# SCATTERING OF STATIONARY WAVES
# FROM BURIED
# THREE-DIMENSIONAL INHOMOGENEITIES

BY

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FÖR AVLÄGGANDE AV FILOSOFIE DOKTORSEXAMEN

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ABSTRACT

The T-matrix method (also called the "extended boundary condition method" or "null field approach") is a formalism, which applies to scattering of linear classical waves (i.e. acoustic, electromagnetic and elastic waves). The object of this thesis is to extend the T-matrix formalism to a geometry consisting of two half-spaces, one of which contains a three-dimensional inhomogeneity. The two half-spaces, separated by an infinite interface, are otherwise homogeneous, linear and isotropic. The source that excites the scatterer can be located in any of the half-spaces or inside the inhomogeneity, though explicit equations and numerical computations are presented only for sources outside the inhomogeneity (above or below the interface). The assumptions on the sources are fairly weak, and in the numerical computations we consider a monopole or dipole source or an incoming Rayleigh surface wave. Furthermore the formalism applies to a large class of inhomogeneities, and we illustrate some: a sphere, a spheroid (both oblate and prolate) and two spheres. The orientation of the obstacle is arbitrary both with respect to the surface and the source. The scattered field in the general formalism is in a natural way separated in two terms, one of which is the directly scattered field as if no buried inhomogeneity were present. The other term reflects the presence of the inhomogeneity and that is the one which has been the quantity of primary interest in the numerical computations.

# INTRODUCTION

This introduction consists of two sections. In the first section we give a general introduction and a short presentation of the subject of the thesis. This first section also contains a review of the historical development and a discussion of the pertinent litterature. The second section presents more thoroughly the results and gives a summary of the following five papers which constitute the thesis:

I. Scattering from buried inhomogeneities - a general three-dimensional formalism, (together with S. Ström), J. Acoust. Soc. Am. 64, 917-936 (1978).

II. Electromagnetic Scattering from Buried Inhomogeneities - a General Three-dimensional Formalism. Rep. 78-42, Inst. Theoretical Physics, Göteborg (1978), to appear in J. Appl. Phys.

III. Electromagnetic scattering from a buried three-dimensional inhomogeneity in a lossy ground. Rep. 79-29, Inst. Theoretical Physics, Göteborg (1979).

IV. Elastic wave scattering by a three-dimensional inhomogeneity in an elastic half-space, (together with A. Boström). Rep. 79-33, Inst. Theoretical Physics, Göteborg (1979).

V. A uniqueness theorem for the Helmholtz equation: Penetrable media with an infinite interface. Rep. 79-10, Inst. Theoretical Physics, Göteborg (1979).

## GENERAL REVIEW

The subject of this thesis is scattering of stationary classical waves (i.e. linear acoustic, electromagnetic and elastic waves) from buried inhomogeneities. An extension of the T-matrix method (also called the "extended boundary condition method" or the "null field approach"), originally given by Waterman [1-3] (for the elastic case see also Varatharajulu and Pao [4]) for finite scatterers, is presented. In addition to the finite scatterer we also consider a surface of infinite extent, which is bounded by two parallel planes. This geometry is of great interest in modelling a number of prospecting situations.

Integral formulations have been used frequently to solve diverse scattering problems. Extensive work has been done in the field of volume and surface integral equation formulations, and for a review of the theoretical aspects of the surface integral formulation see Kleinman and Roach [5]. A common property of the integral formulations is that they are based upon an appropriate integral representation of the field. In the T-matrix method, however, there is no reduction to a surface integral equation in its usual sense (i.e. we do not let the field point approach the bounding surface) and furthermore the "extinction part" of the integral representation is explicitly used.

The T-matrix method is an approach developed for scattering of stationary linear acoustic, electromagnetic and elastic waves, appropriate for both two or three-dimensional, scalar or vector problems. In this method one usually concentrates on the spherical basis projection of the T operator of potential scattering theory [6]. However, projections on other basis systems can also be used. In the case when the scatterer is a homogeneous body a useful explicit expression is obtained for the T-matrix. This T-matrix method has been applied to a large number of scattering problems during the last decade, mostly scattering of classical waves by a finite obstacle [1-4), 7-29)]. The T-matrix approach provides a formalism for the different types of wave propagation, acoustic electromagnetic and elastic waves, which is very similar in structure, and this is a valuable feature of the method. Thus the scattering of scalar and vector waves can be treated analogously and much of the formal structure developed in the scalar case can simply be taken over, with appropriate changes and modifications, to the vector cases. Furthermore, the formalism has been applied to various boundary conditions, geometries and physical situations [10-11), 16-17), 20-22), 24-25), 28-29)]. All these examples emphasize the generality and versatility of the approach. Extensions in various directions have lead to a systematic treatment of layered scatterers [14-15), 23)], several scatterers [12-13), 23)] and waveguide problems [26)].

The constituent fields are expanded in elementary waves appropriate to the problem, and the formalism gives an algorithm for computing the linear relation between the incoming and scattered fields - the T-matrix. The formalism makes use of the free space Green's function expanded in a suitable system of basis functions. There are a number of acceptable systems of basis functions available, but in the application discussed here we use the spherical and plane wave systems. The first, discrete, set is convenient to describe the finite inhomogeneity, the other, characterized by continuous parameters, is more suited to the infinite surface. The transformation between these two systems of basis functions is then an important tool in the formalism.

The introduction of an inhomogeneity in one of the half-spaces leads to a more intricate scattering problem. The interaction between the infinite surface and the buried obstacle must be taken into account in the general case, and only in some special situations can we ignore the multiple-scattering effects between the surfaces, e.g. for a deeply buried obstacle or a highly conducting ground. This interaction couples the wave which propagates in the ground and the wave which is scattered upon the finite inhomogeneity. One difficulty in the theoretical analysis of the interaction phenomenon is that of finding suitable representations of the fields and furthermore to match these various representations. Useful representations should be found for both the infinite interface and the bounded obstacle. This dilemma will be discussed further in connection with the presentation of the papers.

To put the thesis in its proper perspective, we will here give a short presentation of various formulations, which treat a geometry similar to the one encountered in this thesis. The intention is not to give a complete list of references, but to mention some of the most important ones. Some examples of two-dimensional formulations are found in e.g. Refs. 30-38. They are all specialized to a specific geometry of the inhomogeneity (cylinder or half-plane). This is also the case in a number of papers, see Refs. 39-42, which analyse the response from a three-dimensional obstacle (mostly spherical).

Raiche [43] has adopted a somewhat different method, capable of treating a more general geometry and we will therefore discuss it in some detail. In the T-matrix method we make use of the free space Green's function. However, there is another alternative, namely to employ the Green's function, which satisfies the appropriate boundary conditions of the problem. In practice, to determine this Green's function is as difficult as solving the entire problem. Thus, consider, as Raiche does, the Green's function that satisfies the appropriate boundary conditions for a stratified homogeneous ground. This Green's function can now be applied to solve a scattering problem with a stratified earth and an inhomogeneity present. The appropriate boundary conditions in the stratified earth are already taken into account and we have to apply the boundary conditions on the inhomogeneity to achieve the final solution. This last step is in principle analogous to determining the scattered field from a finite scatterer, using the Green's function discussed above instead of the free-space Green's function. This approach is very general and can be applied to a great number of different scattering geometries. One can even, in principle, consider the situation where the inhomogeneity cuts one or more of the discontinuous layers, i.e. the inhomogeneity can be half-buried.

A somewhat similar approach is used by Mei et al.[44)]. The final solution of the problem is expanded in a set of basis functions, each of which satisfies the appropriate boundary condition on the infinite surface. The wave propagation outside the buried (or half-buried) obstacle is thus in principle determined, and the eventual fit of the solution to the obstacle is done by a numerical approach - The Unimoment method. The inhomogeneity can be quite general, even obstacles with continuously varying material parameters can be considered. For the most recent calculations using this method see K.K. Mei, J.F. Hunka and S-K. Chang, "Recent developments of the Unimoment method" in Ref. 29.

Still another approach to the problem has been used. One then takes as a starting point a finite configuration and the infinite interface problem is considered as a limiting case of the finite one. We mention this approach here since it contains many quantities and building blocks which can be identified with special cases of the T-matrix results for layered scatterers, see e.g. Refs. 14-15, 23. More explicitly: take two excentric spheres, and let the distance from the centre of the smaller sphere to the surface of the larger be constant as the radius of the larger sphere approaches infinity. In the vicinity of the smaller sphere the scattering geometry approaches that of a flat earth and a buried sphere at finite depth. D'Yakonov [45-48] and his followers [49-50] have studied a completely rotational symmetric configuration, i.e. a vertical dipole exciting a buried sphere, located directly below the dipole. The close relation to the T-matrix results is revealed by the use of the translation properties of the cylindrical and spherical basis functions and furthermore by quantities, which can be identified as the T-matrix for the smaller sphere.

In connection with the prospecting problem discussed above, we will now elucidate a related problem, which in some respects can be considered as a special case of our formulation, namely the propagation of electromagnetic waves over a plane homogeneous ground, excited by e.g. a dipole. The problem has been studied intensively for a long period and most of the important results are summarized in the books by Baños [51)] and Wait [52)]. In this case one frequently encounters integrals containing a Bessel function (Sommerfeld integrals), which gives a slowly converging and alternating contribution in the integrals. There is a demand for fast numerical routines for computation of these integrals, and we have noticed an intensified research on this subject, see Refs. 53-56. The calculations of the Sommerfeld integrals reported in the present thesis are based upon a computational scheme in which the alternating parts are separated and the convergence is accelerated by the use of an Euler transformation.

In the approaches, which we have discussed above, the infinite interface between the two half-spaces is assumed to be flat. A deviation from a completely plane surface affects the scattered field, and in many practical applications it is necessary to be able to compute the magnitude of these effects. A canonical problem is that of a flat surface with a deviation of finite extent. A general formalism applicable to scattering from rough surfaces has been developed by Bahar [57)]. The formalism applies to approximate impedence boundary conditions as well as to the exact boundary conditions at the rough surface. In Ref. 58 a two-dimensional geometry (the hill is infinite in one direction) is considered and a formalism based upon Fourier-analysis is applied. In Ref. 59 an integral equation of the second kind for the surface field amplitude is derived using the present formalism for the case of a finite deviation from a plane. This suggests an iteration approach, at least in the long wavelength case.

## SUMMARY OF THE THESIS

The object of this thesis is to study classical scattering phenomena for stationary fields from a buried inhomogeneity, i.e. a three-dimensional scattering geometry of two half-spaces, one of which contains a finite inhomogeneity. The half-spaces are otherwise homogeneous and isotropic. The driving source can be situated in either of the half-spaces or inside the inhomogeneity. The requirement on the incoming field is that it can be expanded in the relevant regions in a suitable set of basis functions. This set is chosen appropriately for the geometry considered (e.g. plane or regular spherical waves). The formalism, which is adopted, was first discussed and studied by Waterman [1-3]. Scattering of stationary waves from a finite obstacle in a homogeneous space was first considered and we summerize some of the main characteristics of the formalism: 1) The field is represented by a surface field integral, which also provides the "extinction part" - a most vital part in the formalism. 2) The free space Green's function is expanded in a suitable set of basis functions. 3) The surface fields are expanded in an appropriate complete set of basis functions. 4) The boundary conditions are introduced into the integral representation. 5) The linear transformation between the expansions of the incoming and the scattered fields - called the T-matrix - is found by eliminating the expansion coefficients of the surface fields. This matrix describes the scattering process, i.e. it contains the effects of the shape and material parameters of the obstacle as well as the boundary conditions.

