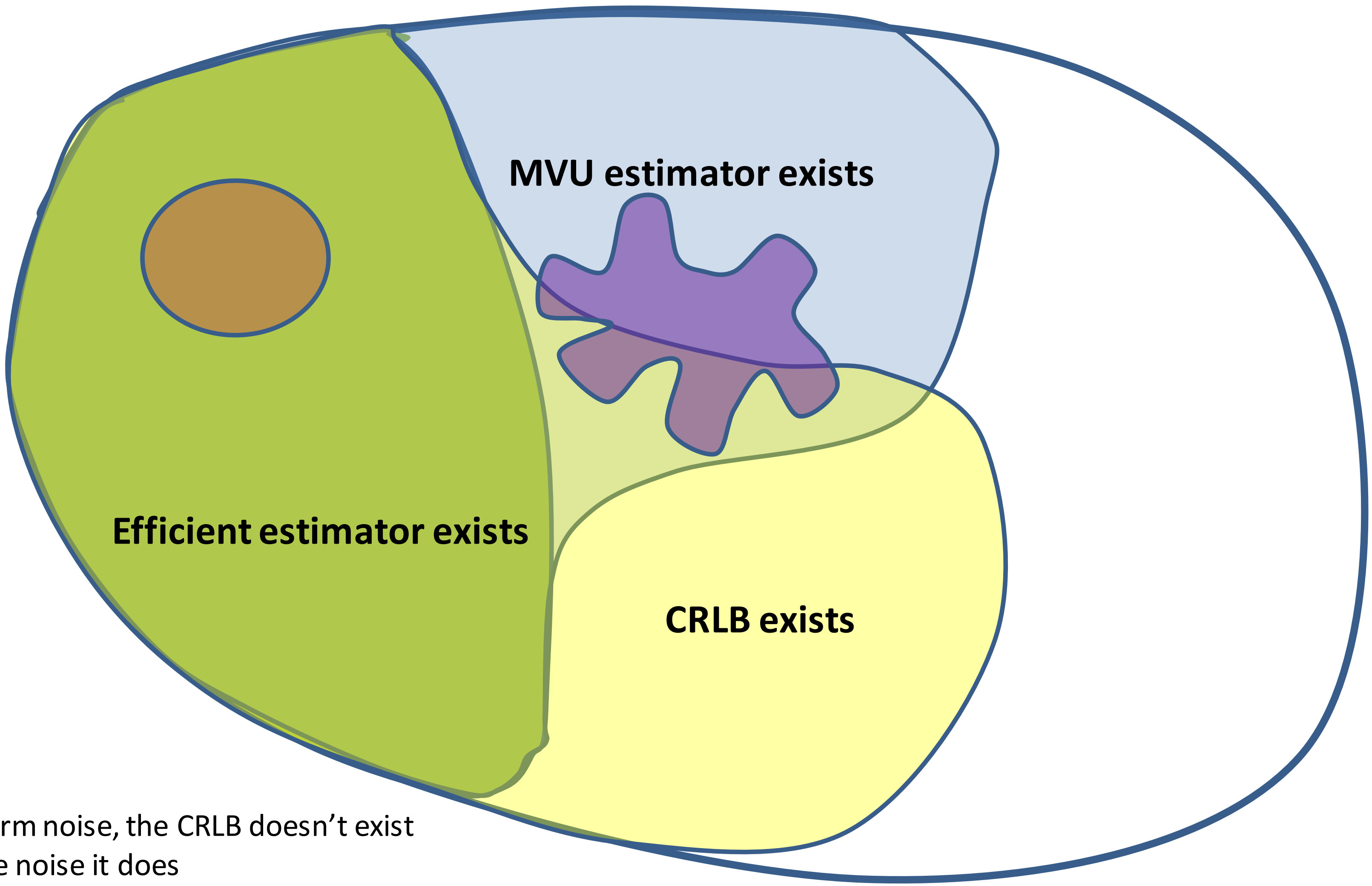
1. Based on calculus of the variations
2. Based on link between Fisher information and Kullback-Leibler divergence (information theory)

Summary


All estimation problems
MVU estimator exists
Efficient estimator exists
CRLB exists
Linear signal model,
Gaussian noise
Where to put??
Linear signal in non-Gaussian noise

We do not have explicit info

Summary ????



