Asymptotic CRLB for Gaussian WSS processes

WSS process

If x[n] is WSS, then it holds that

$$E\{x[n]\} = \mu, \ \forall n$$

$$E\{x[n]x[n+k]\} = \rho_k, \ \forall n$$

If x[n] is Gaussian WSS, then x[n] is Gaussian

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But since it is WSS Gaussian, $\mathbf{x} \sim \mathcal{N}\left(\mu\mathbf{1}, \mathbf{C}_{\mathrm{Tp}}\right)$

Toeplitz

$$\mathbf{C}_{\mathrm{Tp}} = \begin{bmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \cdots & \cdots & 0 \\ \rho_{-1} & \rho_0 & \rho_1 & \rho_2 & \cdots & \cdots & 0 \\ \rho_{-2} & \rho_{-1} & \rho_0 & \rho_1 & \rho_2 \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \end{bmatrix}$$

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CRLB

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = \left[\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_i}\right]^T \mathbf{C}^{-1}(\boldsymbol{\theta}) \left[\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_j}\right] + \frac{1}{2}\operatorname{tr}\left[\mathbf{C}^{-1}(\boldsymbol{\theta})\frac{\partial \mathbf{C}(\boldsymbol{\theta})}{\partial \theta_i}\mathbf{C}^{-1}(\boldsymbol{\theta})\frac{\partial \mathbf{C}(\boldsymbol{\theta})}{\partial \theta_j}\right]$$

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Toeplitz

Depends only on ρ, μ, N
Simpler formula possible

$$\mathbf{C}_{\mathrm{Tp}} = \begin{bmatrix} \rho_0 & \rho_1 & \rho_2 & \cdots & \cdots & \cdots & 0 \\ \rho_{-1} & \rho_0 & \rho_1 & \rho_2 & \cdots & \cdots & 0 \\ \rho_{-2} & \rho_{-1} & \rho_0 & \rho_1 & \rho_2 \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \end{bmatrix}$$

Asymptotic CRLB for Gaussian WSS processes

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Section 3.10

Set μ = 0, N-> ∞

CRLB

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = \cancel{\left[\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_i}\right]^T \mathbf{C}^{-1}(\boldsymbol{\theta}) \left[\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_j}\right]} + \frac{1}{2}\mathrm{tr}\left[\mathbf{C}^{-1}(\boldsymbol{\theta})\frac{\partial \mathbf{C}(\boldsymbol{\theta})}{\partial \theta_i}\mathbf{C}^{-1}(\boldsymbol{\theta})\frac{\partial \mathbf{C}(\boldsymbol{\theta})}{\partial \theta_j}\right]$$

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Asymptotic CRLB for Gaussian WSS processes

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Simpler formula possible

Clearly, as N->∞, I(θ) ->∞, CRLB ->0

Asymptotic CRLB for Gaussian WSS processes

WSS process

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Simpler formula possible

Clearly, as N->∞, I(θ) ->∞, CRLB ->0

Makes sense to study (1/N) I(θ) as N->∞

Asymptotic CRLB for Gaussian WSS processes

WSS process

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But since it is WSS Gaussian, $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_{\mathrm{Tp}})$

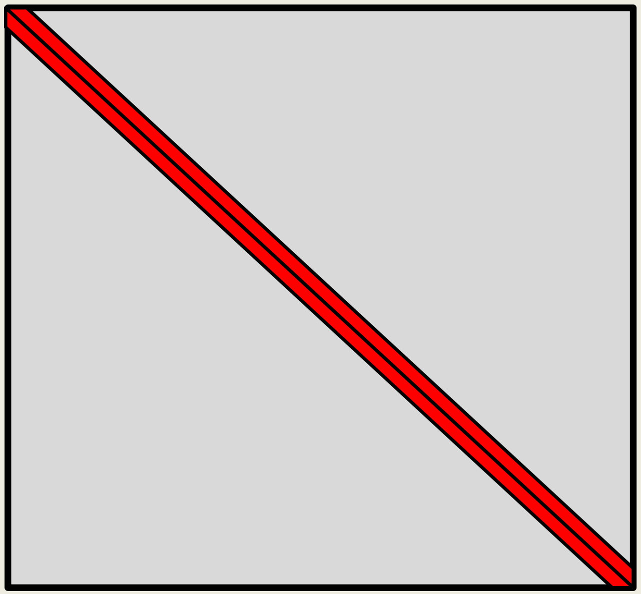
Depends only on ρ, N
Simpler formula possible

Problem formulation: *Simplify* the Fisher information formula for the case of a Topelitz covariance matrix of very large size

Asymptotic CRLB for Gaussian WSS processes

Loose Derivations

Consider a Toeplitz matrix of very large size



Asymptotic CRLB for Gaussian WSS processes

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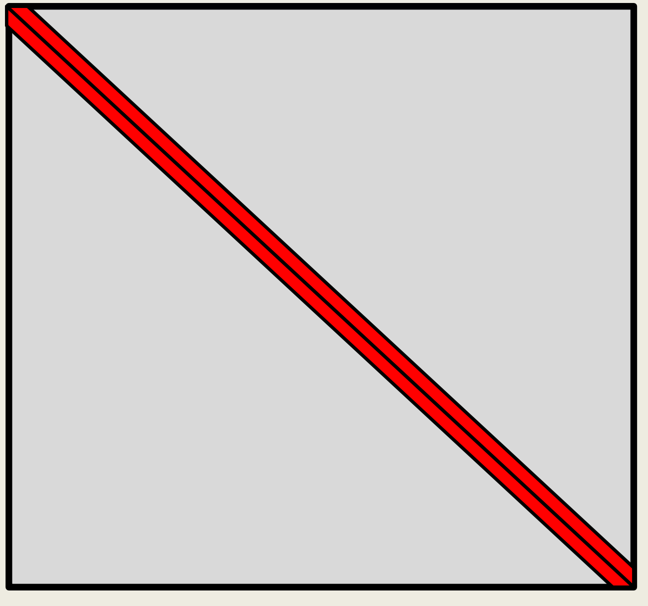
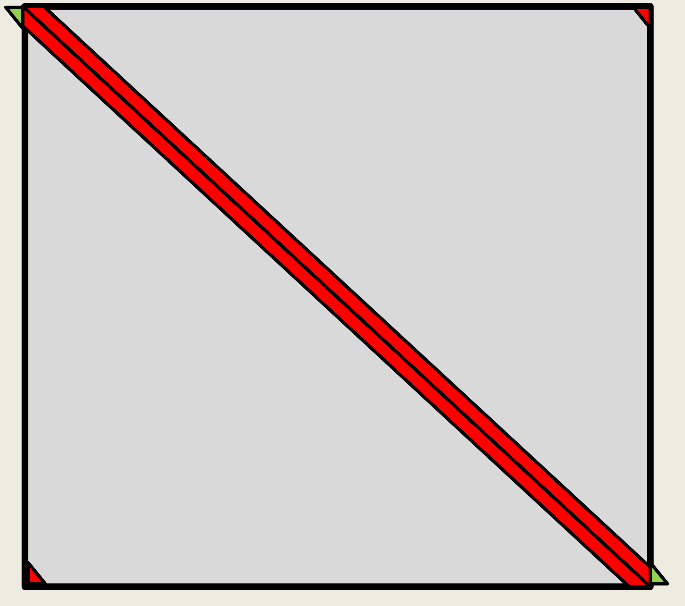