The numerical effectiveness of this approach has been demonstrated in the long wavelength and resonance regions for not too extreme shapes of the scatterers [1-2), 27), 60-61)]. However, we emphasize that considerable work is being done at present on the numerical aspects. In particular, several new combinations of the T-matrix ideas and other numerical techniques have been introduced recently and a rapid development is expected to take place in the near future concerning these aspects, see e.g. Ref. 22 and the following contribution in Ref. 29: D.J.N. Wall, "T-Matrix Method and Numerical Methods of Overcoming Computational Instabilities", P. Barber, "Scattering and Absorption by Homogeneous and Layered Dielectrics", P.C. Waterman, "Survey of T-matrix methods and surface field representations", V.N. Bringi and T.A. Seliga, "Surface Currents and Near Zone Fields using the T-matrix".

The T-matrix method has successfully been applied to scattering by finite obstacles, and in this thesis we extend these results to a geometry, which contains an infinite surface in addition to the finite one. This extension can, in principle, be performed in two different ways. We have already, in the preceding section, discussed the approach suggested by D'Yakonov [45-48)] and its connection with the T-matrix formalism. This limiting procedure can be extended to an arbitrary finite scatterer below the ground, but we have found it more convenient to solve the problem directly for an infinite interface. However, some details and comments on the limiting process are found in paper I.

Thus, we have adopted a scattering geometry, which should be of interest for a large class of prospecting situations. The infinite surface can be general in shape, but it is assumed to be bounded by two parallel planes, and the restrictions on the inhomogeneity are fairly weak, see the papers I-IV. Although the general formalism can treat the influence of a non-planar interface, we specialize in our numerical applications to a plane interface. However, we have found it instructive particularly with respect to the physical interpretation to develop the formal structure of the general theory before specializing to a plane interface.

The expansion of the relevant surface fields in a suitable complete set of basis functions is as already mentioned a characteristic property of the formalism. The spherical waves have been extensively used. The completeness of these waves, both the regular and outgoing waves as well as their normal derivatives on a class of finite surfaces, has been shown by Millar [62)] and Müller [63)]. The technique relies on the existence of appropriate uniqueness theorems for the Dirichlet or Neumann problem (exterior or interior problem). We can also apply this technique to other basis functions and for the plane waves see paper I and paper II and Ref. 62. Uniqueness theorems for the homogeneous boundary conditions are available for a large class of finite surfaces, see e.g. Ref. 64, and for corresponding results for some classes of infinite boundary surfaces see Refs. 65-67 (an extension of these results to penetrable media are found in paper V). The completeness of the basis functions for these types of waves are to be interpreted as a fit in a least-square sense. The expansion coefficients will thus in general depend on the truncation order, which depends on the required accuracy of the expansion. These questions are discussed in the excellent review by Millar [62)]. The relation between these expansion coefficients and the ones found by assuming the validity of the Rayleigh hypothesis is also reviewed in the paper by Millar. It should be emphasized that calculations according to the T-matrix scheme have been shown to be stable and rapidly converging in cases where the Rayleigh hypothesis [68-69] is not satisfied [1]. However, considerable work remains to be done on these questions, and we do not pursue this subject further here.

When discussing the expansions of various surface fields in suitable basis functions it is also important to consider the associated expansions of the relevant derivatives of the fields in the corresponding derivatives of the basis functions. In the formalism it often appears as if the expansions of the surface fields can be differentiated formally e.g. in the normal direction. We emphasize that this differentiation property depends very much on the choice of the basis function and for a different choice it will look totally different. For some sets of basis functions (they should satisfy the proper regularity properties) it is possible to show that the same expansion coefficients approximate the surface field and its relevant derivatives with arbitrarily high accuracy. The basic case with the source outside a finite permeable scatterer is discussed in Ref. 2. The complications alluded to above become apparent when one wants to go through the same application of Green's theorem as in Ref. 2 but now with a different system of expansion functions. An extra relation is then necessary to relate the two sets of expansion coefficients. One usually tries to avoid this extra relation by choosing the most suitable set of basis functions. We take an explicit example; the expansion of the surface field and its relevant derivatives (say its normal derivative , as in the scalar case) on the inside of a finite obstacle can both be expanded in regular spherical basis functions and its normal derivatives, respectively. For a situation where the sources are located outside the scatterer these regular basis functions approximate both the surface field and its normal derivative with arbitrarily high accuracy. This will not be true if we adopt the outgoing spherical basis functions in expanding the fields from the inside. These features are very easily illustrated in the spherical permeable scatterer case. It should also be emphasized that in these expansion systems the appropriate wave number must be used. Thus, for each field and its relevant derivatives there are in practice only a limited number of sets of basis functions which lead to the most compact form of the theory. These questions are usually treated by applying the integral representation to the interior (i.e. source-free) region. The relation then obtained determines if the set of basis functions can approximate both the field and its relevant derivatives with arbitrarily high accuracy. If this is not the case the integral representation then gives the extra relation needed to relate the two sets of expansion coefficients. However, we want to stress that considerable analysis remains to be done, before we have a complete understanding of these matters.

We here give a short summary of the papers presented in this thesis. In paper I we analyse the scalar problem of scattering from a buried inhomogeneity. We also discuss the limiting procedure suggested by D'Yakonov, which was considered above. The sources are assumed to be located in the upper half-space and in this case we give explicit expressions for the scattered field above the ground. We also consider the case where the sources are below the interface. In this latter situation no explicit expressions for the fields are given, but they can be achieved with appropriate modifications. In a series of appendices we analyse various mathematical problems which occur in the paper, e.g. the completeness of the plane wave system and convergence of the iteration scheme, which is used to solve the matrix equation, reflecting the interaction between the inhomogeneity and the ground. A number of numerical computations illustrate the use of the formalism.

Paper II extends the results of paper I to the electromagnetic case. The structure of the equations is formally similar. The various theoretical aspects of the required modifications are analysed in detail. We introduce a plane wave expansion of the Green's dyadic and in an appendix the completeness of the spherical and plane vector waves for both finite and infinite surfaces is established. Paper III proves that the formalism can be extended to lossy media. The fundamental transformation properties between plane and spherical vector waves in the lossy case is analysed in an appendix. In both papers II and III we illustrate the formalism with some numerical examples, showing the scattered field on the interface or below the ground in a drillhole. The source in these applications is an oscillating dipole, located on the ground.

Paper IV considers the extension of the formalism to elastic waves. The formal structure of the equations in the formalism are very similar in all types of classical wave motions, and this feature makes generalizations easier to perform. In the elastic case when we have two different types of wave motions - shear and pressure waves - the coupling effects imply a more complicated structure of the formalism, and the explicit expressions of the fields become rather lengthy, but in most respects the formalism developed in the scalar and the electromagnetic cases apply with appropriate modifications. In the elastic case an incoming surface wave, the Rayleigh wave, is of particular interest, and therefore this case is given a more detailed analysis, see the appendix in paper IV. In a number of numerical illustrations a Rayleigh wave excites a buried inhomogeneity (spheres or spheroids) and the field is computed on the interface, showing various components of the scattered or total field.

The last paper in this thesis, paper V, contains a uniqueness theorem for scalar fields, proving the uniqueness of the solution to Helmholtz equation satisfying a prescribed radiation condition. An extension of some of the results given by Kato [70)] is obtained. We consider an interface, which for sufficiently large distances is a cone of arbitrary cross-section, and adopt a class of boundary conditions, which allow discontinuities both in the field and in its normal derivative. We have briefly emphasized the importance of uniqueness theorems of the above type when discussing the completeness properties of the basis functions, and we refer to papers I and II for more details. In these applications it is the results for the homogeneous boundary conditions (Dirichlet and Neumann) which are relevant. Paper V is an extension of already existing uniqueness theorems for the homogeneous boundary conditions [65-67)] to the case with penetrable media (the homogeneous boundary conditions become a special case). It also explicitly considers the case when finite inhomogeneities are present, and thus applies directly to the situation one encounters in proving the completeness properties in papers I and II.

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# Paper I

# Scattering from buried inhomogeneities—a general three-dimensional formalism

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In the present article we give a general three-dimensional formalism for scattering in two half spaces, one of which contains a bounded inhomogeneity. Our formalism consists of an extension of the transition matrix method which has been given by Waterman, a method which applies equally well to acoustic, electromagnetic, and elastic scattering. The formalism is here developed in detail for the case when the source and inhomogeneity are situated in different half spaces. However, the same method works for other source positions as well, and the basic equations are given also for the case when the source and the inhomogeneity lie in the same half space. In the final expression for the total scattered field, that part (the so-called anomalous scattered field) which depends on the presence of the inhomogeneity, can be separated, and the physical meaning of the various quantities which determine this anomalous scattered field can be identified. The inhomogeneity enters through its $T$ matrix, and previous results on various bounded configurations of scatterers can therefore be inserted and used in the present formalism. Numerical results are given for inhomogeneities consisting of one or two spheres.

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## INTRODUCTION

In the present article we adapt the transition matrix formulation of stationary scattering to the problem of determining the scattering from an inhomogeneity which is buried in a half space. The transition matrix formulation was introduced in Refs. 1–4 (and earlier references given there) for a single scatterer, and it has also been extended to situations involving several (multilayered) scatterers[5–9] and to infinite periodic surfaces.[10] We shall here consider the scattering of a scalar field. However, the extension to the electromagnetic or elastic cases is not expected to present any difficulties and will be treated in subsequent work. As in the above-mentioned cases for which the $T$ matrix formalism has been developed it can also in the present case be given in a general three-dimensional form which is independent of any symmetries characterizing the surface which separates the two half spaces or the inhomogeneity. However, the simplifications which occur in the presence of specific symmetries are easily identified within the framework of the general formalism.

A key element in the transition matrix formulation is the use of certain expansions for the surface fields, and in the present work we introduce corresponding expansions, relevant to the configuration under consideration. Thus the geometrical and physical characteristics of the surface, separating the two half spaces, and of the inhomogeneity are limited mainly by the condition that these expansions shall be allowed. However, complete systems, suitable for approximation of the surface fields, can be found for very general classes of surfaces.[11] Considerable experience has by now been accumulated concerning the numerical effectiveness of the transition matrix formalism for finite scatterers, and it is reviewed, e.g., in Refs. 12–14.

Results concerning scattering from buried inhomogeneities obviously have applications in many geophysical applications, such as prospecting by means of acoustic, electromagnetic, or elastic waves. The electromagnetic case has been treated extensively by different analytical and numerical techniques by several authors (see, e.g., Refs. 15–25), and below we shall make some comments on the relation between some of these papers and the present approach. However, relatively few treatments of truly three-dimensional cases exist.

The plan of the present article is as follows. In Sec. I we derive the basic equations of the transition matrix formalism in a form which is adapted to the scattered configuration of Fig. 1. In Sec. II we consider the structure of the equations and we explain the physical meaning of the various quantities which appear in the process of solving the equations, such as transmission and reflexion coefficients, and transformations between different wave types. The solution for the case of a plane interface is given in some detail. The inhomogeneity is here still of a general nature, characterized by its $T$ matrix. In Sec. III we describe some numerical applications. We consider first the simplest case of a spherical homogeneous inhomogeneity. The $T$ matrix for the inhomogeneity is in this case diagonal, which leads to further simplifications of the solution. That part of the scattered field which is due to the presence of the inhomogeneity is calculated for several different positions of a source lying on the plane $S_0$. In

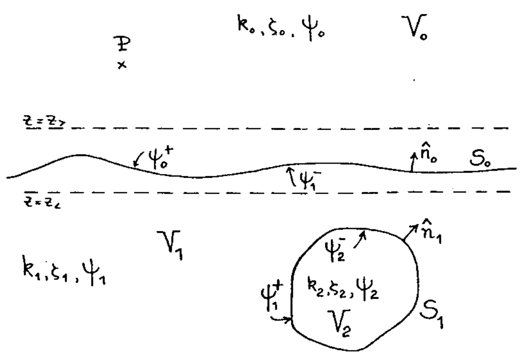

FIG. 1. Geometry and notations of the scattering problem.