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- For Laplace noise it does
- No efficient estimator seems to exist – see my ”Italian proof”
- Perhaps MVU does not always exist….?

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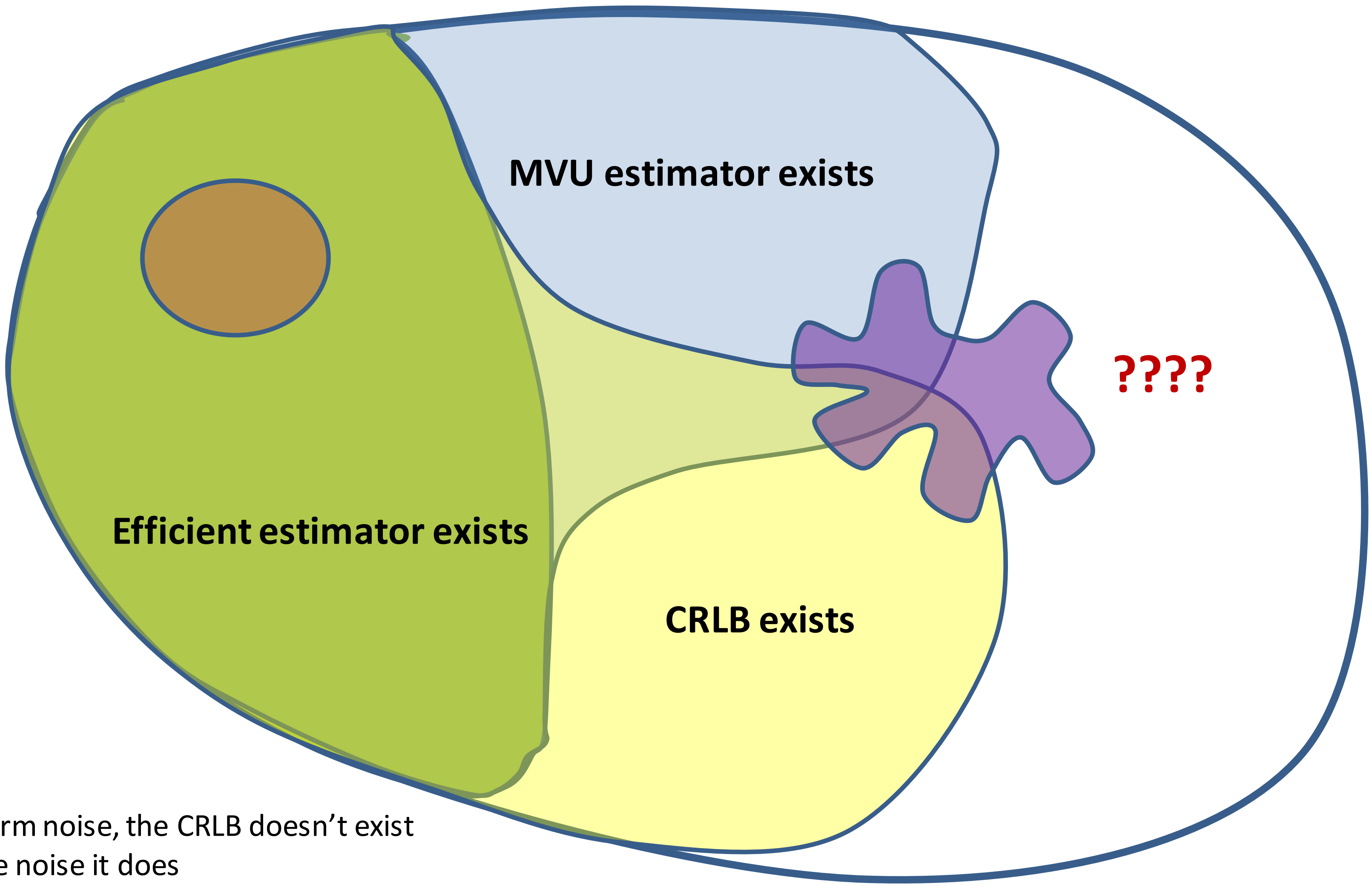
Summary ????



