### Estimation Theory Fredrik Rusek

Introduction Chapters 1-3.4

## Schedule, tentative

- Lectures: Tuesdays 10-12 in E:3139
  - Easter break?
  - 19/2 cancelled
  - Possibly 26/2 needs to be moved
- Seminars: (about) every 2-3 weeks

## Contents

- Chapters 1-13 in the book (+ some extra minor material)
- I don't know how many lectures that we need, depends on the amount of discussion at the lectures
- 8-12 lectures is a good guess

## Examination

- Reqs for passing degree (Alt. 1)
  - About 80% attendance at lectures
  - About 80% attendance at seminars
  - Presented 2 problems during seminars
- Reqs for passing degree (Alt. 2)
  - Attend as much as you can
  - Solve a lot of homework problems and hand them in
- 9 ECTS

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If it was a detection problem, we would have tried to estimate the **presence** of an airplane (0/1)

In Detection Theory, we try to estimate the value of a discrete variable

Simply put,

In detection theory, we are either right or wrong In estimation theory, we are <u>always</u> wrong



**Remark:** When the cardinality of the set is large,

discrete problems are usually classified as estimation theory problems

Two famous examples:

- German tank problem
  - Estimate number of produced german tanks per month based on the number of tanks you observe at battlefield
- Doomsday problem
  - Estimate how many more humans that will be born in the future from the number of humans born so far (around 10<sup>11</sup>)

#### Some words about the book

- Simple to read
- For engineers, not for mathematicians
- More than half of the book is examples
- Many examples are used throughout the book
- Almost no proofs
- 99% based on discrete time. The reader is assumed to be able to convert continuous time to discrete on his own. Possibly, we will add one lecture at the end dealing with this shortcoming
- First part is assuming deterministic parameters to estimate, second part is assuming random parameters

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May be confusing if you have background on the subject, for example knowledge of the MMSE estimator for MIMO communications

#### Mathematical formulation of the problem we'll study

$$\hat{\theta} = g(x[0], x[1], \dots, x[N-1])$$

- x[n] is a sequence of, possibly dependent, observations. These observations carry information about a parameter θ that we would like to estimate
- We do this by constructing a (deterministic) function g(x) that produces an estimate of θ



#### Likelihood functions vs conditional probabilities

- The data x=x[0]...x[N-1] is of course dependent on the parameter we would like to estimate, θ, in some way
- We denote by p(x;θ) a family of PDFs parameterized by θ. In words,
  "This is the pdf that x will abide if the the unknown parameter is θ"
- Note that  $\theta$  is, on the most basic level, not a random variable
- If  $\theta$  was the realization of a random process, then we have the conditional pdf p(**x**| $\theta$ )

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- If  $\theta$  was the realization of a random process, then we have the conditional pdf  $p(\mathbf{x}|\theta)$
- Note that a likelihood and a conditional pdf have the same formulas, it is only the interpretation of them that differ

#### Likelihood functions vs conditional probabilities



Given the functional form of the family of likelihoods,  $p(x[0];\theta)$ , we can infer the value of  $\theta$  from an observation x[0]

For example, if x[0]<0, then it is unlikely that  $\theta = \theta_2$ 

**Unbiased estimators** 

$$\hat{\theta} = g(x[0], x[1], \dots, x[N-1])$$

Recall that the estimator is

- A function only of **x**
- Random, since **x** is random

Since  $\hat{\theta}$  is random, it has an expectation  $E(\hat{\theta})$ .

The estimator is unbiased if  $E(\hat{\theta}) = \theta, \forall \theta$ 

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$$E(\hat{\theta}) = \int g(\mathbf{x}) p(\mathbf{x}; \theta) \, d\mathbf{x} = \theta$$
 for all  $\theta$ .

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Note: The variance of the estimator depends on  $\boldsymbol{\theta}$ 