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order to illustrate the generality of the formalism, we also apply it to a situation where we have two different inhomogeneities. In this case the total $T$ matrix corresponding to the two inhomogeneities is calculated, using earlier results, and then inserted into the present formalism. In Sec. IV we have collected some concluding remarks on the extension of the present results. Various technical questions which occur in the development of the formalism are treated in three appendices.

## 1. $T$ MATRIX FORMALISM IN THE PRESENCE OF AN INFINITE SURFACE

We shall consider the type of scattering situation which is depicted in Fig. 1, where we have two half spaces separated by a surface $S_0$ which need not be a plane but which can be enclosed by two parallel planes. The normal to these planes then defines a direction which will be taken as the $z$ axis. The upper half space is completely homogeneous and the lower half space is homogeneous except for a region of finite extent–bounded by the surface $S_1$–which has different propagation characteristics. In order to simplify certain steps in the derivations below we shall also assume that the region inside $S_1$ is homogeneous. However, this restriction is in no way essential and can be relaxed.[9]

Consider a wave field with its source in a point $P$ in the upper half space. In the three different regions $V_0$, $V_1$, and $V_2$ this wave field is assumed to satisfy [a time factor $\exp(-i\omega t)$ is suppressed].

$$(\nabla^2 + k_r^2)\psi_r = 0 , \quad r = 0, 1, 2 . \tag{1}$$

As usual, we assume that the driving field $\psi_0^{\text{inc}}$ which is generated at the point $P$ can be treated as a prescribed incoming field and we shall obtain the total field in the three different regions. In the $T$ matrix formalism this is done by means of application of the following integral representation (together with its accompanying "extinction theorem") for a wave field in terms of the boundary values from the outside $\psi^+$ and $\hat{n}\cdot\nabla\psi^+$ on a closed surface $S$ (and here we assume the Sommerfeld radiation condition to hold):

$$\left.\begin{matrix}\psi(\mathbf{r})\\ 0\end{matrix}\right\} = \psi^{\text{inc}}(\mathbf{r}) + \int_S \hat{n}\cdot(\psi^+\nabla' G - G\nabla'\psi^+)\, dS', \quad \text{for}\begin{cases}\mathbf{r} \text{ outside } S\\ \mathbf{r} \text{ inside } S .\end{cases} \tag{2}$$

Here $G$ is a free-space Green's function, normalized according to

$$(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{r}'; k) = -\delta(\mathbf{r} - \mathbf{r}') . \tag{3}$$

In the present problem we use this formula for the regions $V_0$ and $V_1$ (cf. the derivation of the $T$ matrix for a multilayered scatterer[7,8]). The surface fields are denoted $\psi_r^{\pm}$ $r = 0, 1, 2$ according to Fig. 1. We now consider the case when the surface $S$ in Eq. (2) consists of a finite part of $S_0$ and a lower half sphere. By letting the radius go to infinity and assuming that the surface integral over $S_0$ exists, we get

$$\left.\begin{matrix}\psi_0(\mathbf{r})\\ 0\end{matrix}\right\} = \psi_0^{\text{inc}}(\mathbf{r}) + \int_{S_0} \hat{n}_0\cdot[\psi_0^+(\mathbf{r}')\nabla' G(\mathbf{r}, \mathbf{r}'; k_0) - G(\mathbf{r}, \mathbf{r}'; k_0)\nabla'\psi_0^+(\mathbf{r}')]\, dS', \quad \text{for}\begin{cases}\mathbf{r} \text{ in } V_0\\ \mathbf{r} \text{ outside } V_0 .\end{cases} \tag{4}$$

The relevant Green's function is that corresponding to outgoing waves and we proceed by introducing suitable expansions for $G(\mathbf{r}, \mathbf{r}'; k_0)$ and for the surface field $\psi_0^+(\mathbf{r})$. The main consideration in choosing the expansion for the Green's function is that one should be able to use one and the same expansion in the whole integral over $S_0$ in the right-hand side of Eq. (4). This can be achieved if we consider, not all $\mathbf{r}$ inside and outside $V_0$, respectively, but only those $\mathbf{r}$ in $V_0$ for which $z > z_>$ and only those $\mathbf{r}$ outside $V_0$ for which $z < z_<$ and if, furthermore, we expand the Green's function in terms of harmonic and evanescent plane waves:

$$G(\mathbf{r}, \mathbf{r}'; k_0) = \frac{ik_0}{8\pi^2}\int_0^{2\pi} d\beta \int_{C_\pm} d\alpha\, \sin\alpha\, \exp[i\mathbf{k}_0\cdot(\mathbf{r} - \mathbf{r}')] , \tag{5}$$

where $\mathbf{k}_0 \equiv k_0(\sin\alpha\cos\beta,\ \sin\alpha\sin\beta,\ \cos\alpha)$ and where the integration is over $C_+$ if $z - z' > 0$ and over $C_-$ if $z - z' < 0$. The contours $C_\pm$ are given in Fig. 2. (See, e.g., Ref. 26 where integral representations of this type for $G$ are reviewed.) By considering an $\mathbf{r}$ with $z > z_>$ we thus have a representation of $G(\mathbf{r}, \mathbf{r}'; k_0)$ as an integral over $C_+$ for all $\mathbf{r}'$ on $S_0$. In this way the scattered field $\psi_0^{\text{sc}}$ which is given by the integral over $S_0$ on the right-hand side of Eq. (4), is given in terms of upwards-traveling plane harmonic waves and evanescent waves which decrease exponentially with increasing $z$. Thus the representation (5) for the Green's function leads to waves which propagate and decrease, respectively, in specific ways in half spaces and in a geometry of scattering surfaces like the one considered here; this representation therefore seems preferable to representations which use harmonic waves traveling in all directions. (See, e.g., Refs. 26 and 27.)

Using Eq. (4) for an $\mathbf{r}$ in $V_0$ with $z > z_>$ we thus have, after a change of the order of integration, the following representation of the scattered field:

$$\psi_0^{\text{sc}}(\mathbf{r}) = \int_0^{2\pi} d\beta_0 \int_{C_+} f(k_0)\, e^{i\mathbf{k}_0\cdot\mathbf{r}} \sin\alpha_0\, d\alpha_0 , \tag{6}$$

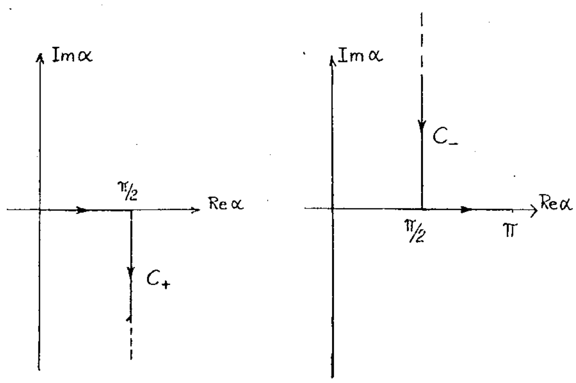


FIG. 2. The integration contours $C_+$ and $C_-$.

where

$$f(\mathbf{k}_0)=\frac{ik_0}{8\pi^2}\int_{S_0}\hat{n}_0\cdot[\psi_0^+(\mathbf{r}')\nabla'(e^{-i\mathbf{k}_0\cdot\mathbf{r}'}) - e^{-i\mathbf{k}_0\cdot\mathbf{r}'}\nabla'\psi_0^+(\mathbf{r}')]\,dS_0' , \quad \hat{k}_0\in C_+ . \tag{7}$$

The prescribed incoming field $\psi_0^{inc}$ is assumed to be generated at the point $P$ with position vector $\mathbf{r}_p$ in $V_0$. It is natural to consider a $\psi_0^{inc}$ which is a superposition of multipole fields emanating from $\mathbf{r}_p$, i.e.,

$$\psi_0^{inc}(\mathbf{r})=\sum_n a_n\psi_n[k_0(\mathbf{r}-\mathbf{r}_p)] . \tag{8}$$

Here we use the same notation as in earlier work on the $T$ matrix formalism,[1-9] i.e., the regular and outgoing spherical waves are denoted $\mathrm{Re}\psi_n$ and $\psi_n$, respectively, where $n$ is an abbreviation of a multi-index $n\equiv\sigma mn$ and where $\mathrm{Re}\psi_n$ is chosen as a real function. For $z>z_p$ and $z<z_p$, $\psi_0^{inc}$ can then be expanded in terms of harmonic and evanescent plane waves by means of the relation (see, e.g., Refs. 28 and 29)

$$\psi_n(k\mathbf{r})=\frac{1}{2\pi i^n}\int_0^{2\pi}d\beta\int_{C_\pm}Y_n(\hat{k})\,e^{i\mathbf{k}\cdot\mathbf{r}}\sin\alpha\,d\alpha , \quad z\gtrless 0 . \tag{9}$$

For $z<z_p$ we thus have [after a change of order between summation and integration, which is always allowed for any finite expansion (8)]

$$\psi_0^{inc}(\mathbf{r})=\int_0^{2\pi}d\beta_0\int_{C_-}a(\mathbf{k}_0)\,e^{i\mathbf{k}_0\cdot\mathbf{r}}\sin\alpha_0\,d\alpha_0 , \tag{10}$$

where $a(\mathbf{k}_0)$ is a known function obtained from (8) and (9). We now apply Eq. (4) for $\mathbf{r}$ outside $V_0$ and consider furthermore an $\mathbf{r}$ with $z<z_<$. We can then use a representation (5) of $G(\mathbf{r}, \mathbf{r}'; k_0)$ as an integral over $C_-$ for all $\mathbf{r}'$ on $S_0$. The integral on the right-hand side of Eq. (4) in this case gives a superposition of down-going harmonic plane waves and evanescent waves which decrease as $z$ decreases.

In this way we get (again after a change of order of integrations) the following relation between $a(\mathbf{k}_0)$ and the surface fields $\psi_0^+$ on $S_0$:

$$a(\mathbf{k}_0)=-\frac{ik_0}{8\pi^2}\int_{S_0}\hat{n}_0\cdot[\psi_0^+(\mathbf{r}')\nabla'(e^{-i\mathbf{k}_0\cdot\mathbf{r}'}) - e^{-i\mathbf{k}_0\cdot\mathbf{r}'}\nabla'\psi_0^+(\mathbf{r}')]\,dS_0' , \quad \hat{k}_0\in C_- . \tag{11}$$

Note that the integrals on the right-hand sides of (11) and (7) differ only in that $\hat{k}_0\in C_-$ in (11), whereas $\hat{k}_0\in C_+$ in (7).