Add a tiny piece to it



Asymptotic CRLB for Gaussian WSS processes

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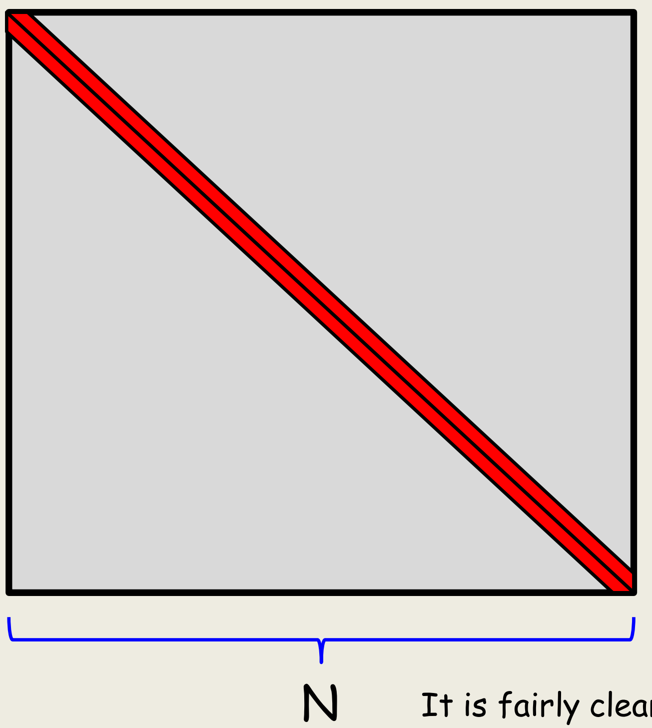

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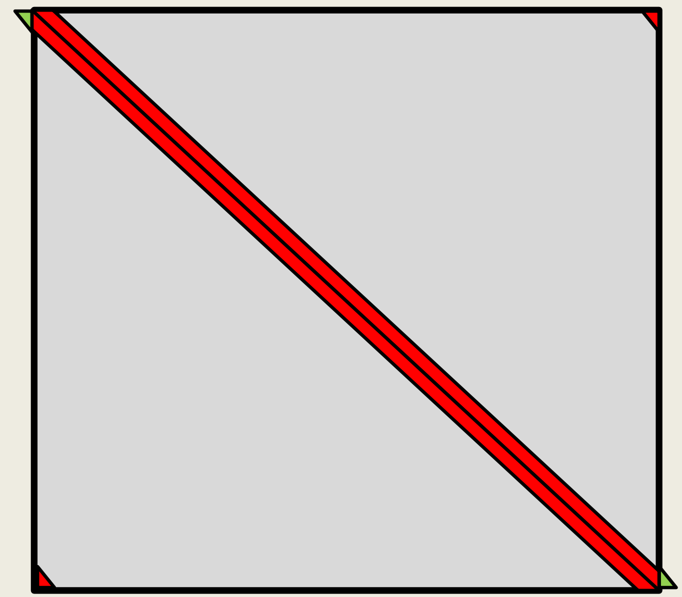

It is fairly clear that with large N, the difference between the two matrices is, in a relative sense, vanishing

Asymptotic CRLB for Gaussian WSS processes

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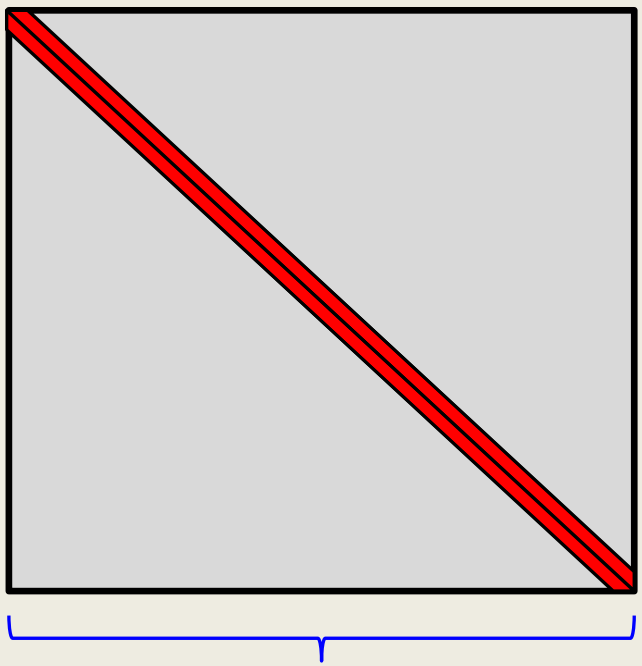
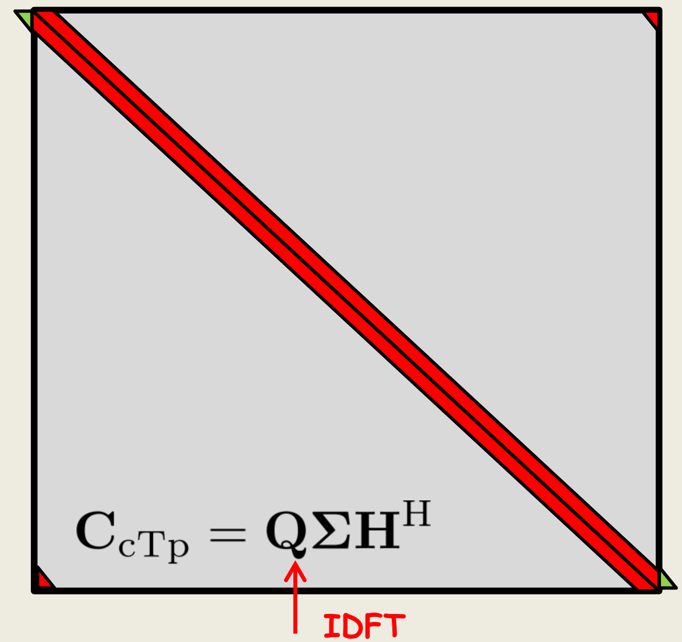