**Quality of estimator** 

A natural criterion may be the mean square error

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However,

$$\mathbf{mse}(\hat{\theta}) = E\left\{\left[\left(\hat{\theta} - E(\hat{\theta})\right) + \left(E(\hat{\theta}) - \theta\right)\right]^2\right\}$$
$$= \operatorname{var}(\hat{\theta}) + \left[E(\hat{\theta}) - \theta\right]^2$$
$$= \operatorname{var}(\hat{\theta}) + b^2(\theta)$$

where  $b(\theta)$  is the *bias*:  $b(\theta) = E(\hat{\theta}) - \theta$ 

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However,

$$\mathbf{mse}(\hat{\theta}) = E\left\{\left[\left(\hat{\theta} - E(\hat{\theta})\right) + \left(E(\hat{\theta}) - \theta\right)\right]^2\right\}$$
$$= \operatorname{var}(\hat{\theta}) + \left[E(\hat{\theta}) - \theta\right]^2 + 2E\left(\left[\hat{\theta} - E(\hat{\theta})\right]b(\theta)\right)$$
$$= \operatorname{var}(\hat{\theta}) + b^2(\theta) = 0$$

where  $b(\theta)$  is the bias:  $b(\theta) = E(\hat{\theta}) - \theta$ 

The dependence of the bias on  $\theta$  is **bad news**, as is shown with an example next

Example 2.1 - Unbiased Estimator for DC Level in White Gaussian Noise

$$x[n] = A + w[n]$$
  $n = 0, 1, ..., N - 1$ 

A: DC level to be estimated

- w[n], zero mean white Gaussian noise

Proposed estimator:  $\hat{A} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$ 

Unbiased

$$E(\hat{A}) = E\left[\frac{1}{N}\sum_{n=0}^{N-1}x[n]\right] = A$$

Variance

$$\operatorname{var}(\hat{A}) = \operatorname{var}\left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right) = \frac{1}{N^2}\sum_{n=0}^{N-1} \operatorname{var}(x[n]) = \frac{\sigma^2}{N}$$

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Modified estimator: 
$$\check{A} = a \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

bias (a –

$$(a-1)A$$

$$\operatorname{mse}(\check{A}) = \operatorname{var}(\hat{\theta}) + b^2(\theta) = \frac{a^2\sigma^2}{N} + (a-1)^2A^2.$$

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**Modified estimator:** 
$$\check{A} = a \frac{1}{N} \sum_{n=0}^{N-1} x[n]$$

$$\frac{d\operatorname{mse}(\check{A})}{da} = \frac{2a\sigma^2}{N} + 2(a-1)A^2 \qquad \qquad a_{\operatorname{opt}} = \frac{A^2}{A^2 + \sigma^2/N}$$

$$\operatorname{mse}(\check{A}) = \operatorname{var}(\hat{\theta}) + b^2(\theta) = \frac{a^2 \sigma^2}{N} + (a-1)^2 A^2.$$

#### The optimal estimator depends on A -> Not realizable!!

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We seek a function  $g(\mathbf{x})$  that estimates a parameter  $\theta$  well

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- The natural performance metric is mse  $\operatorname{mse}(\hat{\theta}) = E\left[(\hat{\theta} \theta)^2\right]$ 
  - But this does not work as the bias term b( $\theta$ ) makes the optimal function g(**x**) dependent on  $\theta$
- Restrict to the class of unbiased estimators, i.e.,

$$E(\hat{\theta}) = \int g(\mathbf{x})p(\mathbf{x};\theta) d\mathbf{x} = \theta$$
 for all  $\theta$ .

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GOAL: Find unbiased  $g(\mathbf{x})$  with as small  $\operatorname{var}(\hat{\theta})$  as possible

**Check point** 

It is easy to find  $g(\mathbf{x})$  with a small variance,  $var(\hat{\theta})$ , for example  $g(\mathbf{x})=0$  has zero variance

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But g(x)=0 is not unbiased

Minimum variance unbiased estimator (MVU)

Recall, the variance of the estimator,  $var(\hat{\theta})$ , depends on  $\theta$ , and to be the optimal estimator, it must provide the smallest variance for all  $\theta$ 



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The MVU exists only in some cases, see example 2.3 for a case where it does not exist Sometimes, there is not even any unbiased estimator at all

Unbiasedness for Vector parameters  $\boldsymbol{\theta} = [\theta_1 \theta_2 \dots \theta_p]^T$ 

$$E(\hat{\theta}_i) = \theta_i$$
 or  $E(\hat{\theta}) = \theta$ 

#### **MVU for Vector parameters**

 $var(\hat{\theta}_i)$  for i = 1, 2, ..., p is minimum among all unbiased estimators.