One way of solving the scattering problem, and this is the one used in the $T$ matrix formalism, is to eliminate the surface field $\psi_0^+$ between (7) and (11) so as to obtain $f(\mathbf{k}_0)$ in terms of $a(\mathbf{k}_0)$. $\psi_0^+$ is influenced by the lower half space and the inhomogeneity in a way which is determined by the boundary conditions on $S_0$ and $S_1$. In the next step we therefore apply Eq. (2) to the case when $S$ consists of $S_0$ and $S_1$ and when outside $S$ corresponds to inside $V_1$. Since we have assumed that the only source present is the one at $\mathbf{r}_p$ above $S_0$, we get in this case

$$\left.\begin{matrix}\psi_1(\mathbf{r})\\ 0\end{matrix}\right\}=-\int_{S_0}\hat{n}_0\cdot[\psi_1^-(\mathbf{r}')\nabla'G(\mathbf{r}, \mathbf{r}'; k_1) - G(\mathbf{r}, \mathbf{r}'; k_1)\nabla'\psi_1^-(\mathbf{r}')]\,dS_0' + \int_{S_1}\hat{n}_1\cdot[\psi_1^+(\mathbf{r}')\times\nabla'G(\mathbf{r}, \mathbf{r}'; k_1) - G(\mathbf{r}, \mathbf{r}'; k_1)\nabla'\psi_1^+(\mathbf{r}')]\,dS_1' ,$$

$$\text{for}\begin{cases}\mathbf{r}\text{ in }V_1\\ \mathbf{r}\text{ in }V_0\text{ or }V_2 .\end{cases} \tag{12}$$

Here $\psi_1^-$ and $\psi_1^+$ are related to $\psi_0^+$ and $\psi_2^-$, respectively, by the boundary conditions. As in the case of a bounded two-layered scatterer we retain $\psi_1^-$ and $\psi_2^-$ as unknowns which are to be eliminated.[7,8] The equations needed for this elimination are obtained from a consideration of $\mathbf{r}$ in $V_0$ and $V_2$ in Eq. (12). The case of $\mathbf{r}$ in $V_1$ can be used to obtain different expansions for $\psi_1$ in different parts of $V_1$. Consider first $\mathbf{r}$ in $V_0$ in Eq. (12). Since $S_1$ is a closed surface we choose to describe the influence of the inhomogeneity by means of its $T$ matrix referring to spherical waves.[1] To this end we use the spherical wave expansion for $G(\mathbf{r}, \mathbf{r}'; k_1)$ in the integral over $S_1$,

$$G(\mathbf{r}, \mathbf{r}'; k_1)=ik_1\sum_n\mathrm{Re}\psi_n(k_1\mathbf{r}_<)\psi_n(k_1\mathbf{r}_>) . \tag{13}$$

In order to be able to identify $\mathbf{r}$, $\mathbf{r}'$ with $\mathbf{r}_>$, $\mathbf{r}_<$ in a unique way for all $\mathbf{r}'$ on $S_1$ we choose an origin $O_2$ inside $S_1$ and consider an $\mathbf{r}$ which, furthermore, is outside the circumscribed sphere of $S_1$ with center at $O_2$ (this sphere may extend into $V_0$). We then have $\mathbf{r}=\mathbf{r}_>$ and $\mathbf{r}'=\mathbf{r}_<$ for $\mathbf{r}'$ on $S_1$. In the integral over $S_0$ it turns out to be advantageous to postpone the expansion of the Green's function. In the present case, Eq. (12) thus takes the form (after changing the order of summation and integration)

$$\int_{S_0}\hat{n}_0\cdot[\psi_1^-(\mathbf{r}')\nabla'G(\mathbf{r}, \mathbf{r}'; k_1) - G(\mathbf{r}, \mathbf{r}'; k_1)\times\nabla'\psi_1^-(\mathbf{r}')]\,dS_0' = ik_1\sum_n\psi_n(k_1\mathbf{r})\int_{S_1}\hat{n}_1\cdot[\psi_1^+(\mathbf{r}')\times\nabla'\,\mathrm{Re}\psi_n(k_1\mathbf{r}') - \mathrm{Re}\psi_n(k_1\mathbf{r}')\nabla'\psi_1^+(\mathbf{r}')]\,dS_1' . \tag{14}$$

Consider next an $\mathbf{r}$ inside $S_1$. We then have $\mathbf{r}=\mathbf{r}_<$ and $\mathbf{r}'=\mathbf{r}_>$ for all $\mathbf{r}'$ on $S_1$ and $S_0$ if we further assume that $\mathbf{r}$ lies inside the inscribed sphere of $S_1$ with center at $O_2$. Again we postpone the expansion of the Green's function in the integral over $S_0$, and in a similar way we then get from Eq. (12)

$$\int_{S_0}\hat{n}_0\cdot[\psi_1^-(\mathbf{r}')\nabla'G(\mathbf{r}, \mathbf{r}'; k_1) - G(\mathbf{r}, \mathbf{r}'; k_1)\times\nabla'\psi_1^-(\mathbf{r}')]\,dS_0' = ik_1\sum_n\mathrm{Re}\psi_n(k_1\mathbf{r})\times\int_{S_1}\hat{n}_1\cdot[\psi_1^+(\mathbf{r}')\nabla'\psi_n(k_1\mathbf{r}') - \psi_n(k_1\mathbf{r}')\nabla'\psi_1^+(\mathbf{r}')]\,dS_1' . \tag{15}$$

We rewrite Eqs. (7), (11), (14), and (15) by means of the boundary conditions so that they contain only the surface fields $\psi_1^-$ and $\psi_2^-$. In order to illustrate the details of this procedure we assume specific boundary condi-

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tions. Let $\psi$ be the velocity potential and $\zeta$ the density, and let $V_0$, $V_1$, and $V_2$ be isotropic permeable media. In this case the boundary conditions require $\zeta\psi$ and $\hat{n}\cdot\nabla\psi$ to be continuous across $S_0$ and $S_1$. In order to eliminate the surface fields and obtain a relation between $f(\mathbf{k}_0)$ and $a(\mathbf{k}_0)$ we shall make use of suitable expansions for these fields. A wide choise of complete systems of functions in which the surface field can be expanded is in principle available here. However, since we eventually want to obtain the $T$ matrix for the inhomogeneity in terms of spherical waves, it is most convenient to use spherical waves also for the expansion of the surface fields. Since $\psi_2(\mathbf{r})$ is regular everywhere inside $S_1$, it is here natural to choose an expansion in terms of regular spherical waves, and we shall use an expansion of the form

$$\psi_2^-(\mathbf{r})=\sum_n \alpha_n^{(2)}\,\mathrm{Re}\psi_n(k_2\mathbf{r}) \ . \tag{16}$$

We note that an expansion for $\psi_2^-$ on $S_2$ in terms of $\{\mathrm{Re}\psi_n(k_2\mathbf{r})\}$ is also obtained if an "inner Rayleigh hypothesis" is valid, i.e., if the expansion for $\psi_2(\mathbf{r})$ in terms of $\{\mathrm{Re}\psi_n(k_2\mathbf{r})\}$, which always exists inside the inscribed sphere of $S_1$, is furthermore valid everywhere *on* $S_1$. In general, however, the use of the expansion (16) in the $T$ matrix formalism does not rely on such an inner Rayleigh hypothesis.[1,11]

The surface field $\psi_1^-$ is treated analogously. Since $S_0$ is an infinite surface and the plane waves are not square integrable over $S_0$, a slight extension of the completeness properties given in Ref. 11 is required, and it is given in Appendix A. As is shown there, we can on $S_0$ use an expansion for $\psi_1^-(\mathbf{r})$ which contains plane waves corresponding to both the $C_+$ and $C_-$ contours:

$$\psi_1^-(\mathbf{r})=\int_0^{2\pi} d\beta_1\left[\int_{C_-}\alpha(\mathbf{k}_1)+\int_{C_+}\beta(\mathbf{k}_1)\right]\times e^{i\mathbf{k}_1\cdot\mathbf{r}}\sin\alpha_1\,d\alpha_1 \ . \tag{17}$$

The use of an expansion (17) is again independent of the Rayleigh hypothesis.

If (16) and (17) are introduced, together with the boundary conditions $\psi_1^+=\zeta_1^{-1}\zeta_2\psi_2^-\equiv C_{12}\psi_2^-$ and $\hat{n}\cdot\nabla\psi_1^+=\hat{n}\cdot\nabla\psi_2^-$, into (14) and (15), we thus obtain

$$\int_0^{2\pi} d\beta_1\left[\int_{C_-}\alpha(\mathbf{k}_1)+\int_{C_+}\beta(\mathbf{k}_1)\right]I(\mathbf{k}_1,\mathbf{r})\sin\alpha_1\,d\alpha_1 =-i\sum_{n,n'}\mathrm{Re}\psi_n(k_1\mathbf{r})Q_{nn'}(\mathrm{Out},\mathrm{Re})\alpha_{n'}^{(2)} \tag{18}$$

for $\mathbf{r}$ inside the inscribed sphere of $S_1$, and

$$\int_0^{2\pi} d\beta_1\left[\int_{C_-}\alpha(\mathbf{k}_1)+\int_{C_+}\beta(\mathbf{k}_1)\right]I(\mathbf{k}_1,\mathbf{r})\sin\alpha_1\,d\alpha_1 =-i\sum_{n,n'}\psi_n(k_1\mathbf{r})Q_{nn'}(\mathrm{Re},\mathrm{Re})\alpha_{n'}^{(2)} \tag{19}$$

for $\mathbf{r}$ above $S_0$ and $\mathbf{r}$ outside the circumscribed sphere of $S_1$. Here we have used the notation

$$Q_{nn'}(\mathrm{Out},\mathrm{Re})\equiv k_1\int_{S_1}\{\psi_n(k_1\mathbf{r}')\nabla'\,\mathrm{Re}\psi_{n'}(k_2\mathbf{r}') - C_{12}[\nabla'\psi_n(k_1\mathbf{r}')]\,\mathrm{Re}\psi_{n'}(k_2\mathbf{r}')\}\cdot d\mathbf{S}_1' . \tag{20}$$

$Q_{nn'}(\mathrm{Re},\mathrm{Re})$ is given by an analogous expression which has regular spherical waves in all four places. Furthermore,

$$I(\mathbf{k},\mathbf{r})\equiv\int_{S_0}[e^{i\mathbf{k}\cdot\mathbf{r}'}\nabla' G(\mathbf{r},\mathbf{r}';k) - (\nabla' e^{i\mathbf{k}\cdot\mathbf{r}'})G(\mathbf{r},\mathbf{r}';k)]\cdot d\mathbf{S}_0' \ . \tag{21}$$

In this context we note that in analogy with the $Q_{nn'}$ matrices (20) of the spherical wave formalism for finite scatterers[1-4] we can introduce a corresponding quantity for $S_0$, which refers to plane waves (cf. also Ref. 10). We put ($C_{01}\equiv\zeta_0^{-1}\zeta_1$)

$$Q(\mathbf{k}_0,\mathbf{k}_1)\equiv\frac{k_0}{8\pi^2}\int_{S_0}[e^{-i\mathbf{k}_0\cdot\mathbf{r}'}\nabla'(e^{i\mathbf{k}_1\cdot\mathbf{r}'}) - C_{01}(\nabla' e^{-i\mathbf{k}_0\cdot\mathbf{r}'})\,e^{i\mathbf{k}_1\cdot\mathbf{r}'}]\cdot d\mathbf{S}_0' \ . \tag{22}$$

Thus, if the surface $S_0$ is a plane, $Q(\mathbf{k}_0,\mathbf{k}_1)$ is "diagonal" in the sense that it is proportional to $\delta^{(2)}[\hat{n}_0\times(\mathbf{k}_0-\mathbf{k}_1)]$. If the boundary conditions on $S_0$ are taken into account and if $\psi_1^-$ is expressed by means of (17), Eqs. (7) and (11) can be written

$$f(\mathbf{k}_0)=-i\int_0^{2\pi} d\beta_1\left[\int_{C_-}\alpha(\mathbf{k}_1)+\int_{C_+}\beta(\mathbf{k}_1)\right]\times Q(\mathbf{k}_0,\mathbf{k}_1)\sin\alpha_1\,d\alpha_1 \ , \quad \hat{k}_0\in C_+ \ , \tag{23}$$

$$a(\mathbf{k}_0)=i\int_0^{2\pi} d\beta_1\left[\int_{C_-}\alpha(\mathbf{k}_1)+\int_{C_+}\beta(\mathbf{k}_1)\right]\times Q(\mathbf{k}_0,\mathbf{k}_1)\sin\alpha_1\,d\alpha_1 \ , \quad \hat{k}_0\in C_- \ . \tag{24}$$