N

Add a tiny piece to it

Circulant Toeplitz



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Asymptotic CRLB for Gaussian WSS processes

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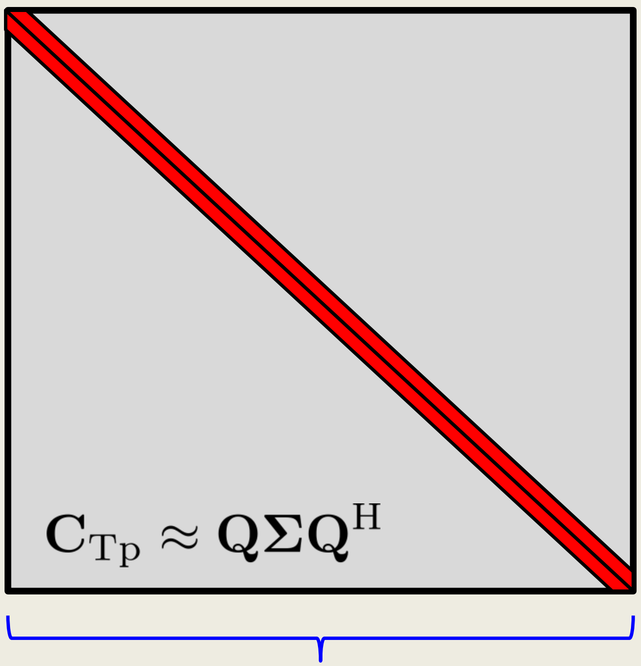

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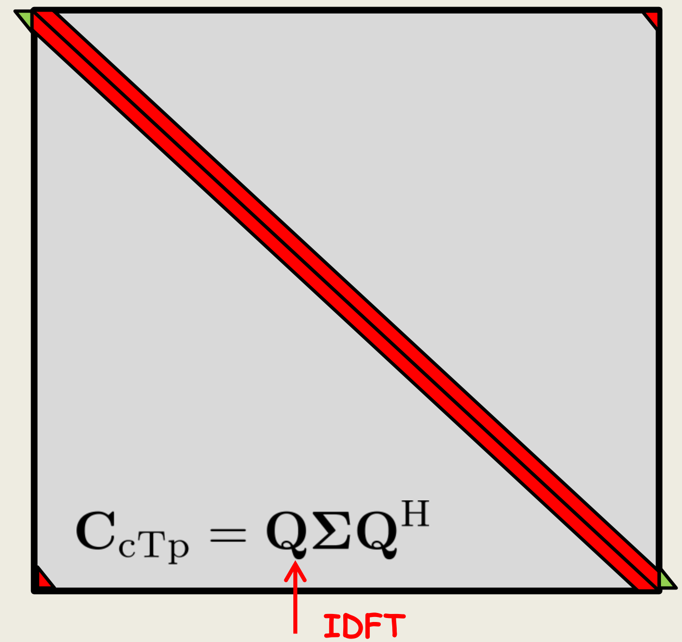

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Asymptotic CRLB for Gaussian WSS processes

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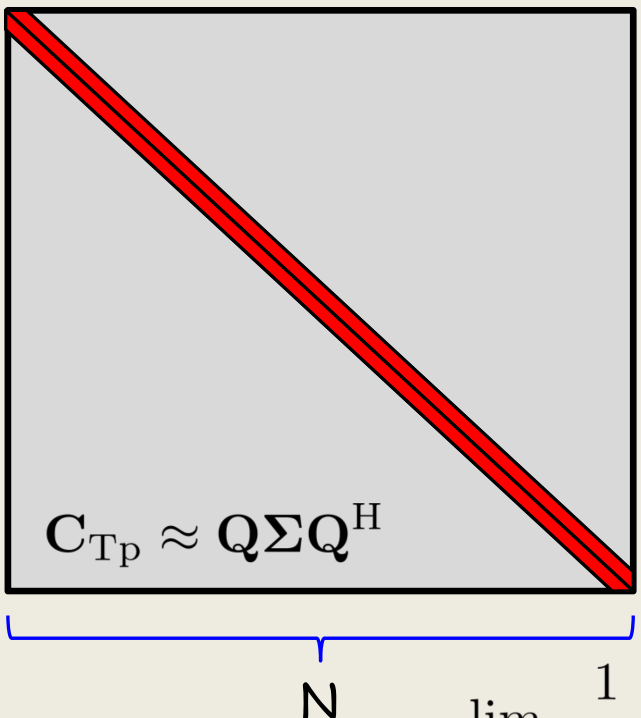