### **Chapter 3 – Cramer-Rao lower bound**

#### The CRLB is useful in the following ways:

- It provides a lower bound on the variance of any unbiased estimator
- If an MVU exists, the function g(x) will fall out as a side result

#### Some properties of CRLB

- Derived for unbiased estimators (can easily be extended to biased, but not in this book)
- The bound is not always reachable (it is only a lower bound)
- A regularity condition must hold, so the CRLB cannot always be applied
- Better bounds exist (more about this later)
- Cramér and Rao proved it independently in the mid 40s
- French mathematician Frechét proved it earlier, but never published



For case (a): the value A=5 is not very likely For case (b): A=5 cannot be ruled out



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In fact, the *sharpness* around x[0]=3 is important If p(x[0];A) is very sharp around A=x[0], then we can expect to be able to infer the value of A with a much better precision


In mathematics, "sharpness" has the name curvature The curvature is measured by the reciprocal of the radius of the circle



But, at the correct value, an unbiased estimator has 0 slope, so that we get



$$\kappa = |y''|$$



Further, multiplicative constants should not affect the precision, e.g.,  $-x^2$  should be as good as the likelihood  $-2x^2$ 

Taking the log of the likelihood removes these constants. Hence, the "sharpness" can be effectively measured by

 $\frac{\partial^2 \ln p(x[0];A)}{\partial A^2}$ 

 $\partial^2 \ln p(x[0];A)$ 



while we get the below plot if it is small









Example 3.1 - PDF Dependence on Unknown Parameter

x[0] = A + w[0] Estimate A $w[0] \sim \mathcal{N}(0,\sigma^2)$ 

$$\hat{A} = x[0]$$
  $\operatorname{var}(\hat{A}) = \sigma^2$ 

Likelihood 
$$p(x[0]; A) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(x[0] - A)^2\right]$$

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**Find curvature** 

$$\begin{aligned} &: \quad \ln p(x[0];A) = -\ln \sqrt{2\pi\sigma^2} - \frac{1}{2\sigma^2} (x[0] - A)^2 \\ &\quad \frac{\partial \ln p(x[0];A)}{\partial A} = \frac{1}{\sigma^2} (x[0] - A) \\ &\quad - \frac{\partial^2 \ln p(x[0];A)}{\partial A^2} = \frac{1}{\sigma^2} \end{aligned}$$

Example 3.1 - PDF Dependence on Unknown Parameter

$$x[0] = A + w[0] \text{ Estimate A}$$

$$w[0] \sim \mathcal{N}(0, \sigma^2)$$

$$\hat{A} = x[0] \quad \text{var}(\hat{A}) = \sigma^2$$

$$\text{Likelihood} \quad p(x[0]; A) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(x[0] - A)^2\right]$$

$$\text{Eikelihood} \quad p(x[0]; A) = -\ln\sqrt{2\pi\sigma^2} - \frac{1}{2\sigma^2}(x[0] - A)^2$$

$$\frac{\partial \ln p(x[0]; A)}{\partial A} = \frac{1}{\sigma^2}(x[0] - A)$$

$$-\frac{\partial^2 \ln p(x[0]; A)}{\partial A^2} = \frac{1}{\sigma^2}$$

#### Interlude

In the previous example, the second derivative did not depend on x[0], but it will in general  $-\frac{\partial^2 \ln p(x[0]; A)}{\partial A^2} = \frac{1}{\sigma^2}$ 

Curvature is a measure of precision, but we cannot have measure of precision that depends on the particular realization of **x** 

**Solution:** Find expected curvature  $-E\left[\frac{\partial^2 \ln p(x[0]; A)}{\partial A^2}\right]$ 

This is independent of **x** 

**Theorem 3.1 (Cramer-Rao Lower Bound - Scalar Parameter)** It is assumed that the PDF  $p(\mathbf{x}; \theta)$  satisfies the "regularity" condition

$$E\left[rac{\partial \ln p(\mathbf{x}; heta)}{\partial heta}
ight] = 0 \qquad \textit{for all } heta$$

where the expectation is taken with respect to  $p(\mathbf{x}; \theta)$ . Then, the variance of any unbiased estimator  $\hat{\theta}$  must satisfy

$$\operatorname{var}(\hat{\theta}) \geq \frac{1}{-E\left[\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2}\right]}$$
(3.6)

where the derivative is evaluated at the true value of  $\theta$  and the expectation is taken with respect to  $p(\mathbf{x}; \theta)$ . Furthermore, an unbiased estimator may be found that attains the bound for all  $\theta$  if and only if

$$\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta} = I(\theta)(g(\mathbf{x}) - \theta)$$
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for some functions g and I. That estimator, which is the MVU estimator, is  $\hat{\theta} = g(\mathbf{x})$ , and the minimum variance is  $1/I(\theta)$ .

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# Reciprocal of curvature bounds precision

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where the <u>derivative is evaluated at the true value of  $\theta$ </u> and the expectation is taken with respect to  $p(\mathbf{x}; \theta)$ .