We want to extract coefficient relations, analogous to (23) and (24), from (18) and (19). To this end we note that the value of $I(\mathbf{k}_1,\mathbf{r})$ depends on whether $\mathbf{r}$ lies above or below $S_0$ and on whether $\hat{k}_1$ belongs to $C_+$ or $C_-$. In order to calculate the explicit value of $I(\mathbf{k}_1,\mathbf{r})$ one introduces a small (positive) imaginary part to $k_1$ and closes $S_0$ in the upper and lower half space, respectively, depending on whether $\hat{k}_1$ belongs to $C_+$ or $C_-$. In this way one obtains (cf. Appendix B for more details)

$$I(\mathbf{k},\mathbf{r})=\begin{Bmatrix} e^{i\mathbf{k}\cdot\mathbf{r}} & \text{for } \hat{k}\in C_+ \\ 0 & \text{for } \hat{k}\in C_- \end{Bmatrix} \text{ for } \mathbf{r} \text{ above } S_0 \ , \tag{25}$$

$$I(\mathbf{k},\mathbf{r})=\begin{Bmatrix} 0 & \text{for } \hat{k}\in C_+ \\ -e^{i\mathbf{k}\cdot\mathbf{r}} & \text{for } \hat{k}\in C_- \end{Bmatrix} \text{ for } \mathbf{r} \text{ below } S_0 \ . \tag{26}$$

From (18), (19), (25), and (26) we thus get

$$\int_0^{2\pi} d\beta_1\int_{C_-}\alpha(\mathbf{k}_1)(-e^{i\mathbf{k}_1\cdot\mathbf{r}})\sin\alpha_1\,d\alpha_1 =-i\sum_{nn'}\mathrm{Re}\psi_n(k_1\mathbf{r})Q_{nn'}(\mathrm{Out},\mathrm{Re})\alpha_{n'}^{(2)} \ , \tag{27}$$

$$\int_0^{2\pi} d\beta_1\int_{C_+}\beta(\mathbf{k}_1)\,e^{i\mathbf{k}_1\cdot\mathbf{r}}\sin\alpha_1\,d\alpha_1 =-i\sum_{nn'}\psi_n(k_1\mathbf{r})Q_{nn'}(\mathrm{Re},\mathrm{Re})\alpha_{n'}^{(2)} \ . \tag{28}$$

In these equations plane wave expansions occur on the left-hand sides, whereas the right-hand sides contain spherical wave expansions. Coefficient relations can

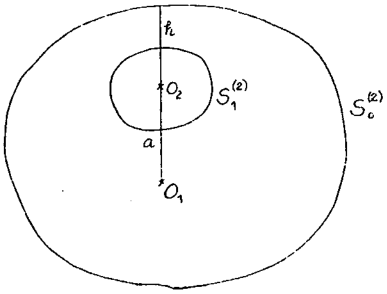


FIG. 3. Geometry for the two-layered situation.

therefore be obtained by introducing the appropriate transformations between these two types of expansions as follows. Multiply Eq. (27) by $Y_n(\hat{r})$ and integrate over a sphere lying inside $S_1$. For real $\hat{k}$ we have (see, e.g., Ref. 28)

$$\int_0^{2\pi} d\varphi \int_0^{\pi} e^{i\mathbf{k}\cdot\mathbf{r}} Y_n(\hat{r}) \sin\vartheta\, d\vartheta = 4\pi i^n j_n(kr) Y_n(\hat{k}) . \quad (29)$$

This is the regular analogue of Eq. (9) and gives the transformation from regular spherical waves to plane waves. Here the left- and right-hand sides of Eq. (29) define analytic functions in the $\cos\alpha$ plane (except for the branch points and cuts). Thus the equality (29) holds for all complex $\hat{k}$ in their common region of analyticity. We thus obtain

$$4\pi i^n \int_0^{2\pi} d\beta_1 \int_{C_-} \alpha(\mathbf{k}_1) Y_n(\hat{k}_1) \sin\alpha_1\, d\alpha_1 = i \sum_{n'} Q_{nn'}(\text{Out, Re}) \alpha_{n'}^{(2)} . \quad (30)$$

In Eq. (28), $\mathbf{r}$ can be taken to lie on a plane $z = z_0 > z_>$ (this plane must also lie above the circumscribed sphere of $S_1$) and therefore we introduce plane waves on the right-hand side of (28). Using (9) for $C_+$ we thus obtain

$$\beta(\mathbf{k}_1) = \sum_{nn'} \frac{1}{2\pi i^{n+1}} Y_n(\hat{k}_1) Q_{nn'}(\text{Re, Re}) \alpha_{n'}^{(2)} . \quad (31)$$

(We can compare coefficients on a plane in these formulas; on a plane $z = \text{const}$ it consists essentially of a slightly modified way of writing an ordinary two-dimensional Fourier integral.[26])

Thus, between the five amplitudes $a(\mathbf{k}_0)$, $f(\mathbf{k}_0)$, $\alpha(\mathbf{k}_1)$, $\beta(\mathbf{k}_1)$, and $\alpha_n^{(2)}$ we have four relations, namely Eqs. (23), (24), (30), and (31), which can be used to extract a relation between any pair of these amplitudes.

Before we discuss how these equations are used in determining the scattering from buried inhomogeneities it is instructive to consider at this stage the relation between these equations and the corresponding equations used in the derivation of the $T$ matrix for a scatterer of finite extension, which consists of layers which consecutively enclose each other.[7,8] Consider the case depicted in Fig. 3. This case is treated in detail in Refs. 7 and 8. We note that Eqs. (23), (24), (30), and (31) above are the analogs of Eqs. (25)–(28) in Ref. 8. Furthermore, we note that the results (25) and (26) above are the analogs of Eqs. (20) and (21) in Ref. 8. Thus the structure of the equations in the two cases is similar, and the main difference is, naturally, that in the present case there occur transformations which describe the change from spherical to plane waves and vice versa.

One might also try to exploit the similarity between the two cases in Figs. 1 and 3 further by considering case 1 (in Fig. 1) as a limit of case 2 when $S_0^{(2)}$ becomes infinitely large, i.e., when $a \to \infty$, with constant $h$ (in Fig. 3). This has been done, e.g., by D'Yakonov in special cases.[15]

The equations which are used in Ref. 15 are essentially those which are obtained from the general equations in Refs. 7 and 8 by explicitly introducing the displacement of the inner surface and then considering a source on the symmetry axis. This limiting procedure does not appear to be particularly useful for actual computations. It contains nevertheless some interesting aspects of the problem which warrant a few additional remarks.

The free space Green's function $G(\mathbf{r}, \mathbf{r}'; k)$ and its expansions in various complete function systems play a central role in the $T$ matrix formalism, and it is instructive to first consider the effect of the limit on this function. In case 2 the relevant expansion for $G(\mathbf{r}, \mathbf{r}'; k)$ is in terms of spherical waves, as given by Eq. (13). Consider therefore the effect of the limiting procedure on this expansion. The limit $a \to \infty$ transforms the portion of $S_0^{(2)}$ which lies in the vicinity of $S_1^{(2)}$ into a plane and the spherical coordinates around $0_1$ becomes a cylindrical coordinate system in the vicinity of $0_2$: If $\theta$ and $\varphi$ are spherical angles at $0_1$ we have in the limit $a \to \infty$ that $r_1' \to \infty$, $\theta \to 0$ so that $\theta r_1' \equiv \zeta$, $r_1' - a \equiv z'$ where $(\zeta, \varphi, z)$ forms a cylindrical coordinate system with origin at $0_2$. However, if one simply lets $r_1' \to \infty$ and $\theta \to 0$, the arguments of the spherical waves degenerate. In order to get a meaningful result the limit $a \to \infty$ must be combined with a limit $n \to \infty$. The way in which the limit for $n$ should be introduced is suggested by the notion of Lie group contraction (see, e.g., Refs. 30 and 31): In the limit $a \to \infty$, the group of transformations of the sphere, i.e., the three-dimensional rotation group $R(3)$, is "contracted" to the group of transformations of the plane, i.e., the two-dimensional Euclidean group $E(2)$.

The notion of contraction has been extended to group representations by means of additional prescriptions. In the present case these prescriptions[30,31] require that the limit $\theta \to 0$ is to be coupled to the limit $n \to \infty$ by the relation $\theta = \mu/n$ where $\mu$ is arbitrary but fixed. Taking this limit in the spherical harmonics (see, e.g., Ref. 30), we have

$$\lim_{n\to\infty} (2n+1)^{-1/2} Y_{nm}(\mu/n, \varphi) = (-1)^m J_m(\mu) e^{im\varphi} . \quad (32)$$

Since the limit $\theta \to 0$ is coupled to $r_1' \to \infty$ by means of $\theta r_1' = \zeta$ we have $r_1' n^{-1} = \zeta\mu^{-1} \equiv \lambda^{-1}$, where $\lambda$ is arbitrary but fixed during the limiting procedure. The relevant limit in the spherical Bessel functions is therefore

$$\lim_{n\to\infty} [j_n(k\lambda^{-1}n),\ h_n(k\lambda^{-1}n)] ,$$

where $\lambda$ is arbitrary, i.e., both $k\lambda^{-1} < 1$ and $k\lambda^{-1} > 1$ appear. Here we use the asymptotic series for $J_n(\kappa n)$ and

$H_n(\kappa n)$ and we keep only the first term in the series. (See, e.g., Refs. 32 and 33. The required formulas are given in simple form in Ref. 32, p. 631; the full asymptotic series, valid uniformly in $\kappa$, is given in Ref. 33.) From $\lambda = r^{-1}n$ we have $\Delta\lambda = r^{-1}\Delta n$ (where $\Delta n = 1$ in the sum). Collecting the above observations and using the explicit form of the asymptotic series for $J_n$ and $H_n$ we arrive at the following formal limit ($\theta = \xi\lambda n^{-1}$, $r = n\lambda^{-1}$, etc.):

$$j_n(kr')\,h_n(kr)Y_{nm}(\theta,\varphi)\bar{Y}_{nm}(\theta',\varphi') \xrightarrow[n\to\infty]{} (-i)\frac{\lambda d\lambda}{(\lambda^2-k^2)^{1/2}} \times \exp[-(\lambda^2-k^2)^{1/2}|z-z'|]\,J_m(\lambda\xi)J_m(\lambda\xi')\,e^{im(\varphi-\varphi')} . \qquad (33)$$

The sum over $n$ goes over to an integral over $\lambda$, and thus we get that the spherical wave expansion (13) is, by the limiting procedure briefly described above, contracted into a cylinder wave expansion which can be written

$$G(\mathbf{r},\mathbf{r}';k) = \sum_m \frac{2-\delta_{0,m}}{4\pi}\cos m(\varphi-\varphi')\int_0^\infty \frac{\lambda\, d\lambda}{(\lambda^2-k^2)^{1/2}} \times \exp[-(\lambda^2-k^2)^{1/2}|z-z'|]\,J_m(\lambda\xi)J_m(\lambda\xi') \qquad (34)$$

(cf., e.g., Ref. 32; note that this expansion contains both harmonic and evanscent waves in the $z$ direction.)

One could now proceed and consider the effect of the limit in other contexts in the $T$ matrix formalism. As is indicated by the above result, one would then expect to get a cylinder wave version of, e.g., the previous equations (23), (24), (30), and (31). However, we shall find it more useful to stay within the plane wave formalism. Furthermore, if cylindrical wave expansions are the most useful ones in a particular application, the cylindrical wave analogs of Eqs. (23), (24), (30), and (31) can, of course, also be derived directly, and therefore we do not go into further details of the limiting procedure here.

So far we have explicitly assumed that the source is situated above the surface $S_0$, and this fact partly determines the structure of the basic equations. However, we can just as well treat the case when the source lies in $V_1$ or $V_2$ and develop a similar formalism for these cases. The starting point is again the integral representation (2), and the main difference is that the source term will appear in different places in the equations.