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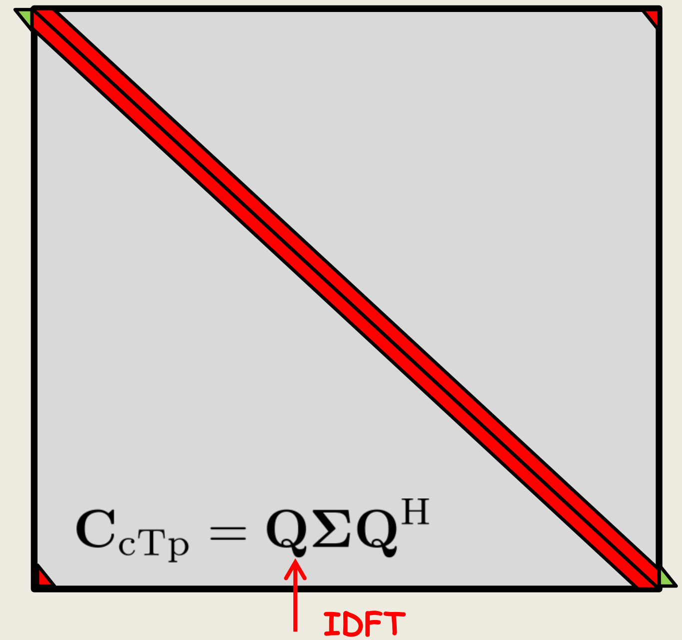
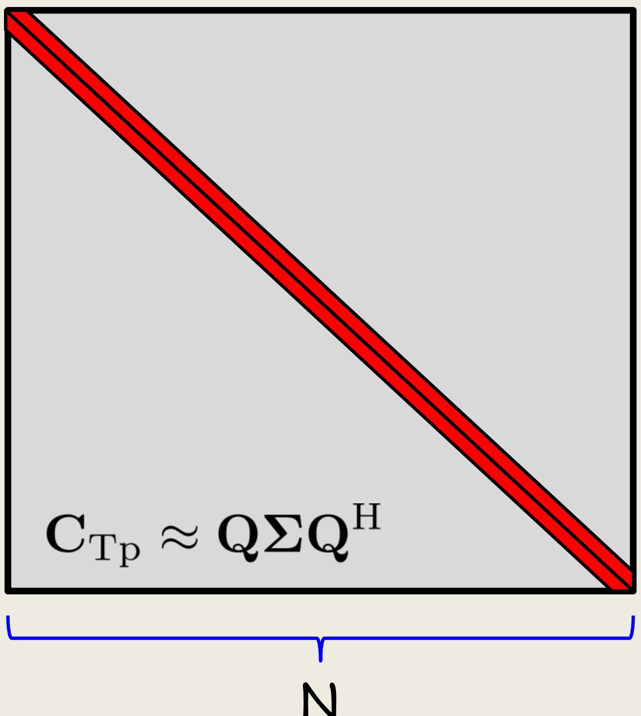


$$\lim_{N\to\infty} \frac{1}{N} \|\mathbf{C}_{\mathrm{Tp}} - \mathbf{Q}\boldsymbol{\Sigma}\mathbf{Q}^{\mathrm{H}}\|^{\mathbf{2}} = \mathbf{0}$$

Asymptotic CRLB for Gaussian WSS processes

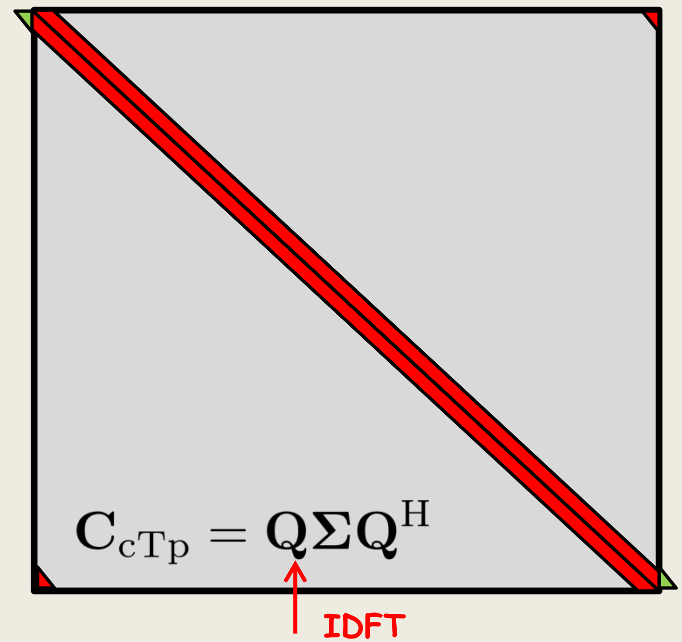
Loose Derivations

Consider a Toeplitz matrix of very large size



Add a tiny piece to it

Circulant Toeplitz



Important: For a large Toeplitz, the matrix Q is always an IDFT, and *cannot depend on θ*

Asymptotic CRLB for Gaussian WSS processes

Loose Derivations

We have $\mathbf{x} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{C}(\boldsymbol{\theta})\right)$

We seek $$[I(\boldsymbol{\theta})]_{ij} = \frac{1}{2}\mathrm{Tr}\left[\mathbf{C}^{-\mathbf{1}}(\boldsymbol{\theta})\frac{\partial\mathbf{C}(\boldsymbol{\theta})}{\partial\theta_{\mathbf{i}}}\mathbf{C}^{-\mathbf{1}}(\boldsymbol{\theta})\frac{\partial\mathbf{C}(\boldsymbol{\theta})}{\partial\theta_{\mathbf{j}}}\right]$$

We know $\mathbf{C}(\boldsymbol{\theta}) = \mathbf{Q}\boldsymbol{\Sigma}(\boldsymbol{\theta})\mathbf{Q}^{\mathrm{H}}$

Asymptotic CRLB for Gaussian WSS processes

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Argumentation

- $[I(\boldsymbol{\theta})]_{ij}$ Linear in N

Asymptotic CRLB for Gaussian WSS processes

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Argumentation

- $[I(\boldsymbol{\theta})]_{ij}$ Linear in N
- Study $\frac{1}{N}[I(\boldsymbol{\theta})]_{ij}$

Asymptotic CRLB for Gaussian WSS processes

Loose Derivations

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Argumentation

- $[I(\boldsymbol{\theta})]_{ij}$ Linear in N
- Study $\frac{1}{N}[I(\boldsymbol{\theta})]_{ij}$
- But $\|\mathbf{C}_{\mathrm{Tp}} - \mathbf{C}_{\mathrm{cTp}}\|^{\mathbf{2}}$ is constant

Asymptotic CRLB for Gaussian WSS processes

Loose Derivations

We have $\mathbf{x} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{C}(\boldsymbol{\theta})\right)$