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Chapter 7 – Maximum Likelihood

$$\hat{\theta} = \underset{\theta}{arg\ max}\ p(\mathbf{x};\theta)$$

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Theorem. If an efficient estimator exists, then it is given by the MLE

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Proof.

$$\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta} = I(\theta)(g(\mathbf{x}) - \theta)$$

Chapter 7 – Maximum Likelihood

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Proof.

$$0 = \left.\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right|_{\theta_{ML}} = I(\theta)(g(\mathbf{x}) - \theta)\Big|_{\theta_{ML}}$$

Chapter 7 – Maximum Likelihood

Example 7.3. DC level in white Gaussian noise *with variance = DC-level*

$$p(\mathbf{x}; A) = \frac{1}{(2\pi A)^{\frac{N}{2}}} \exp\left[-\frac{1}{2A}\sum_{n=0}^{N-1}(x[n]-A)^2\right]$$

$$\frac{\partial \ln p(\mathbf{x}; A)}{\partial A} = -\frac{N}{2A} + \frac{1}{A}\sum_{n=0}^{N-1}(x[n]-A) + \frac{1}{2A^2}\sum_{n=0}^{N-1}(x[n]-A)^2$$

Chapter 7 – Maximum Likelihood

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Check point 1: Try the CRLB $\quad \dfrac{\partial \ln p(\mathbf{x}; A)}{\partial A} = I(A)(\hat{A} - A)$

Chapter 7 – Maximum Likelihood

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Check point 2: Try Neyman-Fisher

$$p(\mathbf{x}; A) = \underbrace{\frac{1}{(2\pi A)^{\frac{N}{2}}} \exp\left[-\frac{1}{2}\left(\frac{1}{A}\sum_{n=0}^{N-1}x^2[n] + NA\right)\right]}_{g\left(\sum_{n=0}^{N-1}x^2[n],\, A\right)} \underbrace{\exp(N\bar{x})}_{h(\mathbf{x})}$$

Chapter 7 – Maximum Likelihood

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If $T(\mathbf{x}) = \sum_{n=0}^{N-1} x^2[n]$ *is complete, then an unbiased function of T(**x**) is MVU*

Chapter 7 – Maximum Likelihood

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If $T(\mathbf{x}) = \sum_{n=0}^{N-1} x^2[n]$ *is complete, then an unbiased function of T(**x**) is MVU*

$$\begin{aligned} E\left[\sum_{n=0}^{N-1} x^2[n]\right] &= NE[x^2[n]] \\ &= N\left[\operatorname{var}(x[n]) + E^2(x[n])\right] \\ &= N(A + A^2) \end{aligned}$$

Not clear what function to choose

Chapter 7 – Maximum Likelihood

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No more strategies to find the MVU – proceed to MLE

$$\hat{A} = -\frac{1}{2} + \sqrt{\frac{1}{N}\sum_{n=0}^{N-1}x^2[n] + \frac{1}{4}}$$

Chapter 7 – Maximum Likelihood

Example 7.3. DC level in white Gaussian noise *with variance = DC-level*

Possible to show:

- As N →∞, $E(\hat{A}) \rightarrow A$
- As N →∞, $\mathrm{var}(\hat{A}) \rightarrow \mathrm{CRLB}$

Asymptotically efficient

Chapter 7 – Maximum Likelihood

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We saw this notation before, when we claimed that as N grows, we can estimate a transformation of a variable as the transformation of the estimate

Chapter 7 – Maximum Likelihood

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- By the CLT, we have that $\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$ is Gaussian as N grows.