In order to illustrate the modifications which occur, we consider the case when the source lies in $V_1$. Equation (2), applied to $V_0$, then gives [cf. (23) and (24)]

$$f(\mathbf{k}_0) = -i\int_0^{2\pi} d\beta_1\left[\int_{C_-}\alpha(\mathbf{k}_1) + \int_{C_+}\beta(\mathbf{k}_1)\right] \times Q(\mathbf{k}_0,\mathbf{k}_1)\sin\alpha_1\, d\alpha_1 , \quad \hat{k}_0 \in C_+ , \qquad (35)$$

$$0 = \int_0^{2\pi} d\beta_1\left[\int_{C_-}\alpha(\mathbf{k}_1) + \int_{C_+}\beta(\mathbf{k}_1)\right] \times Q(\mathbf{k}_0,\mathbf{k}_1)\sin\alpha_1\, d\alpha_1 , \quad \hat{k}_0 \in C_- . \qquad (36)$$

The source term appears when Eq. (2) is applied to $V_1$. Consider a source at $P_1$ (see Fig. 4) which generates a field which can be expanded in outgoing spherical waves

$$\psi_1^{\rm inc} = \sum_n a_n \psi_n(k_1\mathbf{r}_1) . \qquad (37)$$

For an $\mathbf{r}$ inside (the inscribed sphere of) $S_1$ we use[5,6,28]

$$\psi_n(k_1\mathbf{r}) = \psi_n[k_1(\mathbf{b}+\mathbf{r})] = \sum_{n'} \sigma_{nn'}(\mathbf{b})\,\mathrm{Re}\psi_{n'}(k_1\mathbf{r}) \quad (b > r) \qquad (38)$$

to obtain an expansion for $\psi_1^{\rm inc}$ in terms of regular spherical waves. Writing the resulting expansion for $\psi_1^{\rm inc}$ as

$$\psi_1^{\rm inc}(\mathbf{r}) = \sum_n a_n'\,\mathrm{Re}\psi_n(k_1\mathbf{r}) , \qquad (39)$$

we get, in the same way as the derivation of Eq. (30), the equation

$$4\pi i^n \int_0^{2\pi} d\beta_1 \int_{C_-} \alpha(\mathbf{k}_1)Y_n(\mathbf{k}_1)\sin\alpha_1\, d\alpha_1 = -a_n' + i\sum_{n'} Q_{nn'}(\mathrm{Out},\mathrm{Re})\alpha_{n'}^{(2)} . \qquad (40)$$

For $\mathbf{r}$ in $V_0$ and above $z = z_>$, we use Eq. (9) to express $\psi_1^{\rm inc}$ in terms of harmonic and evanescent plane waves. Writing this expansion as

$$\psi_1^{\rm inc}(\mathbf{r}) = \int_0^{2\pi} d\beta_1 \int_{C_+} a^1(\mathbf{k}_1)\,e^{i\mathbf{k}_1\cdot\mathbf{r}}\sin\alpha_1\, d\alpha_1 , \qquad (41)$$

we get, in analogy with Eq. (31),

$$\beta(\mathbf{k}_1) = a^1(\mathbf{k}_1) + \sum_{nn'}\frac{1}{2\pi i^{n+1}}Y_n(\hat{k}_1) \times Q_{nn'}(\mathrm{Re},\mathrm{Re})\alpha_{n'}^{(2)} . \qquad (42)$$

Thus, Eqs. (35), (36), (40), and (42) constitute the basic equations when the source lies in $V_1$. It is clear that the equations for a source in $V_2$ can be derived in a similar way. The details of this are left to the reader.

## II. CALCULATION OF THE SCATTERED FIELD

The four basic Eqs. (23), (24), (30), and (31) will now be studied in more detail. As was mentioned before, these equations can in principle be used to obtain a relation between any two of the quantities $a(\mathbf{k}_0)$, $f(\mathbf{k}_0)$, $\alpha(\mathbf{k}_1)$, $\beta(\mathbf{k}_1)$, and $\alpha_n^{(2)}$. In view of the applications we have in mind, namely to calculate the effect of the inhomogeneity on the scattered field in $V_0$, we shall in this section concentrate on the relation between $f(\mathbf{k}_0)$ and $a(\mathbf{k}_0)$. Since the scattering response from the finite inhomo-

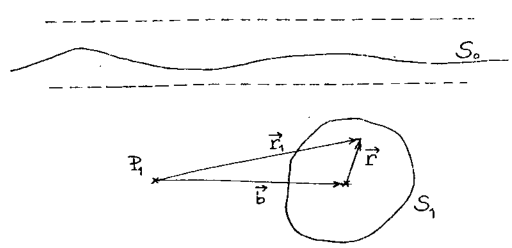


FIG. 4. Geometry and notations for a source location in $V_1$.

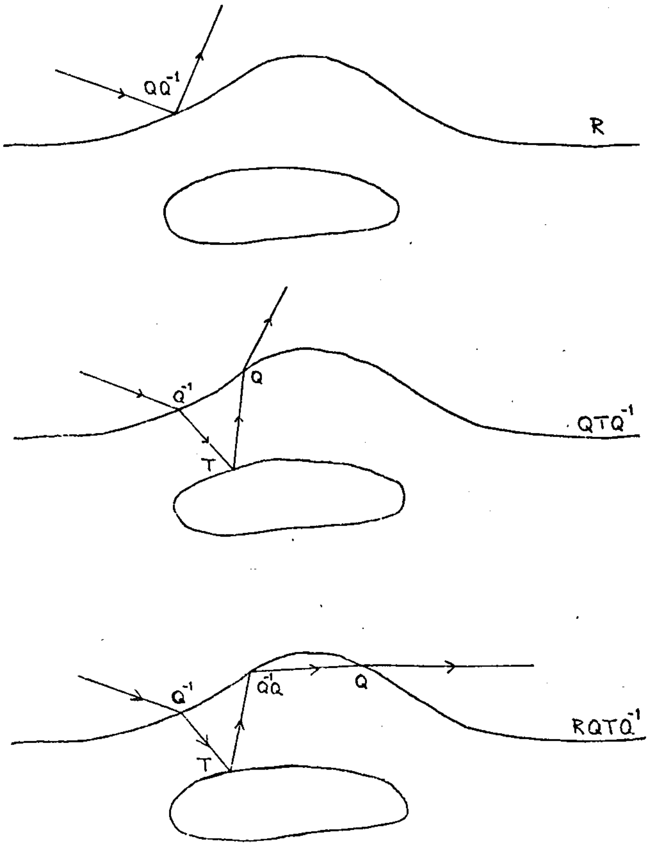


FIG. 5. Multiple-scattering interpretation of (44).

geneity is determined by its $T$ matrix and the prescribed incoming field, we solve the equations in such a way that this $T$ matrix is brought into the description. This is done by solving Eq. (30) for $\alpha_n^{(2)}$. The $T$ matrix $T_{nn'}(1)$ of the inhomogeneity is given by[1]

$$T_{nn'}(1) = -\sum Q_{nn''}(\mathrm{Re},\mathrm{Re})[Q^{-1}(\mathrm{Out},\mathrm{Re})]_{n''n'} .$$

Then one gets

$$\beta(\mathbf{k}_1) = 2\sum_{nn'} i^{n'-n} Y_n(\hat{k}_1) T_{nn'}(1) \int_0^{2\pi} d\beta_1' \int_{C_-} \alpha(\mathbf{k}_1') \times Y_{n'}(\hat{k}_1') \sin\alpha_1' \, d\alpha_1', \quad \hat{k}_1 \in C_+ . \tag{43}$$

This type of relation between $\beta(\mathbf{k}_1)$ and $\alpha(\mathbf{k}_1)$ was to be expected. $\alpha(\mathbf{k}_1)$ and $\beta(\mathbf{k}_1)$ are the amplitudes of up- and down-going plane waves, respectively, and these amplitudes can be expected to be related by the quantity $T_{nn'}(1)$, which expresses the scattering characteristics of the inhomogeneity. However, $\alpha(\mathbf{k}_1)$ and $\beta(\mathbf{k}_1)$ are both plane wave expansion coefficients, while $T_{nn'}(1)$ refers to spherical waves. Therefore the transformation functions between spherical and plane waves, namely $Y_n(\hat{k})$, appear in Eq. (43) together with the appropriate summations and integrations. In the above it was explicitly assumed that the inhomogeneity was homogeneous. However, if a more complicated inhomogeneity is considered—e.g., several homogeneous or layered scatterers or scatterers with continuously varying material parameters—a similar analysis can be carried out.[6,8,9] In this case the basic equations will be the same—Eqs. (23), (24), and (43)—but with the difference that the $T$ matrix in Eq. (43) should be the $T$ matrix for this more complicated inhomogeneity.

Since scattering occurs at both $S_0$ and $S_1$, we expect the full solution to contain multiple scattering effects. In order to get some insight into these multiple scattering phenomena it is instructive to make a formal analysis of the problem by simply inverting the integral relations (23), (24), and (43). Symbolically, we may write these equations as follows:

$$\mathbf{f} = -iQ\boldsymbol{\alpha} - iQ\boldsymbol{\beta} ,$$

$$\mathbf{a} = iQ\boldsymbol{\alpha} + iQ\boldsymbol{\beta} ,$$

$$\boldsymbol{\beta} = T\boldsymbol{\alpha} ,$$

which formally gives

$$\mathbf{f} = (R - QTQ^{-1})(1 + QTQ^{-1})^{-1}\mathbf{a} . \tag{44}$$

The different character of the $Q$'s is not made explicit here; this character is determined by the vectors on which they operate. Thus the $T$ operator for the surface $S_0$, i.e., the reflection coefficient for $S_0$, reads in this abbreviated notation simply $R = -QQ^{-1}$. A physical interpretation of Eq. (44) in terms of multiple scattering contributions is obtained by expanding the inverse in powers of $T$. (See Fig. 5.)

We note that Eqs. (23) and (24) relate plane wave expansion coefficients directly. In order to obtain the solution in a form which is more amenable to a numerical treatment, we want to rewrite (24) in matrix form. This is done by introducing the spherical wave projections of $\alpha(\mathbf{k}_1)$. Thus we introduce the vector $\{c_n\}$ defined by

$$c_n \equiv \int_0^{2\pi} d\beta_1 \int_{C_-} \alpha(\mathbf{k}_1) Y_n(\hat{k}_1) \sin\alpha_1 \, d\alpha_1 . \tag{45}$$

$\mathbf{c} \equiv \{c_n\}$ are unknown quantities since they are just the spherical wave projections of the unknown plane wave amplitude $\alpha(\mathbf{k}_1)$. Equations (43) and (24) can then be written

$$\beta(\mathbf{k}_1) = 2\sum_{nn'} i^{n'-n} Y_n(\hat{k}_1) T_{nn'}(1) c_{n'} , \quad \hat{k}_1 \in C_+ , \tag{46}$$

$$a(\mathbf{k}_0) = i\int_0^{2\pi} d\beta_1 \left[ \int_{C_-} \alpha(\mathbf{k}_1) + \int_{C_+} 2\sum_{nn'} i^{n'-n} Y_n(\hat{k}_1) \times T_{nn'}(1) c_{n'} \right] Q(\mathbf{k}_0, \mathbf{k}_1) \sin\alpha_1 \, d\alpha_1 , \quad \hat{k}_0 \in C_- . \tag{47}$$

As already mentioned, $Q(\mathbf{k}_0, \mathbf{k}_1)$ is in general not a function in the usual sense and relations involving $Q(\mathbf{k}_0, \mathbf{k}_1)$ should be understood in a distribution sense. However, we note that Eq. (47) in the case of a flat surface $S_0$ degenerates into an algebraic relation. We now rewrite Eq. (47) with the purpose of isolating $\alpha(\mathbf{k}_1)$. Thus we formally introduce the inverse of $Q(\mathbf{k}_0, \mathbf{k}_1)$, $\hat{k}_0, \hat{k}_1 \in C_-$, which is analogous to the inverse of the matrix $Q_{nn'}$(Out, Re) in the case of finite scatterers. From Eq. (47) we then get formally

$$\alpha(\mathbf{k}_1) = -i \int_0^{2\pi} d\beta_0 \int_{C_-} Q^{-1}(\mathbf{k}_0, \mathbf{k}_1) \Big[ a(\mathbf{k}_0) - 2i \sum_{nn'} i^{n'-n} T_{nn'}(1) c_{n'} \int_0^{2\pi} d\beta_1' \int_{C_+} Q(\mathbf{k}_0, \mathbf{k}_1') \times Y_n(\hat{k}_1') \sin\alpha_1' \, d\alpha_1' \Big] \sin\alpha_0 \, d\alpha_0 \ , \quad \hat{k}_1 \in C_- \ . \tag{48}$$

