Cryptology – something old and something new

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Cryptology, what is that?

- the study of *cryptographic primitives*
- a cryptographic primitive - low-level cryptographic algorithms (black-box description), building block, ...
- a cryptographic primitive contains a mathematically oriented description
- cryptographic primitives are used to build *secure systems*

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Cryptographic primitives

- Cryptographic hash function, (compute a hash value for a message (SHA-256))
- Symmetric-key encryption (encrypt/decrypt with a shared key (AES))
- Public-key encryption (encrypt with a public key and decrypt with a different secret key (RSA))
- Digital signatures (confirm the authorship and authenticity of a message by its signature (DSS))
- Mix network (pool communications from many users to anonymize what came from whom)
- Private information retrieval (get database information without server knowing which item was requested)
- Commitment scheme (commit to a chosen value hidden to others, with the ability to reveal it later)

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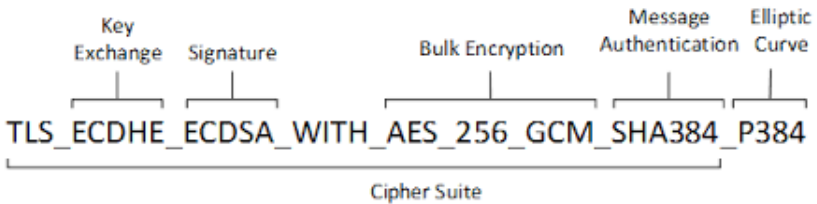
Example of an application

- HTTPS protocol through TLS
- secure communication over a computer network
- you need a Public Key Infrastructure, key agreement, signatures, encryption and data integrity



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Cryptographers, who are they?

- mathematicians, theoretical computer scientists, electrical engineers
- International Association for Cryptologic Research, IACR, iacr.org

discrete algebraic structures, probability theory and statistics

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Historic view

- secrecy

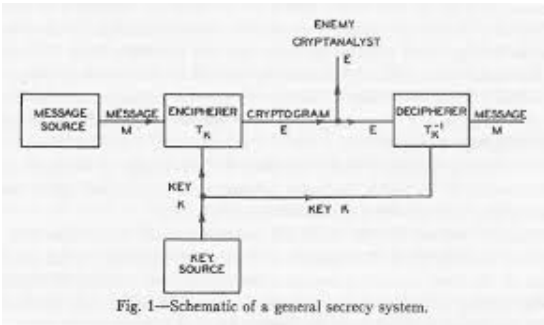

Fig. 1—Schematic of a general secrecy system.

(Shannon, 1949)

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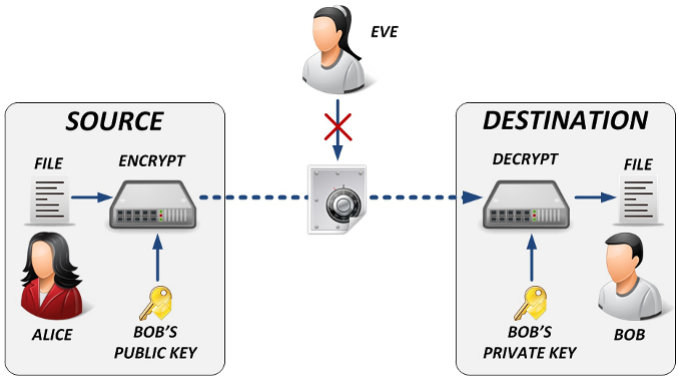
The big game changer I

- the birth of Public Key Cryptography (Diffie, Hellman 1976 and R. Merkle)



Public-key encryption scheme

- A set of encryption transformations $\{E_e : e \in \mathcal{K}\}$ and a set of decryption transformations $\{D_d : d \in \mathcal{K}\}$. For each $e \in \mathcal{K}$ there is a corresponding $d \in \mathcal{K}$ such that $D_d(E_e(M)) = M$, $\forall M$.
- Furthermore, after choosing such a pair (e, d), the *public key* e (or the public parameter) is made public, while the associated *secret key* d is kept secret.
- For the scheme to be secure, it must be computationally infeasible to compute d as well as computing $E_e^{-1}(C)$, knowing the public value e.