  Asymptotically, one can show that the MLE is a linear transformation of this Gaussian variable. Thus, the estimator is Gaussian distributed

$$\hat{A} = -\frac{1}{2} + \sqrt{\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] + \frac{1}{4}}$$

Linearity is not easy to see, but possible…..

Chapter 7 – Maximum Likelihood

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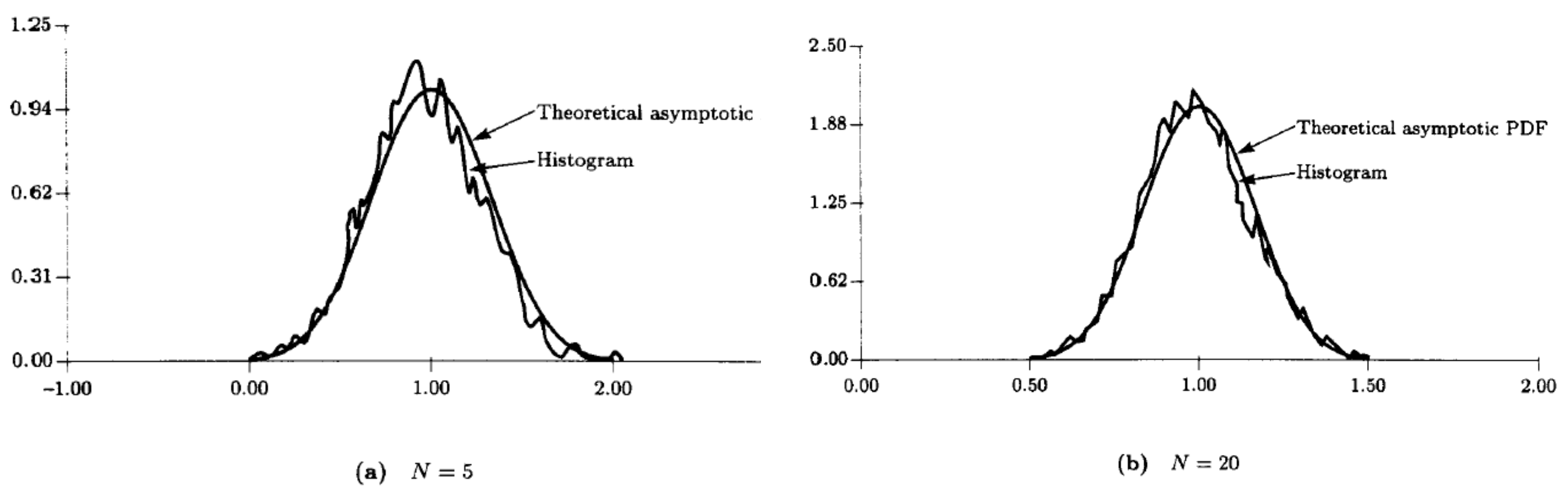
Asymptotically, one can show that the MLE is a linear transformation of this Gaussian variable. Thus, the estimator is Gaussian distributed

But since the variance of the estimator is given by the CRLB (asympt.),

$$\boxed{\hat{\theta} \overset{a}{\sim} \mathcal{N}(\theta, I^{-1}(\theta))}$$

Chapter 7 – Maximum Likelihood

”Effect of N”



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Chapter 7 – Maximum Likelihood

Theorem 7.1 (Asymptotic Properties of the MLE) *If the PDF $p(\mathbf{x};\theta)$ of the data $\mathbf{x}$ satisfies some "regularity" conditions, then the MLE of the unknown parameter θ is asymptotically distributed (for large data records) according to*

$$\hat{\theta} \stackrel{a}{\sim} \mathcal{N}(\theta, I^{-1}(\theta)) \tag{7.11}$$

where $I(\theta)$ is the Fisher information evaluated at the true value of the unknown parameter.