Multiply Eq. (48) by $Y_n(\hat{k}_1)$ and integrate over $\beta_1$ and $\alpha_1$, where $\alpha_1$ is taken along the $C_-$ contour. Then the left-hand side becomes equal to $c_n$ and Eq. (48) can be written as a matrix equation as follows:

$$c_n = d_n - \sum_{n'} A_{nn'} c_{n'} \ , \tag{49}$$

where

$$d_n \equiv -i \int_0^{2\pi} d\beta_1 \int_{C_-} Y_n(\hat{k}_1) \int_0^{2\pi} d\beta_0 \times \int_{C_-} Q^{-1}(\mathbf{k}_0, \mathbf{k}_1) a(\mathbf{k}_0) \sin\alpha_0 \, d\alpha_0 \sin\alpha_1 \, d\alpha_1 \ , \tag{50}$$

$$A_{nn'} \equiv -2 \sum_{n''} i^{n'-n''} T_{n''n'}(1) \int_0^{2\pi} d\beta_1 \int_{C_-} Y_n(\hat{k}_1) \times \int_0^{2\pi} d\beta_1' \int_{C_+} R(\mathbf{k}_1, \mathbf{k}_1') Y_{n''}(\hat{k}_1') \sin\alpha_1' \, d\alpha_1' \sin\alpha_1 \, d\alpha_1 \ , \tag{51}$$

$$R(\mathbf{k}_1, \mathbf{k}_1') \equiv -\int_0^{2\pi} d\beta_0 \int_{C_-} Q^{-1}(\mathbf{k}_0, \mathbf{k}_1) Q(\mathbf{k}_0, \mathbf{k}_1') \times \sin\alpha_0 \, d\alpha_0 \ , \quad \hat{k}_1 \in C_- \ , \quad \hat{k}_1' \in C_+ \ . \tag{52}$$

Here $R(\mathbf{k}_1, \mathbf{k}_1^1)$ is the reflection coefficient from underneath the surface $S_0$. Similarly $Q^{-1}(\mathbf{k}_0, \mathbf{k}_1)$ in Eq. (50) is the transmission coefficient from $V_0$ to $V_1$ through $S_0$ (cf. the analogous meaning of the corresponding $Q$ matrices for a finite surface[8]). Thus the coefficients $d_n$ are the spherical wave projections of the plane wave amplitudes of the incoming field after it has passed through $S_0$. In Eq. (49), $d_n$ and $A_{nn'}$ are known quantities; i.e., this equation can be used to determine $c_n$. We also note that the matrix $A_{nn'}$ is independent of the field. Equation (49) can be written in the form

$$\mathbf{c} = (1 + A)^{-1} \mathbf{d} \ . \tag{53}$$

Sufficient conditions on the infinite matrix $A$ for a solution to this equation to exist can be found in Ref. 34. Appendix C contains more details on the nature of these conditions for an $A$ matrix given by (51), for a simple choice of $T_{nn'}$. We note that the multiple scattering effects are contained in Eq. (53). $A_{nn'}$ contains a product of $T_{n''n}$ and $R(\mathbf{k}_1, \mathbf{k}_1')$ and therefore describes a reflection at $S_0$ from underneath and a scattering from $S_1$. If $(1 + A)^{-1}$ is expanded in powers of $A$, the terms in such an expansion can thus be interpreted as multiple scattering contributions containing an increasing number of reflections back and forth between $S_0$ and $S_1$.

When $\mathbf{c}$ has been determined, $\alpha(\mathbf{k}_1)$ and $\beta(\mathbf{k}_1)$ are obtained from Eqs. (48) and (46), respectively, and the amplitude $f(\mathbf{k}_0)$ of the scattered field in $V_0$ is obtained by inserting Eqs. (46) and (48) in Eq. (23). In this way we obtain

$$f(\mathbf{k}_0) = \int_0^{2\pi} d\beta_0' \int_{C_-} \tilde{R}(\mathbf{k}_0, \mathbf{k}_0') a(\mathbf{k}_0') \sin\alpha_0' \, d\alpha_0' - 2i \sum_{nn'} i^{n'-n} T_{nn'}(1) c_{n'} \int_0^{2\pi} d\beta_1 \int_{C_+} Y_n(\hat{k}_1) \times \Big[ Q(\mathbf{k}_0, \mathbf{k}_1) + \int_0^{2\pi} d\beta_1' \int_{C_-} R(\mathbf{k}_1', \mathbf{k}_1) Q(\mathbf{k}_0, \mathbf{k}_1') \times \sin\alpha_1' \, d\alpha_1' \Big] \sin\alpha_1 \, d\alpha_1 \ , \quad \hat{k}_0 \in C_+ \ . \tag{54}$$

In analogy with the $T$ matrix for a finite scatterer we have here introduced the "$T$ matrix" for the infinite surface $S_0$, which is in this case usually called the reflection coefficient and which is denoted $\tilde{R}(\mathbf{k}_0, \mathbf{k}_0')$. $\tilde{R}(\mathbf{k}_0, \mathbf{k}_0')$ is defined by [cf. (52)]

$$\tilde{R}(\mathbf{k}_0, \mathbf{k}_0') \equiv -\int_0^{2\pi} d\beta_1 \int_{C_-} Q(\mathbf{k}_0, \mathbf{k}_1) \times Q^{-1}(\mathbf{k}_0', \mathbf{k}_1) \sin\alpha_1 \, d\alpha_1 \ , \quad \hat{k}_0 \in C_+ \ , \quad \hat{k}_0' \in C_- \ . \tag{55}$$

The scattered field $\psi_0^{sc}(\mathbf{r})$ is then finally obtained from (54) and (6);

$$\psi_0^{sc}(\mathbf{r}) = \int_0^{2\pi} d\beta_0 \int_{C_+} \Big\{ \int_0^{2\pi} d\beta_0' \int_{C_-} \tilde{R}(\mathbf{k}_0, \mathbf{k}_0') a(\mathbf{k}_0') \times \sin\alpha_0' \, d\alpha_0' + 2 \sum_{nn'} i^{n'-n-1} T_{nn'}(1) c_{n'} \int_0^{2\pi} d\beta_1 \times \int_{C_+} Y_n(\hat{k}_1) \Big[ Q(\mathbf{k}_0, \mathbf{k}_1) + \int_0^{2\pi} d\beta_1' \int_{C_-} R(\mathbf{k}_1', \mathbf{k}_1) \times Q(\mathbf{k}_0, \mathbf{k}_1') \sin\alpha_1' \, d\alpha_1' \Big] \sin\alpha_1 \, d\alpha_1 \Big\} e^{i\mathbf{k}_0 \cdot \mathbf{r}} \sin\alpha_0 \, d\alpha_0 \ . \tag{56}$$

The first term in (56) is a direct reflection at the infinite surface $S_0$, as if no inhomogeneity were present in $V_1$. The remaining terms represent the anomalous scattered field, i.e., that part of $\psi_0^{sc}(\mathbf{r})$ which contains information about the scattering properties of the buried inhomogeneity (and since $c_n$ appears here all the multiple scatterings between $S_0$ and $S_1$ are included in this part). Thus, although the above treatment of the basic equations has been mainly of a formal nature, it provides some insight into the meaning of the various quantities which appear.

In order to see explicitly how this scheme works we specialize to a plane surface $S_0$. In this case $Q(\mathbf{k}_0, \mathbf{k}_1)$ [cf. (22)] becomes a δ function and the integral relations (23) and (24) degenerate into simple algebraic expressions. (Note that we do not yet introduce any additional assumptions concerning the inhomogeneity.) Thus let $S_0$ be the plane $z = z_0$. The Eqs. (52), (55), (50), (51), and (56) then simplify into

$$R(\mathbf{k}_1, \mathbf{k}_1') = -R(\lambda_1) \exp[2iz_0(k_1^2 - \lambda_1^2)^{1/2}] \, \delta(\beta_1 - \beta_1') \times \delta(\alpha_1 + \alpha_1' - \pi)/\sin\alpha_1 \ , \quad \hat{k}_1 \in C_- \ , \quad \hat{k}_1' \in C_+ \ , \tag{57}$$

$$\tilde{R}(\mathbf{k}_0, \mathbf{k}_0') = R(\lambda_0) \exp[-2iz_0(k_0^2 - \lambda_0^2)^{1/2}] \delta(\beta_0 - \beta_0') \times \delta(\alpha_0 + \alpha_0' - \pi)/\sin\alpha_0 \ , \quad \hat{k}_0 \in C_+ \ , \quad \hat{k}_0' \in C_- \ , \tag{58}$$

$$d_n = \frac{k_1}{k_0} \int_0^{2\pi} d\beta_1 \int_{C_-} Y_n(\hat{k}_1)[1 - R(\lambda_1)]$$

$$\times \exp\{iz_0[(k_1^2-\lambda_1^2)^{1/2}-(k_0^2-\lambda_1^2)^{1/2}]\}\, a(\bar{\mathbf{k}}_0)\sin\alpha_1\, d\alpha_1\,, \tag{59}$$

$$A_{nn'} = 2\sum_{n''} i^{n'-n''}T_{n''n'}(1)\int_0^{2\pi} d\beta_1 \int_{C_-} Y_n(\hat{k}_1) \times R(\lambda_1)Y_{n''}(\hat{k}_1^\dagger)\exp[2iz_0(k_1^2-\lambda_1^2)]\sin\alpha_1\, d\alpha_1\,, \tag{60}$$

$$\psi_0^{sc}(\mathbf{r}) = \int_0^{2\pi} d\beta_0 \int_{C_+} R(\lambda_0)\exp[-2iz_0(k_0^2-\lambda_0^2)^{1/2}]\, a(\mathbf{k}_0^\dagger) \times e^{i\mathbf{k}_0\cdot\mathbf{r}}\sin\alpha_0\, d\alpha_0 + \frac{2k_0}{k_1}\sum_{nn'} i^{n'-n}T_{nn'}(1)\,c_{n'} \times \int_0^{2\pi} d\beta_0\int_{C_+} Y_n(\bar{\bar{k}}_0)\exp\{iz_0[(k_1^2-\lambda_0^2)^{1/2}-(k_0^2-\lambda_0^2)^{1/2}]\} \times[1+R(\lambda_0)]\,e^{i\mathbf{k}_0\cdot\mathbf{r}}\sin\alpha_0\, d\alpha_0 \equiv \psi_0^{sc,dir}(\mathbf{r})+\psi_0^{sc,anom}(\mathbf{r})\,, \tag{61}$$

where

$$R(\lambda) = \frac{C_{01}(k_0^2-\lambda^2)^{1/2}-(k_1^2-\lambda^2)^{1/2}}{C_{01}(k_0^2-\lambda^2)^{1/2}+(k_1^2-\lambda^2)^{1/2}}$$

is the well-known reflection coefficient of the plane. Similarly, $1+R(\lambda)$ and $1-R(\lambda)$ are both transmission coefficients through $S_0$. $1-R(\lambda_1)$ refers to transmission from $V_0$ to $V_1$ and $1+R(\lambda_0)$ to transmission from $V_1$ to $V_0$. (Additional phase factors also appear, since the coordinate origin lies at a distance $z_0$ below $S_0$.) We have also introduced the notations

$$\lambda_i \equiv k_i\sin\alpha_i\,,\quad i=0,1\,,$$
$$\bar{\mathbf{k}}_0 \equiv [k_1\sin\alpha_1\cos\beta_1,\ k_1\sin\alpha_1\sin\beta_1,\ -(k_0^2-\lambda_1^2)^{1/2}],$$
$$\mathbf{k}_0^\dagger \equiv k_0(\sin\alpha_0\cos\beta_0,\ \sin\alpha_0\sin\beta_0,\ -\cos\alpha_0)\,,$$
$$\bar{\bar{k}}_0 \equiv k_0k_1^{-1}[\sin\alpha_0\cos\beta_0,\ \sin\alpha_0\sin\beta_0,\ (k_1^2k_0^{-2}-\sin^2\alpha_0)^{1/2}],$$
$$\hat{k}_1^\dagger \equiv (\sin\alpha_1\cos\beta_1,\ \sin\alpha_1\sin\beta_1\,,\ -\cos\alpha_1)\,,$$

and the square roots are specified by the conditions $\mathrm{Im}(k_i^2-\lambda_i^2)^{1/2}\geq 0$.