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The RSA public-key encryption scheme

Let $n = pq$, where p and q are two large primes. Let $\mathcal{M} = \mathcal{C} = \mathbb{Z}_n$. Pick a number e relatively prime to $\phi(n)$ and calculate a number d such that $ed = 1 \bmod \phi(n)$. The public key is the two numbers (n, e) and the (public) encryption transformation $E(M)$ is

$$E(M) = M^e \bmod n.$$

The secret key is the number d (as well as p, q and $\phi(n)$) and the secret decryption transformation $D(C)$ is

$$D(C) = C^d \bmod n.$$

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Verify that decrypting a ciphertext returns the encrypted plaintext.

$$D(C) = C^d = (M^e)^d = M^{ed} \bmod n.$$

Now we note that $ed = 1 \bmod \phi(n)$, which means that we can write

$$ed = 1 + t \cdot \phi(n),$$

for some integer t. S we can continue

$$D(C) = M^{ed} = M^{(1+t\cdot\phi(n))} = M \cdot M^{t\cdot\phi(n)} \bmod n.$$

From Euler's formula we know that $x^{\phi(n)} = 1$ for any $x \in \mathbb{Z}_n^*$. So assuming that M is invertible we have

$$D(C) = M \cdot M^{t\cdot\phi(n)} = M \cdot 1 \bmod n.$$

Security of the RSA cryptosystem

- If we can factor n then we can compute $\phi(n)$ and d.
- RSA relies on the *factoring problem*.
- This does not mean that breaking RSA is equivalent to solving a factorization problem. It is not known whether RSA can be broken without factoring n.

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The computational complexity of factoring

- We often use the function

$$L_n(\alpha, \beta) = \exp((\beta + o(1))(\log n)^{\alpha}(\log \log n)^{1-\alpha}).$$

- An algorithm with complexity $O(L_n(\alpha, \beta))$ for $0 < \alpha < 1$ is said to have *sub-exponential* behaviour.
- Number Field Sieve: This is currently the most successful method for numbers with more than 100 decimal digits. It can factor numbers of the size of 2^{512} and has complexity $L_n(1/3, 1.923)$.
- For long-term security one would need to take a size of over 2048 bits.
- RSA-768 was factored in 2009.

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How fast can we compute a ciphertext?

- Complexity of computing $x^e \bmod n$?
- Complexity of computing $xy \bmod n$?
- Basic school book n^2; Karatsuba $n^{1.58}$; ...

$$n = 279978339112213278708294676387226016210704467869554285375600099293261284001076093456710529553608560618223519109513657886371059544820065767750985805576135790987349501441788631789462951872378692218239 83.$$

It still requires many operations to compute $M^e \bmod n$ ...

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The El Gamal cryptosystem (1985)

- Domain parameters: p large prime; g an element in $\mathbb{Z}_p^*$.
- Public key: $h = g^x \bmod p$
- Encryption: Pick a random $k \in \mathbb{Z}_p$. For message $m \in \mathbb{Z}_p$ generate the ciphertext

$$(c_1, c_2) = (g^k, m \cdot h^k).$$

- Decryption:

$$\frac{c_2}{c_1^x} = \frac{m \cdot h^k}{g^{kx}} = m.$$

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The Discrete Log Problem

- Domain parameters: p large prime; g generator for a large subgroup in $\mathbb{Z}_p^*$.
- Given
$$h = g^x \bmod p,$$
compute x.
- Index calculus algorithms: can use methods similar to the NFS
- Complexity: subexponential $L_p(1/3, c)$ requires p to be 1024-bit or 2048-bit.

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The El Gamal cryptosystem again

- El Gamal works for any finite group!
- The discrete log problem in a group G:
- Given a finite group $h = g^x$ compute x.
- The computational complexity for solving the DLP depends on the group!
- $< \mathbb{Z}_p, + >$ easy; $< \mathbb{Z}_p^*, \cdot >$ subexponential; elliptic curve groups - exponential (?)

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Elliptic curve crypto

- Suggested in the 80s, wide-spread from 2004-.
- Example:

$$E : Y^2 = X^3 + X + 3$$

  over the field $\mathbb{Z}_p = \mathbb{Z}_7$.
  Points on the curve: $\mathrm{O}, (4,1), (6,6), (5,0), (6,1), (4,6)$
  Group law for adding points, e.g., $(4,1) + (6,6) = (5,0)$
- The computational complexity for the best algorithms solving the DLP on some elliptic curve groups is exponential.
- Modulus p can be much smaller! (top breaking record is 112-bit ECDLP)

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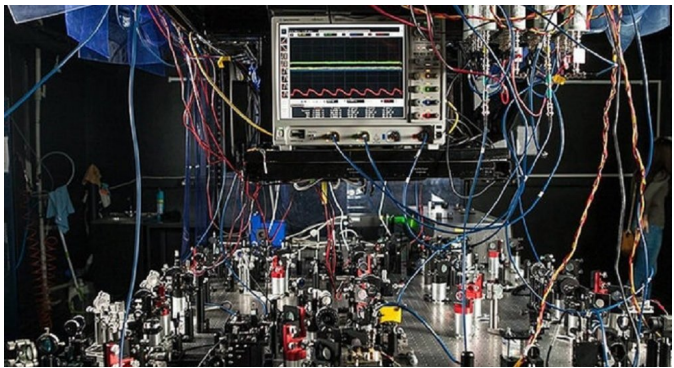
The big game changer II ?