1. The first-order and second-order derivatives of the log-likelihood function are well defined.
2. $$E\left[\frac{\partial \ln p(x[n];\theta)}{\partial \theta}\right] = 0.$$

Chapter 7 – Maximum Likelihood

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Proof of mean (conistency):

$$\frac{1}{N}\ln p(\mathbf{x};\theta) = \frac{1}{N}\sum_{n=0}^{N-1}\ln p(x[n];\theta)$$

Let θ_0 be the true value of θ

Chapter 7 – Maximum Likelihood

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Chapter 7 – Maximum Likelihood

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Chapter 7 – Maximum Likelihood

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Chapter 7 – Maximum Likelihood

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Chapter 7 – Maximum Likelihood

Neyman-Fisher factorization

$$\boxed{p(\mathbf{x};\theta) = g(T(\mathbf{x}),\theta)h(\mathbf{x})}$$

MLE

$$\boxed{\max_{\theta} \; p(\mathbf{x};\theta)}$$

Chapter 7 – Maximum Likelihood

Neyman-Fisher factorization

$$\boxed{p(\mathbf{x};\theta) = g(T(\mathbf{x}),\theta)h(\mathbf{x})}$$

MLE

$$\boxed{\max_{\theta}\; p(\mathbf{x};\theta)}$$

To find the ML, it is sufficient to optimize the function g(T(x),θ)

Hence, MLE can be made on the basis of the sufficient statistic(s) only

Chapter 7 – Maximum Likelihood

Transformation of parameters

Theorem 7.2 (Invariance Property of the MLE) *The MLE of the parameter* $\alpha = g(\theta)$, *where the PDF* $p(\mathbf{x};\theta)$ *is parameterized by* θ, *is given by*

$$\hat{\alpha} = g(\hat{\theta})$$

where $\hat{\theta}$ *is the MLE of* θ. *The MLE of* $\hat{\theta}$ *is obtained by maximizing* $p(\mathbf{x};\theta)$. *If* g *is not a one-to-one function, then* $\hat{\alpha}$ *maximizes the modified likelihood function* $\bar{p}_T(\mathbf{x};\alpha)$, *defined as*

$$\bar{p}_T(\mathbf{x};\alpha) = \max_{\{\theta:\alpha=g(\theta)\}} p(\mathbf{x};\theta).$$

Chapter 7 – Maximum Likelihood

Transformation of parameters

Let us tie things together

- The MLE of a transformed parameter is the transformed MLE of the parameter

Chapter 7 – Maximum Likelihood

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Chapter 7 – Maximum Likelihood

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- **From before:** The (non-linear) transformed estimate of an efficient estimator does not preserve efficiency

$$\hat{\alpha} = g(\hat{\theta})$$

efficient

Not efficient

Chapter 7 – Maximum Likelihood

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Not efficient

This argument didn't say that an efficient estimator for α does not exist

Chapter 7 – Maximum Likelihood

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$$\hat{\alpha} = g(\hat{\theta})$$

Not efficient

efficient

- **We can deduce:** A non-linear function of a parameter that can be efficiently estimated ***cannot be efficiently estimated***

$$\hat{\alpha} = g(\hat{\theta})$$

Not efficient

(but would have been if an efficient existed since it is the transformed MLE)

efficient

Chapter 7 – Maximum Likelihood

Transformation of parameters

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$$\underbrace{\hat{\alpha}}_{\text{Not efficient}} = g(\underbrace{\hat{\theta}}_{\text{efficient}})$$

- **We can deduce:** A non-linear function of a parameter that can be efficiently estimated ***<u>cannot be efficiently estimated</u>***
- Due to previous linearization argument, the transformed estimate is asymptotically efficient.

Chapter 7 – Maximum Likelihood

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efficient

Not efficient

- **We can deduce:** A non-linear function of a parameter that can be efficiently estimated ***cannot be efficiently estimated***
- Due to previous linearization argument, the transformed estimate is asymptotically efficient.
- This could also have been realized by the observation that

  transformed estimate = MLE = asymptotically efficient

Chapter 7 – Maximum Likelihood

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$$\bar{p}_T(\mathbf{x};\alpha) = \boxed{\max_{\{\theta:\alpha=g(\theta)\}}} p(\mathbf{x};\theta).$$

What is this?