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We know $\mathbf{C}(\boldsymbol{\theta}) = \mathbf{Q}\boldsymbol{\Sigma}(\boldsymbol{\theta})\mathbf{Q}^{\mathrm{H}}$

- So $[I_{\mathrm{cTp}}(\boldsymbol{\theta})]_{ij} - [I_{\mathrm{Tp}}(\boldsymbol{\theta})]_{ij}$

  does not depend on N

Argumentation

- $[I(\boldsymbol{\theta})]_{ij}$ Linear in N
- Study $\dfrac{1}{N}[I(\boldsymbol{\theta})]_{ij}$
- But $\|\mathbf{C}_{\mathrm{Tp}} - \mathbf{C}_{\mathrm{cTp}}\|^{2}$ is constant

Asymptotic CRLB for Gaussian WSS processes

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- So $[I_{\mathrm{cTp}}(\boldsymbol{\theta})]_{ij} - [I_{\mathrm{Tp}}(\boldsymbol{\theta})]_{ij}$ does not depend on N

$$\lim_{N\to\infty} \frac{1}{N}\left[[I_{\mathrm{cTp}}(\boldsymbol{\theta})]_{ij} - [I_{\mathrm{Tp}}(\boldsymbol{\theta})]_{ij}\right] = 0$$

Argumentation

- $[I(\boldsymbol{\theta})]_{ij}$ Linear in N
- Study $\frac{1}{N}[I(\boldsymbol{\theta})]_{ij}$
- But $\left\|\mathbf{C}_{\mathrm{Tp}} - \mathbf{C}_{\mathrm{cTp}}\right\|^{\mathbf{2}}$ is constant

Asymptotic CRLB for Gaussian WSS processes

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We have $\mathbf{x} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{C}(\boldsymbol{\theta})\right)$

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We obtain $$[I(\boldsymbol{\theta})]_{ij} = \frac{1}{2}\mathrm{Tr}\left[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta})\frac{\partial \boldsymbol{\Sigma}(\boldsymbol{\theta})}{\partial \theta_i}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta})\frac{\partial \boldsymbol{\Sigma}(\boldsymbol{\theta})}{\partial \theta_j}\right]$$

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Manipulation $$[I(\boldsymbol{\theta})]_{ij} = \frac{1}{2}\sum_{n=0}^{N-1}\left[\frac{1}{\Sigma_n^2(\boldsymbol{\theta})}\frac{\partial \Sigma_n(\boldsymbol{\theta})}{\partial \theta_i}\frac{\partial \Sigma_n(\boldsymbol{\theta})}{\partial \theta_j}\right]$$

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Some function

Manipulation $$[I(\boldsymbol{\theta})]_{ij} = \frac{1}{2}\sum_{n=0}^{N-1}\left[\frac{1}{\Sigma_n^2(\boldsymbol{\theta})}\frac{\partial \Sigma_n(\boldsymbol{\theta})}{\partial \theta_i}\frac{\partial \Sigma_n(\boldsymbol{\theta})}{\partial \theta_j}\right] = \frac{1}{2}\sum_{n=0}^{N-1} F(\Sigma_n)$$

Asymptotic CRLB for Gaussian WSS processes

Robert M. Gray: Toeplitz and circulant matrices: A review

Toeplitz and Circulant Matrices: A review

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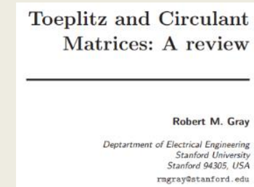
Asymptotic CRLB for Gaussian WSS processes

Robert M. Gray: Toeplitz and circulant matrices: A review


Toeplitz and Circulant Matrices: A review

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Theorem 4.2. Let $T_n(f)$ be a sequence of Toeplitz matrices such that $f(\lambda)$ is in the Wiener class or, equivalently, that $\{t_k\}$ is absolutely summable. Let $\tau_{n,k}$ be the eigenvalues of $T_n(f)$ and s be any positive integer. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \tau_{n,k}^s = \frac{1}{2\pi} \int_0^{2\pi} f(\lambda)^s \, d\lambda. \tag{4.46}$$

Furthermore, if $f(\lambda)$ is real or, equivalently, the matrices $T_n(f)$ are all Hermitian, then for any function $F(x)$ continuous on $[m_f, M_f]$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} F(\tau_{n,k}) = \frac{1}{2\pi} \int_0^{2\pi} F(f(\lambda)) \, d\lambda. \tag{4.47}$$

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Toeplitz and Circulant Matrices: A review

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Theorem 4.2. Let $T_n(f)$ be a sequence of Toeplitz matrices such that $f(\lambda)$ is in the Wiener class or, equivalently, that $\{t_k\}$ is absolutely summable. Let $\tau_{n,k}$ be the eigenvalues of $T_n(f)$ ~~and s be any positive integer. Then~~

$$\cancel{\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \tau_{n,k}^s = \frac{1}{2\pi} \int_0^{2\pi} f(\lambda)^s \, d\lambda.} \qquad (4.46)$$

Furthermore, if $f(\lambda)$ is real or, equivalently, the matrices $T_n(f)$ are all Hermitian, then for any function $F(x)$ continuous on $[m_f, M_f]$

$$\boxed{\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} F(\tau_{n,k}) = \frac{1}{2\pi} \int_0^{2\pi} F(f(\lambda)) \, d\lambda.} \qquad (4.47)$$

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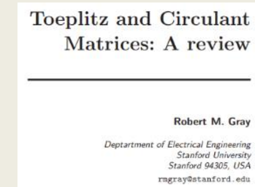
Toeplitz and Circulant Matrices: A review

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$$[I(\boldsymbol{\theta})]_{ij} = \frac{1}{2}\sum_{n=0}^{N-1}\left[\frac{1}{\Sigma_n^2(\boldsymbol{\theta})}\frac{\partial\Sigma_n(\boldsymbol{\theta})}{\partial\theta_i}\frac{\partial\Sigma_n(\boldsymbol{\theta})}{\partial\theta_j}\right] = \frac{N}{2}\frac{1}{N}\sum_{n=0}^{N-1}F(\Sigma_n)$$

Fourier transform of first column of circulant Toeplitz matrix

Theorem 4.2.