- the birth of quantum computing



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The big game changer II ?

Shor's algorithm (1994)

If a large enough (number of qubits) quantum computer is available, then

- The factorization problem can be solved in polynomial time.
- The discrete log problem can be solved in polynomial time.
- Most public-key crypto of today is not secure...
- ...the world is collapsing...

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The Quantum era

- The Grover algorithm - affects symmetric crypto (but not so serious)
- Some things we encrypt today might need to be kept secret for the upcoming 30-50 years.
- we need to find a new solution now...
- the area of crypto in the presence of a quantum computer is called *post-quantum cryptography*

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Post-Quantum Crypto algorithms and future standards

- Most of todays security solutions are based on the difficulty of either *factoring* or solving the *discrete log* problem (RSA, DH, ECDH, . . .).
- If a (large enough) quantum computer can be built in 5 (or 10, or 50) years, it can solve these problems fast (in polynomial time).
- A huge threat that forces everyone to reconsider how to build future security algorithms.
- NIST now has an ongoing post-quantum standardization project.

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Post-Quantum Crypto are usually based on ...

- Lattice-based crypto (Learning with errors)
- Code-based crypto
- Multivariate crypto (solving systems of multi-variate quadratic equations)
- hash-based crypto

Our research investigates how difficult instances from these different problems are.

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Explaining Learning with Errors (LWE)

Basic Linear Algebra in $\mathbb{R}$

$$\begin{pmatrix} 3 & 5 & 2 & -4 & -1 \\ 4 & -1 & 3 & -4 & 3 \\ -2 & 2 & 2 & 3 & -3 \\ 1 & 0 & -4 & -4 & 1 \\ 0 & -5 & 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{pmatrix} = \begin{pmatrix} -6 \\ 20 \\ -9 \\ -5 \\ 20 \end{pmatrix}$$

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A Slight Modification - the LWE problem

- Arithmetic in $\mathbb{Z}_{13} = \{-6, -5, \ldots, 5, 6\}$
- e_i small (for example $e_i \in \{0, 1\}$)

$$\begin{pmatrix} 3 & 5 & 2 & -4 & -1 \\ 4 & -1 & 3 & -4 & 3 \\ -2 & 2 & 2 & 3 & -3 \\ 1 & 0 & -4 & -4 & 1 \\ 0 & -5 & 2 & -2 & 1 \\ -3 & 1 & 2 & -1 & -4 \\ 2 & -1 & 3 & -1 & 3 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{pmatrix} = \begin{pmatrix} -5 \\ -6 \\ 4 \\ -4 \\ -6 \\ 4 \\ 3 \end{pmatrix}$$

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A Slight Modification - the LWE problem

- Arithmetic in $\mathbb{Z}_{13} = \{-6, -5, \ldots, 5, 6\}$
- e_i small (for example $e_i \in \{0, 1\}$)

$$\begin{pmatrix} 3 & 5 & 2 & -4 & -1 \\ 4 & -1 & 3 & -4 & 3 \\ -2 & 2 & 2 & 3 & -3 \\ 1 & 0 & -4 & -4 & 1 \\ 0 & -5 & 2 & -2 & 1 \\ -3 & 1 & 2 & -1 & -4 \\ 2 & -1 & 3 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -6 \\ 4 \\ -4 \\ -6 \\ 4 \\ 3 \end{pmatrix}$$

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Learning with Errors

There is a secret vector **s** in $\mathbb{Z}_q^n$. We can query an oracle:

The LWE Oracle with Parameters $(n, q, \mathcal{X})$:

1. Uniformly picks **a** from $\mathbb{Z}_q^n$.
2. Picks a small 'error' e.
3. Outputs the pair $(\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e)$ as a sample.

Error Distribution

Discrete Gaussian over $\mathbb{Z}_q$ with mean 0 and standard deviation σ.

$D_{\sigma,0}$ returns integers $x \in \mathbb{Z}_q$ with prob. proportional to $\exp(-x^2/(2\sigma^2))$.

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LWE Problem Parameters

The LWE Parameters

1. Dimension n
2. Alphabet size q
3. The standard deviation σ.

How difficult is it to solve an LWE instance?

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Asymptotic Complexity of Solving LWE

There are several reasons to believe the LWE problem is hard (Regev 2005).

- The best known algorithms for LWE run in exponential time (quantum algorithms do not help much).
- LWE (for certain parameters) is known to be hard based on certain assumptions regarding the worst-case hardness of standard lattice problems such as GAPSVP (a decision version of the shortest vector problem)

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Complexity of Solving LWE

- $q = n^{c_q}$
- $\sigma = n^{c_s}$
- Complexity of Solving LWE is usually $2^{(c+o(1))n}$, where constant c depends on c_q and c_s.

Example (Regev instances)

$c_q = 2$ *and* $c_s = 1.5$.