Chapter 7 – Maximum Likelihood

Transformation of parameters

Consider the case $\alpha = A^2$

For some value of α, there are two values of A that produces α, $A = \pm\sqrt{\alpha}$

The likelihood of α, is the largest of the two likelihoods

$$\boxed{\max_{\{\theta:\alpha=g(\theta)\}} p(\mathbf{x};\theta)}$$

Chapter 7 – Maximum Likelihood

Example 7.10 - Power of WGN in dB

We observe N samples of WGN with variance σ^2 whose power in dB is to be estimated. To do so we first find the MLE of σ^2. Then, we use the invariance principle to find the power P in dB, which is defined as

$$P = 10\log_{10}\sigma^2.$$

The PDF is given by

$$p(\mathbf{x};\sigma^2) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right].$$

Differentiating the log-likelihood function produces

$$\begin{aligned}\frac{\partial \ln p(\mathbf{x};\sigma^2)}{\partial\sigma^2} &= \frac{\partial}{\partial\sigma^2}\left[-\frac{N}{2}\ln 2\pi - \frac{N}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right] \\ &= -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{n=0}^{N-1}x^2[n]\end{aligned}$$

Chapter 7 – Maximum Likelihood

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Differentiating the log-likelihood function produces

$$\begin{aligned}\frac{\partial \ln p(\mathbf{x};\sigma^2)}{\partial\sigma^2} &= \frac{\partial}{\partial\sigma^2}\left[-\frac{N}{2}\ln 2\pi - \frac{N}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right]\\ &= -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{n=0}^{N-1}x^2[n]\end{aligned}$$

MLE (linear)

$$\boxed{\hat{\sigma^2} = \frac{1}{N}\sum_{n=0}^{N-1}x^2[n]}$$

Chapter 7 – Maximum Likelihood

Example 7.10 - Power of WGN in dB

We observe N samples of WGN with variance σ^2 whose power in dB is to be estimated. To do so we first find the MLE of σ^2. Then, we use the invariance principle to find the power P in dB, which is defined as

$$P = 10\log_{10}\sigma^2.$$

The PDF is given by

$$p(\mathbf{x};\sigma^2) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right].$$

Differentiating the log-likelihood function produces

$$\begin{aligned}\frac{\partial \ln p(\mathbf{x};\sigma^2)}{\partial\sigma^2} &= \frac{\partial}{\partial\sigma^2}\left[-\frac{N}{2}\ln 2\pi - \frac{N}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right] \\ &= -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{n=0}^{N-1}x^2[n]\end{aligned}$$

MLE (linear)

$$\boxed{\hat{\sigma^2} = \frac{1}{N}\sum_{n=0}^{N-1}x^2[n]}$$

MLE (dB) due to invariance

$$\boxed{\begin{aligned}\hat{P} &= 10\log_{10}\hat{\sigma^2} \\ &= 10\log_{10}\frac{1}{N}\sum_{n=0}^{N-1}x^2[n].\end{aligned}}$$

Chapter 7 – Maximum Likelihood

Efficient ?
Meaningful to compare with CRLB ?

Efficient (Var = $2\sigma^4/N$=CRLB)

MLE (linear)

$$\boxed{\hat{\sigma^2} = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]}$$

MLE (dB) due to invariance

$$\boxed{\begin{aligned} \hat{P} &= 10\log_{10}\hat{\sigma^2} \\ &= 10\log_{10}\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]. \end{aligned}}$$

Chapter 7 – Maximum Likelihood

Efficient ? No! Asymptotically: Yes!
Meaningful to compare with CRLB ? No! Not even unbiased
Meets CRLB asymptotically

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Chapter 7 – Maximum Likelihood

Set N=10.
Let us anyway check the CRLB for the dB case: 1.8861

Measured variance (Matlab): ≈ 4.17

Var=2.21xCRLB

Efficient ? **No! Asymptotically: Yes!**
Meaningful to compare with CRLB ? **No! Not even unbiased**
Meets CRLB asymptotically