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First column of $\mathbf{C}(\boldsymbol{\theta})$: $\mathbf{C}(\boldsymbol{\theta})\mathbf{e}_1$

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First column of $\mathbf{C}(\boldsymbol{\theta})$: $\mathbf{C}(\boldsymbol{\theta})\mathbf{e}_1$

Take Fourier transform of the first column: $\mathbf{Q}^{\mathrm{H}}\mathbf{C}(\boldsymbol{\theta})\mathbf{e}_1$

Fourier transform of first column of circulant Toeplitz matrix

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First column of $\mathbf{C}(\boldsymbol{\theta})$: $\mathbf{C}(\boldsymbol{\theta})\mathbf{e}_1$

Take Fourier transform of the first column: $\mathbf{Q}^{\mathrm{H}}\mathbf{C}(\boldsymbol{\theta})\mathbf{e}_1 = \mathbf{Q}^{\mathrm{H}}\mathbf{Q}\boldsymbol{\Sigma}(\boldsymbol{\theta})\mathbf{Q}^{\mathrm{H}}\mathbf{e}_1$

Fourier transform of first column of circulant Toeplitz matrix

Theorem 4.2.

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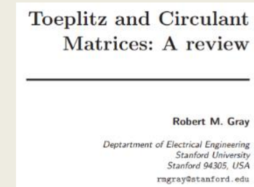
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Fourier transform of first column of circulant Toeplitz matrix

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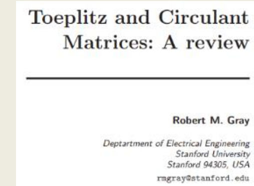
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Fourier transform of impulse

Fourier transform of first column of circulant Toeplitz matrix

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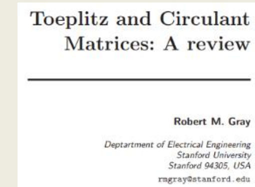
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Take Fourier transform of the first column: $\mathbf{Q}^{\mathrm{H}}\mathbf{C}(\boldsymbol{\theta})\mathbf{e}_1 = \boldsymbol{\Sigma}(\boldsymbol{\theta})\mathbf{1}$

Fourier transform of first column of circulant Toeplitz matrix

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Asymptotic CRLB for Gaussian WSS processes

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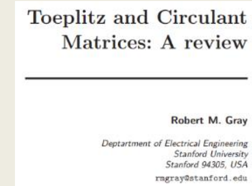
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Take Fourier transform of the first column: $\mathbf{Q}^{\mathrm{H}}\mathbf{C}(\boldsymbol{\theta})\mathbf{e}_1 = \operatorname{diag}(\boldsymbol{\Sigma}(\boldsymbol{\theta}))$

Fourier transform of first column of circulant Toeplitz matrix = Eigenvalues

Theorem 4.2.

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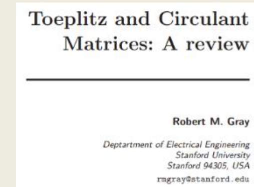
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Eigenvalues of covariance = Fourier transform of first column = PSD of process

Fourier transform of first column of circulant Toeplitz matrix = Eigenvalues

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Eigenvalues of covariance = Fourier transform of first column = PSD of process

$$[I(\boldsymbol{\theta})]_{ij} = \frac{N}{2}\int_{-1/2}^{1/2}\frac{1}{\mathrm{PSD}(f;\boldsymbol{\theta})}\frac{\partial \mathrm{PSD}(f;\boldsymbol{\theta})}{\partial\theta_i}\frac{1}{\mathrm{PSD}(f;\boldsymbol{\theta})}\frac{\partial \mathrm{PSD}(f;\boldsymbol{\theta})}{\partial\theta_j}\mathrm{d}f$$

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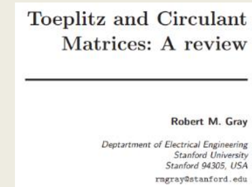
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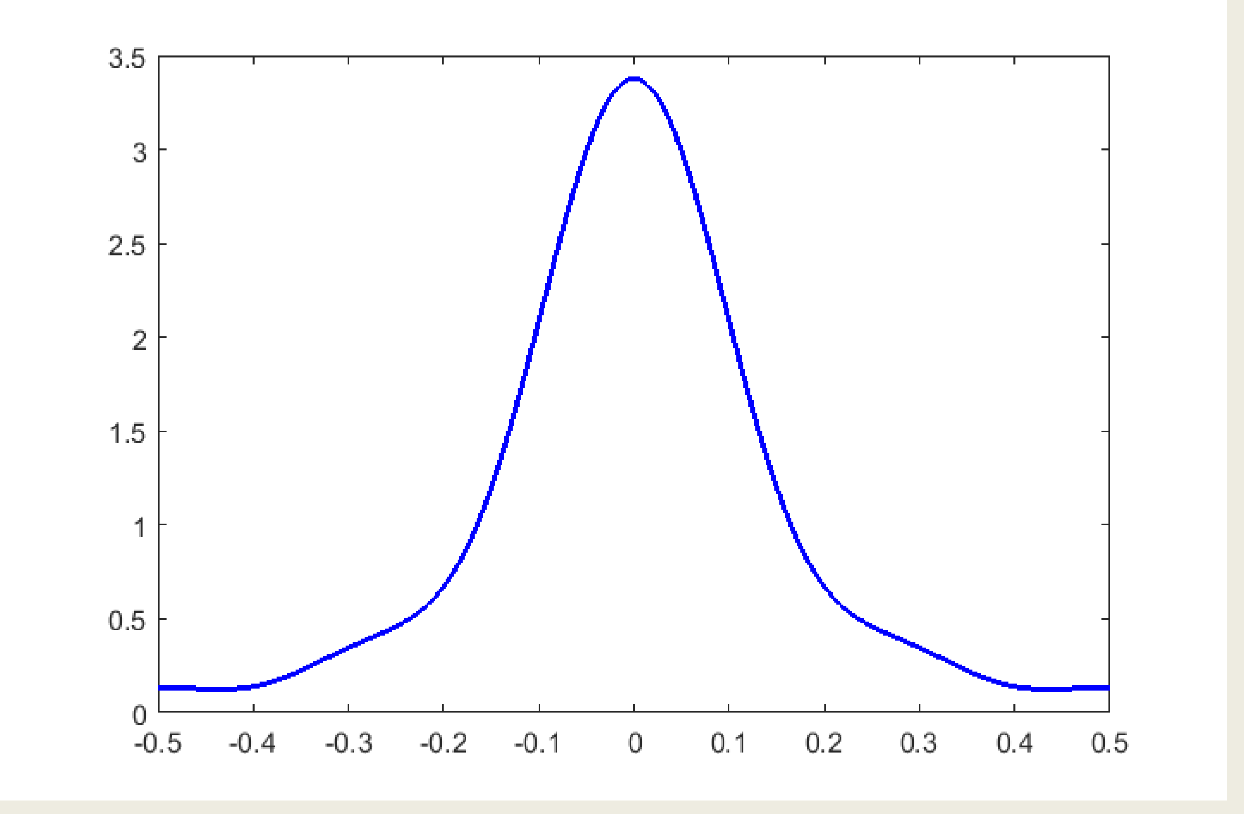
$$= \frac{N}{2}\int_{-1/2}^{1/2}\frac{\partial\ln\mathrm{PSD}(f;\boldsymbol{\theta})}{\partial\theta_i}\frac{\partial\ln\mathrm{PSD}(f;\boldsymbol{\theta})}{\partial\theta_j}\mathrm{d}f$$