# The CRLB *only* takes into account what the likelihood looks like around the correct value

#### This is fine, since it is a lower bound

#### A complicated pdf with many peaks will lead to estimators very far away from the CRLB

**Examples of better bounds:** Weiss-Weinstein, Bobrovsky-Zakai etc etc

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#### **Proof.**

Check the regularity condition

$$E\left[\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right] = 0$$

#### **Proof.**

$$E\left[\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right] = \int \frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta} p(\mathbf{x};\theta) \, d\mathbf{x}$$

#### **Proof.**

$$E\left[\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right] = \int \frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta} p(\mathbf{x};\theta) d\mathbf{x}$$
$$= \int \frac{\partial p(\mathbf{x};\theta)}{\partial \theta} d\mathbf{x}$$

#### **Proof.**

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$$= \int \frac{\partial p(\mathbf{x};\theta)}{\partial \theta} d\mathbf{x}$$
$$= \frac{\partial}{\partial \theta} \int p(\mathbf{x};\theta) d\mathbf{x}$$

#### Proof.

Check the regularity condition  $E\left[\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right] = 0$ 

(derivation and integration  $= \frac{\partial}{\partial \theta} \int p(\mathbf{x}; \theta) d\mathbf{x}$ 

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Alternative regularity condition (plus some differentiability conditions that I don't mention)

can be interchanged)

 $B = \{x: p(x; \theta) > 0\}$ Not dependent on  $\theta$ 

#### Proof.

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The regularity condition is in most books that this equality must hold (derivation and integration can be interchanged)

$$= \int \frac{\partial p(\mathbf{x}; \theta)}{\partial \theta} d\mathbf{x}$$
$$= \frac{\partial}{\partial \theta} \int p(\mathbf{x}; \theta) d\mathbf{x}$$

Alternative regularity condition (plus some differentiability conditions that I don't mention)

 $B = \{x: p(x; \theta) > 0\}$ Not dependent on  $\theta$  Example of likelihood that violates the regularity condition:

$$p(x;\theta) = \frac{1}{\theta}, \qquad 0 \le x \le \theta$$

#### **Proof.**

$$E\left[\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right] = \int \frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta} p(\mathbf{x};\theta) d\mathbf{x}$$
$$= \int \frac{\partial p(\mathbf{x};\theta)}{\partial \theta} d\mathbf{x}$$
$$= \frac{\partial}{\partial \theta} \int p(\mathbf{x};\theta) d\mathbf{x}$$
$$= \frac{\partial 1}{\partial \theta}$$
$$= 0$$

#### **Proof.(interlude)**

Interchanging derivation and integration allows us (at least me) to actually understand the regularity condition

$$E\left[\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right] = 0$$

$$E\left[\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right] = \frac{\partial}{\partial \theta} E\left[\ln p(\mathbf{x};\theta)\right] = 0$$

This is independent of x.

Can be interpreted as the expected likelihood if the parameter of interest has a certain value  $\boldsymbol{\theta}$ 

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Or, no matter what  $\theta$  is, you must get the same value of the log-likelihood on average

#### **Proof.(generalization)**

The proof in the book is more general than the statement of the theorem. However, the generalization in the proof will be used later in the book

We assume some underlying parameter  $\theta$ , but that we would like to estimate a function thereof  $\alpha = g(\theta)$ .

For example,  $\theta$  can be the DC level, but we are more interested in the power of the DC level, so that  $\alpha = \theta^2$ , i.e.,  $g(x)=x^2$ 

#### Warning: the book uses g(x) both for the estimator function

$$\hat{\theta} = g(x[0], x[1], \ldots, x[N-1])$$

and for

$$\alpha = g(\theta)$$

#### Proof

or

The unbiased condition now reads

$$E(\hat{\alpha}) = \alpha = g(\theta)$$
$$\int \hat{\alpha} p(\mathbf{x}; \theta) \, d\mathbf{x} = g(\theta)$$

Take differential (and change order of differentiation and integration)

$$\int \hat{\alpha} \frac{\partial p(\mathbf{x}; \theta)}{\partial \theta} d\mathbf{x} = \int \hat{\alpha} \frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta} p(\mathbf{x}; \theta) d\mathbf{x} = \frac{\partial g(\theta)}{\partial \theta}$$

#### Proof

The unbiased condition now reads

or 
$$\int \hat{\alpha} p(\mathbf{x}; \theta) \, d\mathbf{x} = g(\theta);$$

Take differential (and change order of differentiation and integration)