Efficient (Var = $2\sigma^4/N$=CRLB)

MLE (linear)

$$\hat{\sigma^2} = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$$

MLE (dB) due to invariance

$$\begin{aligned} \hat{P} &= 10\log_{10}\hat{\sigma^2} \\ &= 10\log_{10}\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]. \end{aligned}$$

Chapter 7 – Maximum Likelihood

Set N=50.
Let us anyway check the CRLB for the dB case: 0.3771

Measured variance (Matlab): ≈ 0.77

Var=2.04xCRLB

Efficient ? **No! Asymptotically: Yes!**
Meaningful to compare with CRLB ? **No! Not even unbiased**
Meets CRLB asymptotically

Efficient (Var = 2σ^4/N=CRLB)

MLE (linear)

$$\hat{\sigma^2} = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$$

MLE (dB) due to invariance

$$\begin{aligned}\hat{P} &= 10\log_{10}\hat{\sigma^2} \\ &= 10\log_{10}\frac{1}{N}\sum_{n=0}^{N-1} x^2[n].\end{aligned}$$

Chapter 7 – Maximum Likelihood

Set N=150.
Let us anyway check the CRLB for the dB case: 0.126

Measured variance (Matlab): ≈ 0.25

Var=2xCRLB

Efficient ? No! Asymptotically: Yes!
Meaningful to compare with CRLB ? No! Not even unbiased
Meets CRLB asymptotically

Efficient (Var = 2σ⁴/N=CRLB)

MLE (linear)

$$\hat{\sigma^2} = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$$

MLE (dB) due to invariance

$$\begin{aligned}\hat{P} &= 10\log_{10}\hat{\sigma^2} \\ &= 10\log_{10}\frac{1}{N}\sum_{n=0}^{N-1} x^2[n].\end{aligned}$$

Chapter 7 – Maximum Likelihood

Extension to vector parameter: Straightforward

$$\frac{\partial \ln p(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{0}$$

Theorem 7.3 (Asymptotic Properties of the MLE (Vector Parameter)) *If the PDF $p(\mathbf{x};\boldsymbol{\theta})$ of the data $\mathbf{x}$ satisfies some "regularity" conditions, then the MLE of the unknown parameter $\boldsymbol{\theta}$ is asymptotically distributed according to*

$$\hat{\boldsymbol{\theta}} \overset{a}{\sim} \mathcal{N}(\boldsymbol{\theta}, \mathbf{I}^{-1}(\boldsymbol{\theta})) \tag{7.36}$$

where $\mathbf{I}(\boldsymbol{\theta})$ is the Fisher information matrix evaluated at the true value of the unknown parameter.

Theorem 7.4 (Invariance Property of MLE (Vector Parameter)) *The MLE of the parameter $\boldsymbol{\alpha} = \mathbf{g}(\boldsymbol{\theta})$, where $\mathbf{g}$ is an r-dimensional function of the $p \times 1$ parameter $\boldsymbol{\theta}$, and the PDF $p(\mathbf{x};\boldsymbol{\theta})$ is parameterized by $\boldsymbol{\theta}$, is given by*

$$\hat{\boldsymbol{\alpha}} = \mathbf{g}(\hat{\boldsymbol{\theta}})$$

for $\hat{\boldsymbol{\theta}}$, the MLE of $\boldsymbol{\theta}$. If $\mathbf{g}$ is not an invertible function, then $\hat{\boldsymbol{\alpha}}$ maximizes the modified likelihood function $\bar{p}_T(\mathbf{x};\boldsymbol{\alpha})$, defined as

$$\bar{p}_T(\mathbf{x};\boldsymbol{\alpha}) = \max_{\{\boldsymbol{\theta}:\boldsymbol{\alpha}=\mathbf{g}(\boldsymbol{\theta})\}} p(\mathbf{x};\boldsymbol{\theta}).$$

Chapter 7 – Maximum Likelihood

MLE for linear model $\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$ **in Gaussian noise**

With no derivations: How should we argue in order to establish the MLE ?