Asymptotic CRLB for Gaussian WSS processes

Illustration

Fourier transform of first column of circulant Toeplitz

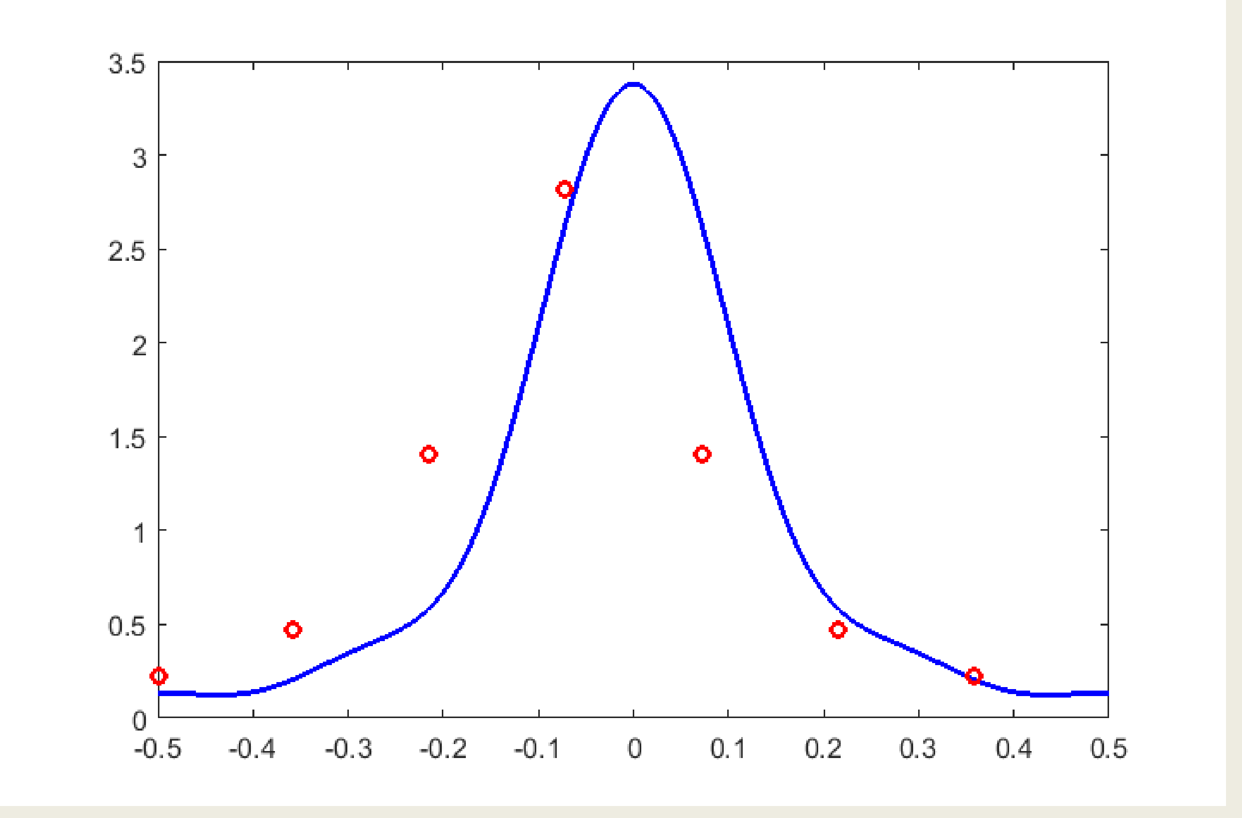


Asymptotic CRLB for Gaussian WSS processes

Illustration

Eigenvalues with size 7x7 Toeplitz

Fourier transform of first column of circulant Toeplitz



Asymptotic CRLB for Gaussian WSS processes

Illustration

Eigenvalues with size 7x7 Toeplitz