 $E(\hat{\alpha}) = \alpha = a(\theta)$ 

$$\int \hat{\alpha} \frac{\partial p(\mathbf{x}; \theta)}{\partial \theta} d\mathbf{x} = \int \hat{\alpha} \frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta} p(\mathbf{x}; \theta) d\mathbf{x} = \frac{\partial g(\theta)}{\partial \theta}$$

Use regularity condition (I added a "0" term to the I.h.s.)

$$\int (\hat{\alpha} - \alpha) \frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta} p(\mathbf{x}; \theta) \, d\mathbf{x} = \frac{\partial g(\theta)}{\partial \theta}$$

**Proof (Interlude: 3-term Cauchy-Schwarz)** 

$$\left[\int w(\mathbf{x})g(\mathbf{x})h(\mathbf{x})\,d\mathbf{x}\right]^2 \leq \int w(\mathbf{x})g^2(\mathbf{x})\,d\mathbf{x}\int w(\mathbf{x})h^2(\mathbf{x})\,d\mathbf{x}$$

equality if and only if  $g(\mathbf{x}) = ch(\mathbf{x})$  for c some constant not dependent on  $\mathbf{x}$ 

#### Proof

$$\left[\int w(\mathbf{x})g(\mathbf{x})h(\mathbf{x})\,d\mathbf{x}\right]^2 \leq \int w(\mathbf{x})g^2(\mathbf{x})\,d\mathbf{x}\int w(\mathbf{x})h^2(\mathbf{x})\,d\mathbf{x}$$

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Compare now with what we had before

$$\int (\hat{\alpha} - \alpha) \frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta} p(\mathbf{x}; \theta) d\mathbf{x} = \frac{\partial g(\theta)}{\partial \theta}$$
$$w(\mathbf{x}) = p(\mathbf{x}; \theta)$$
$$g(\mathbf{x}) = \hat{\alpha} - \alpha$$
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#### Proof

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$$h(\mathbf{x}) = \frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta}$$
$$(\frac{\partial g(\theta)}{\partial \theta})^2 \leq \int (\hat{\alpha} - \alpha)^2 p(\mathbf{x}; \theta) d\mathbf{x} \int \left(\frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta}\right)^2 p(\mathbf{x}; \theta) d\mathbf{x}$$

#### Proof

$$\left(\frac{\partial g(\theta)}{\partial \theta}\right)^2 \leq \int (\hat{\alpha} - \alpha)^2 p(\mathbf{x}; \theta) \, d\mathbf{x} \int \left(\frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta}\right)^2 p(\mathbf{x}; \theta) \, d\mathbf{x}$$

But this can be written as

$$\operatorname{var}(\hat{\alpha}) \geq \frac{\left(\frac{\partial g(\theta)}{\partial \theta}\right)^2}{E\left[\left(\frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta}\right)^2\right]}$$

#### Proof

$$\left(\frac{\partial g(\theta)}{\partial \theta}\right)^2 \leq \int (\hat{\alpha} - \alpha)^2 p(\mathbf{x}; \theta) \, d\mathbf{x} \int \left(\frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta}\right)^2 p(\mathbf{x}; \theta) \, d\mathbf{x}$$

But this can be written as



This is not the same as is stated in the theorem, so we are not done yet

#### Proof

#### We have

$$E\left[\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right] = 0$$

$$\int \frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta} p(\mathbf{x};\theta) d\mathbf{x} = 0$$



#### Proof

We have

$$E\left[\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right] = 0$$

$$\int \frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta} p(\mathbf{x};\theta) d\mathbf{x} = 0$$



Take one more differential, and change order

$$\frac{\partial}{\partial \theta} \int \frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta} p(\mathbf{x}; \theta) \, d\mathbf{x} = 0$$
$$\int \left[ \frac{\partial^2 \ln p(\mathbf{x}; \theta)}{\partial \theta^2} p(\mathbf{x}; \theta) + \frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta} \frac{\partial p(\mathbf{x}; \theta)}{\partial \theta} \right] d\mathbf{x} = 0$$


#### Proof

$$\operatorname{var}(\hat{\alpha}) \geq \frac{\left(\frac{\partial g(\theta)}{\partial \theta}\right)^2}{-E\left[\frac{\partial^2 \ln p(\mathbf{x}; \theta)}{\partial \theta^2}\right]}$$