Chapter 7 – Maximum Likelihood

MLE for linear model $\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$ in Gaussian noise

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- In this case a linear estimator was optimal (Chapter 4)

Chapter 7 – Maximum Likelihood

MLE for linear model $\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$ in Gaussian noise

With no derivations: How should we argue in order to establish the MLE ?

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- This linear estimator was also efficient

Chapter 7 – Maximum Likelihood

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- An efficient estimator is always the MLE

Chapter 7 – Maximum Likelihood

MLE for linear model $\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}$ in Gaussian noise

With no derivations: How should we argue in order to establish the MLE ?

- In this case a linear estimator was optimal (Chapter 4)
- This linear estimator was also efficient
- An efficient estimator is always the MLE
- **MLE is the linear estimator from Chapter 4** $\hat{\boldsymbol{\theta}} = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{C}^{-1}\mathbf{x}$

Chapter 7 – Maximum Likelihood

Asymptotic MLE: some intuition

$$p(\mathbf{x};\boldsymbol{\theta}) = \frac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}}(\mathbf{C}(\boldsymbol{\theta}))} \exp\left[-\frac{1}{2}\mathbf{x}^T \mathbf{C}^{-1}(\boldsymbol{\theta})\mathbf{x}\right]$$

With a stationary Gaussian process, the log-likelihood becomes

$$\ln p(\mathbf{x};\boldsymbol{\theta}) = -\frac{N}{2}\ln 2\pi - \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\ln P_{xx}(f) + \frac{I(f)}{P_{xx}(f)}\right] df$$

$$I(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]\exp(-j2\pi f n)\right|^2$$

Appears strange, but is not if one knows his linear algebra + Szegö's Theorem

Chapter 7 – Maximum Likelihood

Asymptotic MLE: some intuition

- Covariance matrix is Toeplitz
- Elements of C are autocorrelation values of the process (of course)
- Asymptotic Toeplitz matrix admits an Eigenvalue factorization as C=QSQ* (Q=DFT)
- PSD is the Fourier transform of the autocorrelation sequence (e.g. from the Proakis course)

- **not exactly Szegö's Thm, but a consequence thereof:**
  Asymptotically and very loosely speaking, the Eigenvalues of C, i.e. S, converges
  to Fourier transform of the autocorrelation sequence.

Chapter 7 – Maximum Likelihood

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- Consider now log det(C)
  - det(C) is the product of eigenvalues
  - So, log det(C) is the sum of the logarithm of the eigenvalues
  - But: eigenvalues = Fourier transform of autocorrelation = PSD

Chapter 7 – Maximum Likelihood

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$$p(\mathbf{x};\boldsymbol{\theta}) = \frac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}}(\mathbf{C}(\boldsymbol{\theta}))}$$

$$\ln p(\mathbf{x};\boldsymbol{\theta}) = -\frac{N}{2}\ln 2\pi - \frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \ln P_{xx}(f)\, df$$

Chapter 7 – Maximum Likelihood

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- **not exactly Szegö's Thm, but a consequence thereof:**
  Asymptotically and very loosely speaking, the Eigenvalues of C, i.e. S, converges to Fourier transform of the autocorrelation sequence.

- Consider now $-x^T C^{-1} x$
  - $x^T C^{-1} x = x^T Q S^{-1} Q^* x$
  - Q^*x is the Fourier transform of x
  - S^{-1} is "one divided with the PSD"
  - $x^T Q$ together with Q^*x is the periodogram of x

$$p(\mathbf{x};\boldsymbol{\theta}) = \exp\left[-\frac{1}{2}\mathbf{x}^T\mathbf{C}^{-1}(\boldsymbol{\theta})\mathbf{x}\right] \qquad \ln p(\mathbf{x};\boldsymbol{\theta}) = -\frac{N}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{I(f)}{P_{xx}(f)}\,df$$