Eigenvalues with size 20x20 Toeplitz

Fourier transform of first column of circulant Toeplitz

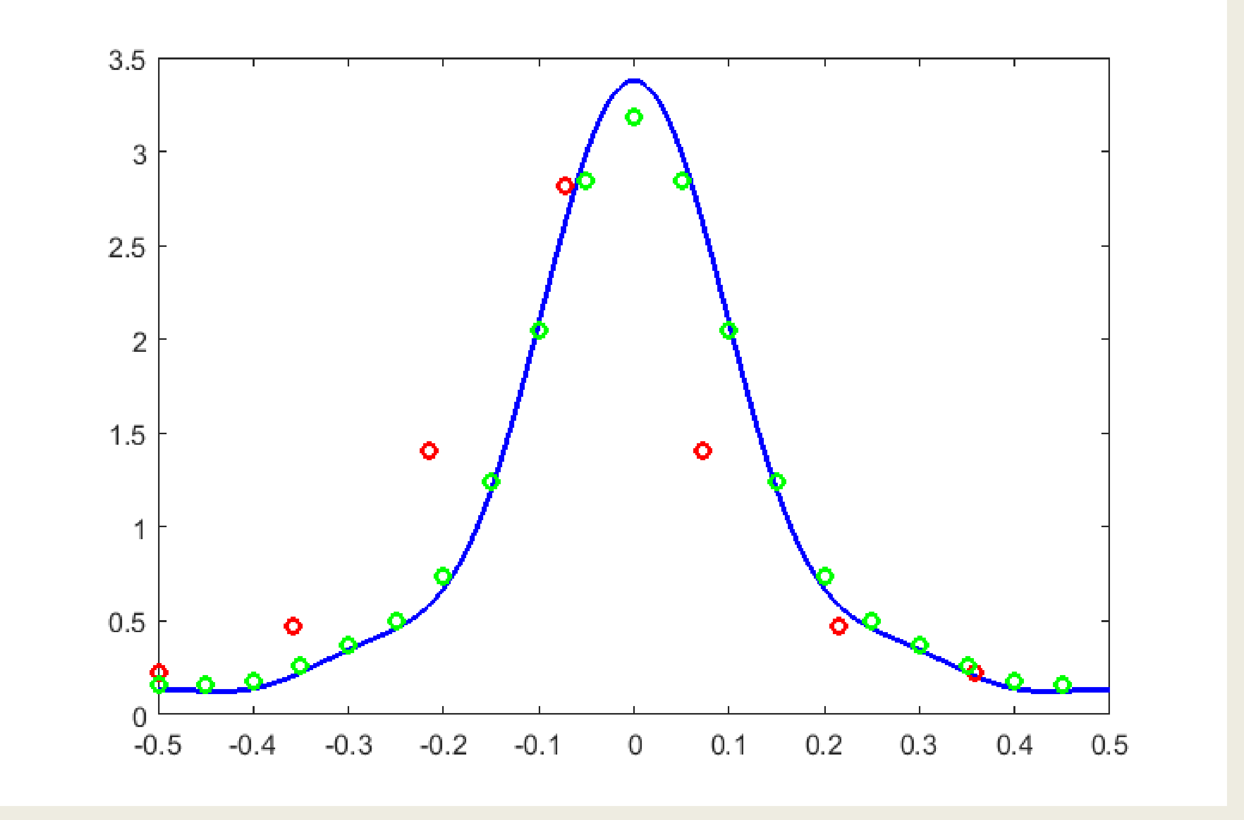


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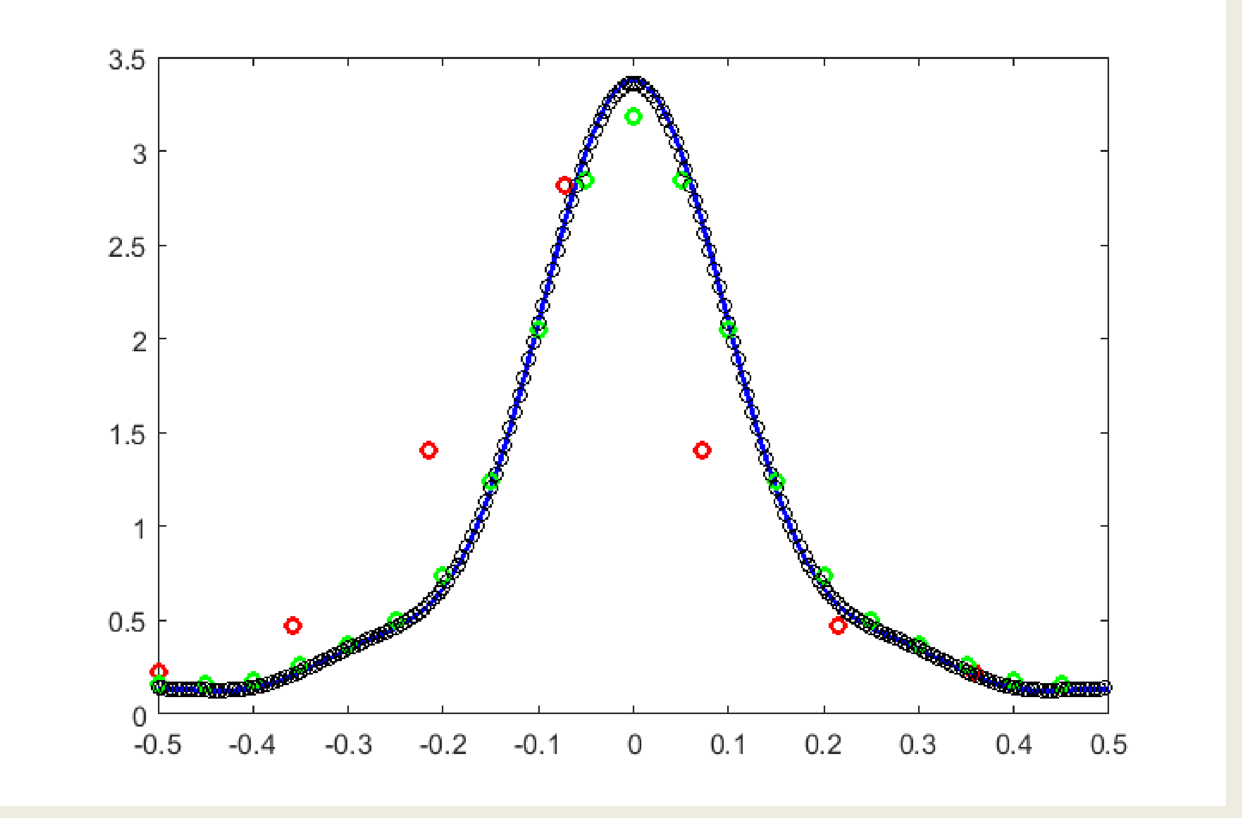
Illustration

Eigenvalues with size 7x7 Toeplitz

Eigenvalues with size 20x20 Toeplitz

Fourier transform of first column of circulant Toeplitz



Eigenvalues with size 200x200 Toeplitz