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Results

Time complexities are on the form $2^{(c+o(1))n}$

Algorithm	Complexity Exponent (c)
Quantum Computer	0.8856
Classical Computer	0.8951
Previous best	0.9299

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Back to crypto - Ring-LWE

$\mathbb{Z}_q/(f(x))$, where $f(x) = x^n + 1$, where $n = 2^m$, is a common choice.
There is a secret polynomial $s(x)$ in $\mathbb{Z}_q/(x^n + 1)$.

The Ring-LWE Oracle with Parameters $(n, q, \mathcal{X})$:

1. Uniformly picks $a(x)$ from $\mathbb{Z}_q/(x^n + 1)$.
2. Picks a small 'error' polynomial $e(x)$ from $D^n_{\sigma,0}$.
3. Outputs $(a(x), b(x)) = (a(x), a(x)s(x) + e(x))$ as a sample.

$D^n_{\sigma,0}$ returns n integers from $D_{\sigma,0}$.

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Distributing keys using Ring-LWE

Think of vectors as polynomials in $\mathbb{Z}_q/(x^n+1)$.
Public key generation by Bob: Chooses random a(x), small $s(x)$, $e(x)$ and form

$$b(x) = a(x)s(x) + e(x).$$

Publish $(b(x), a(x))$.
Alice sends a key to Bob: Chooses small $s'(x)$, $e'(x)$, $e''(x)$ and compute

$$c(x) = a(x)s'(x) + e'(x)$$

and

$$c'(x) = b(x)s'(x) + e''(x) + q/2 \cdot k(x).$$

Here $k(x)$ is a binary polynomial (key). Alice sends $(c(x), c'(x))$ to Bob.

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Distributing keys using LWE

Bob recovering the key: Compute

$$m(x) = c'(x) - c(x)s(x).$$

If m_i closest to $q/2$ then $k_i = 1$ else if m_i closest to 0 then $k_i = 0$.
Explanation:

$$m(x) = c'(x) - c(x)s(x) = b(x)s'(x) + e''(x) + q/2 \cdot k(x) - (a(x)s'(x) + e'(x))s(x)$$

$$= (a(x)s(x) + e(x))s'(x) + e''(x) + q/2 \cdot k(x) - (a(x)s'(x) + e'(x))s(x)$$

$$= q/2 \cdot k(x) + \underbrace{e(x)s'(x) + e''(x) - e'(x)s(x)}_{\textit{small}}.$$

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Example

$\mathbb{Z}_{19}/(x^8+1)$, $\mathbb{Z}_{19} = \{-9, -8, \ldots, 8, 9\}$, small$=\in \{-1, 0, 1\}$
Public key generation by Bob: Chooses random

$$a(x) = 2x^7 - 9x^6 - 4x^5 - 4x^4 - 3x^3 + 5x^2 - 4x + 5,$$

small $s(x) = x^7 + x^3 - x^2 - 1$, $e(x) = -x^6 + x^5 - x^2 + x$ and form
$b(x) = a(x)s(x) + e(x)$

$$b(x) = 3x^7 + 7x^6 + 3x^5 - 1x^4 - 3x^3 + 9x^2 - 8x - 6.$$

Publish $(b(x), a(x))$.

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Example

Alice sends a key to Bob: Let $k = (1, 0, 1, 0, 1, 0, 1, 0)$, or $k(x) = x^7 + x^5 + x^3 + x$.
Chooses small $s'(x) = x^5 + x^4 - x^2$, $e'(x) = -x^7 + x^5 + x^3 + x$,
$e''(x) = x^6 + x^3 - x$ and compute $c(x) = a(x)s'(x) + e'(x)$

$$c(x) = 5x^7 + 5x^6 + 5x^5 - 2x^4 - 7x^3 + 8x^2 - 8x - 2$$

and $c'(x) = b(x)s'(x) + e''(x) + 9 \cdot k(x)$

$$c'(x) = -7x^7 + 3x^6 - 2x^5 + x^4 + 8x^3 - 4x^2 + 9x - 8$$

Alice sends $(c(x), c'(x))$ to Bob.

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Example

Bob recovering the key: Compute

$$m(x) = c'(x) - c(x)s(x).$$

$$m(x) = 7x^7 - 1x^6 - 7x^5 + x^4 - 7x^3 + 9x + 1.$$

If m_i closest to $q/2$ then $k_i = 1$ else if m_i closest to 0 then $k_i = 0$.

$$k = (1, 0, 1, 0, 1, 0, 1, 0)$$

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Current PQ-crypto research topics

basis for future security standards

- NIST standardization project contains 84 different proposals
- Difficulty of underlying problems
- Security of specific proposals (different types of primitives)
- Hardware implementation aspects
- Software implementation for constrained devices
- Side-channel attacks and its protection

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Thank you for your attention!
Any questions?