If  $\, lpha = g( heta) = heta_{,}\,$  the statement of the theorem follows

**Theorem 3.1 (Cramer-Rao Lower Bound - Scalar Parameter)** It is assumed that the PDF  $p(\mathbf{x}; \theta)$  satisfies the "regularity" condition

$$E\left[\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right] = 0 \qquad \text{for all } \theta$$

where the expectation is taken with respect to  $p(\mathbf{x}; \theta)$ . Then, the variance of any unbiased estimator  $\hat{\theta}$  must satisfy

$$\operatorname{var}(\hat{\theta}) \geq \frac{1}{-E\left[\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2}\right]}$$
(3.6)

where the derivative is evaluated at the true value of  $\theta$  and the expectation is taken with respect to  $p(\mathbf{x}; \theta)$ .

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### **Proof (last part)**

Lets go back and check the condition for equiity in the C-S inequality

$$\left[\int w(\mathbf{x})g(\mathbf{x})h(\mathbf{x})\,d\mathbf{x}\right]^2 \leq \int w(\mathbf{x})g^2(\mathbf{x})\,d\mathbf{x}\int w(\mathbf{x})h^2(\mathbf{x})\,d\mathbf{x}$$

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$$w(\mathbf{x}) = p(\mathbf{x}; \theta) = g(\mathbf{x}; \theta)$$

$$g(\mathbf{x}) = \hat{\alpha} - \alpha = \frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta}$$

So, for equality we must have

$$\frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta} = \frac{1}{c} (\hat{\alpha} - \alpha)$$

Where c can depend on  $\theta$ , but *not* on **x** 

#### **Proof (last part)**

Now, assume  $\alpha = g(\theta) = \theta$  (other g() are discussed in later sections) which yields

$$\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta} = \frac{1}{c} (\hat{\alpha} - \alpha) \qquad \longrightarrow \qquad \frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta} = \frac{1}{c(\theta)} (\hat{\theta} - \theta)$$

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Take one more differential:

$$\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2} = -\frac{1}{c(\theta)} + \frac{\partial \left(\frac{1}{c(\theta)}\right)}{\partial \theta} (\hat{\theta} - \theta)$$

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Take expectation with respect to **x** and use the unbiasedness of the estimator

$$-E\left[\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2}\right] = \frac{1}{c(\theta)}$$

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Take expectation with respect to **x** and use the unbiasedness of the estimator

$$-E\left[\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2}\right] = \frac{1}{c(\theta)} \qquad c(\theta) = \frac{1}{-E\left[\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2}\right]} = \frac{1}{I(\theta)}$$

**Theorem 3.1 (Cramer-Rao Lower Bound - Scalar Parameter)** It is assumed that the PDF  $p(\mathbf{x}; \theta)$  satisfies the "regularity" condition

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where the derivative is evaluated at the true value of  $\theta$  and the expectation is taken with respect to  $p(\mathbf{x}; \theta)$ . Furthermore, an unbiased estimator may be found that attains the bound for all  $\theta$  if and only if

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#### Some remarks

- The CRLB depends on  $\theta$
- An estimator that meets the CRLB is said to be *efficient*
- An efficient estimator is not the same as an MVU estimator



### Example (3.2, a look at 3.1 again)

#### Example 3.1 - PDF Dependence on Unknown Parameter

x[0] = A + w[0]	Estimate A
$w[0] \sim \mathcal{N}(0,\sigma^2)$	

In this example, we just guessed that x[0] would be a good estimator for A.

$$\hat{A} = x[0]$$
 var $(\hat{A}) = \sigma^2$ ,

But what if we cannot make such guess?

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Use the CRLB theorem

Furthermore, an unbiased estimator may be found that attains the bound for all  $\theta$  if and only if

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But what if we cannot make such guess? Use the CRLB theorem

$$\frac{\partial \ln p(x[0];A)}{\partial A} = \frac{1}{\sigma^2} (x[0] - A)$$

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Now identify  $\theta = A$  $I(\theta) = \frac{1}{\sigma^2}$ q(x[0]) = x[0]

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#### Example 3.1 - PDF Dependence on Unknown Parameter

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$$\frac{\partial \ln p(x[0];A)}{\partial A} = \frac{1}{\sigma^2} (x[0] - A)$$

Now identify

$$\theta = A$$

$$I(\theta) = \frac{1}{\sigma^2}$$

$$g(x[0]) = x[0]$$

In most cases we cannot express the derivative in this way, and an efficient estimator does not exist in those cases