## III. NUMERICAL APPLICATIONS

In this section we will illustrate how the results can be used for numerical computation of the anomalous scattered field. The simplest possible case is of course that of a spherical inhomogeneity, i.e., a diagonal $T(1)$ matrix, and we give numerical results for a selection of media contrasts, sphere sizes, and source positions for this case. Results concerning nonspherical inhomogeneities are of course of great interest, and therefore we present another set of computations concerning this case. Descriptions of several examples of the numerical calculation of the nondiagonal $T$ matrix corresponding to a single nonspherical scatterer can be found in Refs. 1 and 13. Furthermore, computer programs are available for computation of the $T$ matrix of a single general rotationally symmetric scatterer.[35,36] Our examples concern various configurations of two spheres, having different sizes and scattering characteristics. The computation of the $T(1)$ matrix for this case is given in Refs. 6 and 37.

In the numerical examples the incoming field is that of a simple point source

$$\psi_0^{inc}(\mathbf{r}) = \frac{\exp(ik_0|\mathbf{r}-\mathbf{r}_p|)}{|\mathbf{r}-\mathbf{r}_p|} \tag{62}$$

situated at the point $\mathbf{r}_p = (\zeta_p, 0, z_p)$, where $(\zeta, \varphi, z)$ are cylindrical coordinates. From (62) and (5) we get the plane wave amplitudes of $\psi_0^{inc}$,

$$a(\mathbf{k}_0) = \frac{ik_0}{2\pi}\exp[-ik_0(\zeta_p\sin\alpha_0\cos\beta_0+z_p\cos\alpha_0)]\,. \tag{63}$$

Equation (59) and the second part of Eq. (61) then simplify into

$$d_n = 2k_1 i^{m+1}(-1)^n\gamma_n\int_{i\infty}^1 P_n^m(x)\,J_m[k_1\zeta_p(1-x^2)^{1/2}] \times\frac{\exp\{ik_1z_0x+i(z_p-z_0)[k_0^2-k_1^2(1-x^2)]^{1/2}\}\cdot k_1x\,dx}{C_{01}(k_0^2-k_1^2(1-x^2))^{1/2}+k_1x} \times\begin{pmatrix}1\\0\end{pmatrix},\ \begin{pmatrix}\sigma=e\\ \sigma=o\end{pmatrix}, \tag{64}$$

$$\psi_0^{sc,anom}(\mathbf{r}) = \frac{8\pi C_{01}k_0}{k_1}\sum_{nn'} i^{n'-n}T_{nn'}(1)c_{n'}\gamma_n \times i^m\begin{pmatrix}\cos m\varphi\\ \sin m\varphi\end{pmatrix}\int_{i\infty}^1 P_n^m\{[1-k_0^2k_1^{-2}(1-x^2)]^{1/2}\} \times\exp(i\{(z-z_0)k_0x+z_0[k_1^2-k_0^2(1-x^2)]^{1/2}\}) \times J_m[k_0\zeta(1-x^2)^{1/2}]\{k_0C_{01}x+[k_1^2-k_0^2(1-x^2)]^{1/2}\}^{-1} \times k_0x\,dx\quad\text{for}\begin{pmatrix}\sigma=e\\ \sigma=o\end{pmatrix}, \tag{65}$$

where we have used cylindrical coordinates also for $\mathbf{r} = (\zeta, \varphi, z)$ on the right-hand side of (65). $\gamma_n$ is the normalization factor for the spherical harmonics. (See Appendix C.)

The directly scattered field will not be analyzed any further here since the properties of this field has been thoroughly treated in the literature. (For an analytic treatment see Ref. 26 and references given there and for numerical aspects see, e.g., Ref. 38.) Thus we shall mainly present our results in terms of $|\psi_0^{sc,anom}|$, sometimes divided by $|\psi_0^{inc}|$. Some results on the phase of $\psi_0^{sc,anom}$ are also given. It is clear that in actual applications one would often be more interested in the quotient $|\psi_0^{sc,anom}|/|\psi_0^{sc,dir}|$ and the phase difference between $\psi_0^{sc,dir}$ and $\psi_0^{sc,anom}$. Using the well-known results concerning $\psi_0^{sc,dir}$ these quantities can easily be obtained from the results on $\psi_0^{sc,anom}$ given below, but we shall not go further into this in the present article.

In the numerical calculations the evaluation of the infinite integrals constitute an important part. These integrals [cf. Eqs. (60), (64), and (65)] are analytically well behaved when $z_0>0$, but the rapid oscillations of the integrals cause difficulties in numerical evaluation. One oscillating term can be avoided by putting the source and the measuring point on $S_0$. The contour of integration from $i\infty$ to 1 is split into one infinite part from $i\infty$ to 0 and a finite part from 0 to 1. For the finite part we have used the Romberg iteration scheme. The infinite part was evaluated either by the method of Rom-

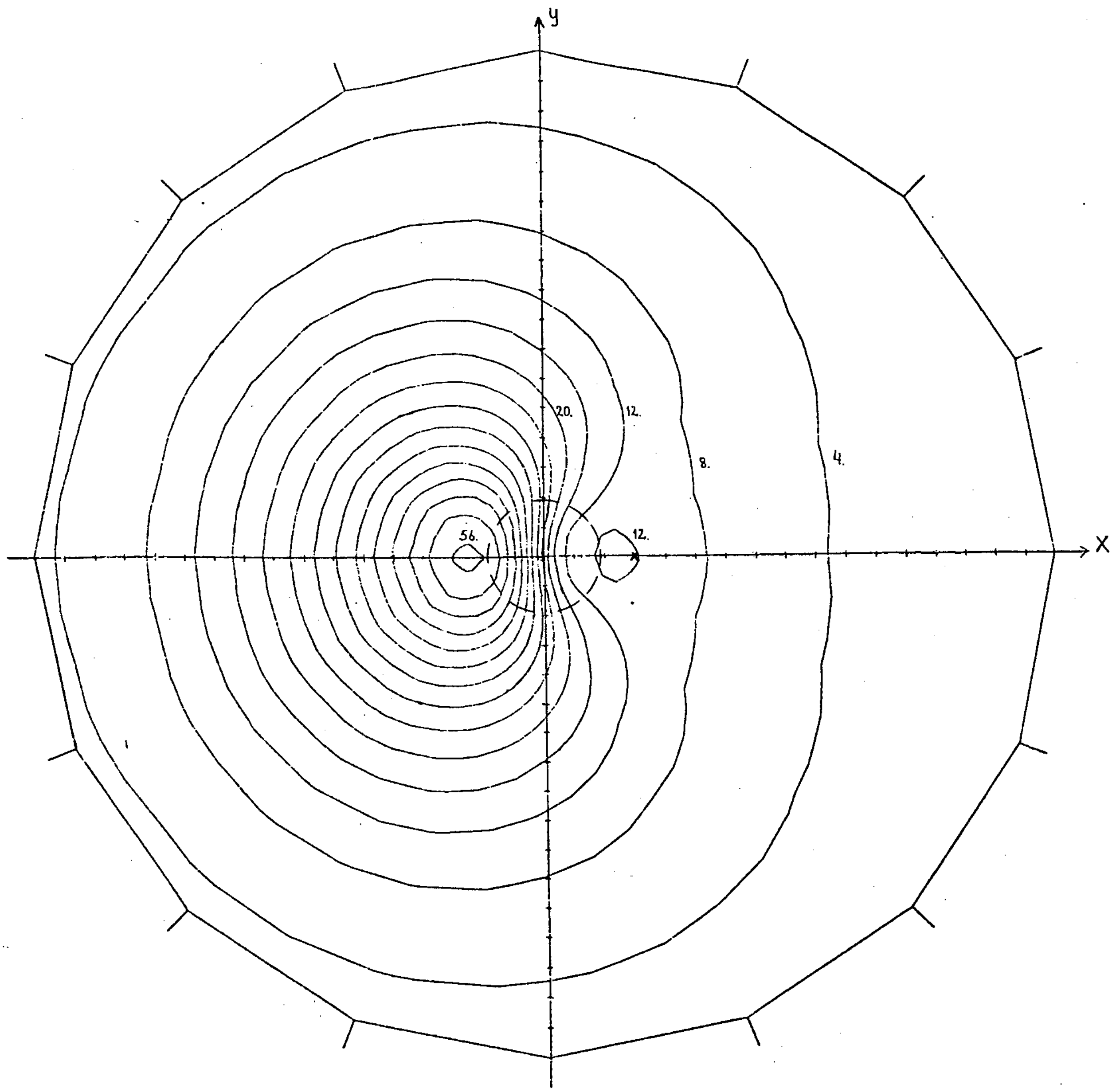


FIG. 6. The amplitude of the anomalous scattered field $|\psi_0^{sc,anom}|$ on the surface $S_0$ $(k_1 z = 2)$ for a buried sphere of radius $k_1 a = 1.25$, $k_2/k_1 = 2$, $\zeta_2/\zeta_1 = 2$. The point source is located at $k_0\zeta_p = 3$. (The scale on the $x$ and $y$ axes are in units of $k_0$. The buried sphere is projected on the surface $S_0$ in units of $k_0$ as a reference.)

berg, after truncation, or a 32-point Gauss–Laguerre quadrature, depending on a cost-effectiveness estimate in each particular case. (See, e.g., 39, 40, and 41.) The numerical procedure is as follows:

(1) Evaluate the $T$ matrix for the inhomogeneity.

(2) Compute the integrals in $A_{nn'}$ and $d_n$ and calculate the matrix $A_{nn'}$.

(3) Obtain the vector $\mathbf{c}$ by interation of Eq. (49).

(4) Evaluate the $\psi_0^{sc,anom}(\mathbf{r})$ in a given measuring point $\mathbf{r}$ by computing the integrals in (65) and sum to sufficient order.

We note that these integrals depend only upon the position vector $\mathbf{r}$ and are independent of the inhomogeneity and the incoming field. Thus these integrals can with advantage be calculated and stored for an array of measuring points. When a new calculation of the scattered field with a different inhomogeneity, but with the same properties of $V_0$ and $V_1$, is desired, one just evaluates the new $T(1)$ matrix and uses this new $T(1)$ matrix together with the old evaluation of the integrals to calculate the new $\psi_0^{sc,anom}$. The evaluation of a solution $\mathbf{c}$ to Eq. (49) can be done in various ways. We have found a simple iteration scheme, the Gauss–Seidel method (see, e.g., Ref. 41), most convenient. For moderate contrasts between the media this method gives fast convergence.

In the calculations we assume that $V_0$, $V_1$, and $V_2$ are all without losses. The difference between the $V_0$ and the $V_1$ parameters is chosen to be moderate, $k_0 = 1.5k_1$, $C_{01} = 2$, and it is the same throughout. The surface $S_0$

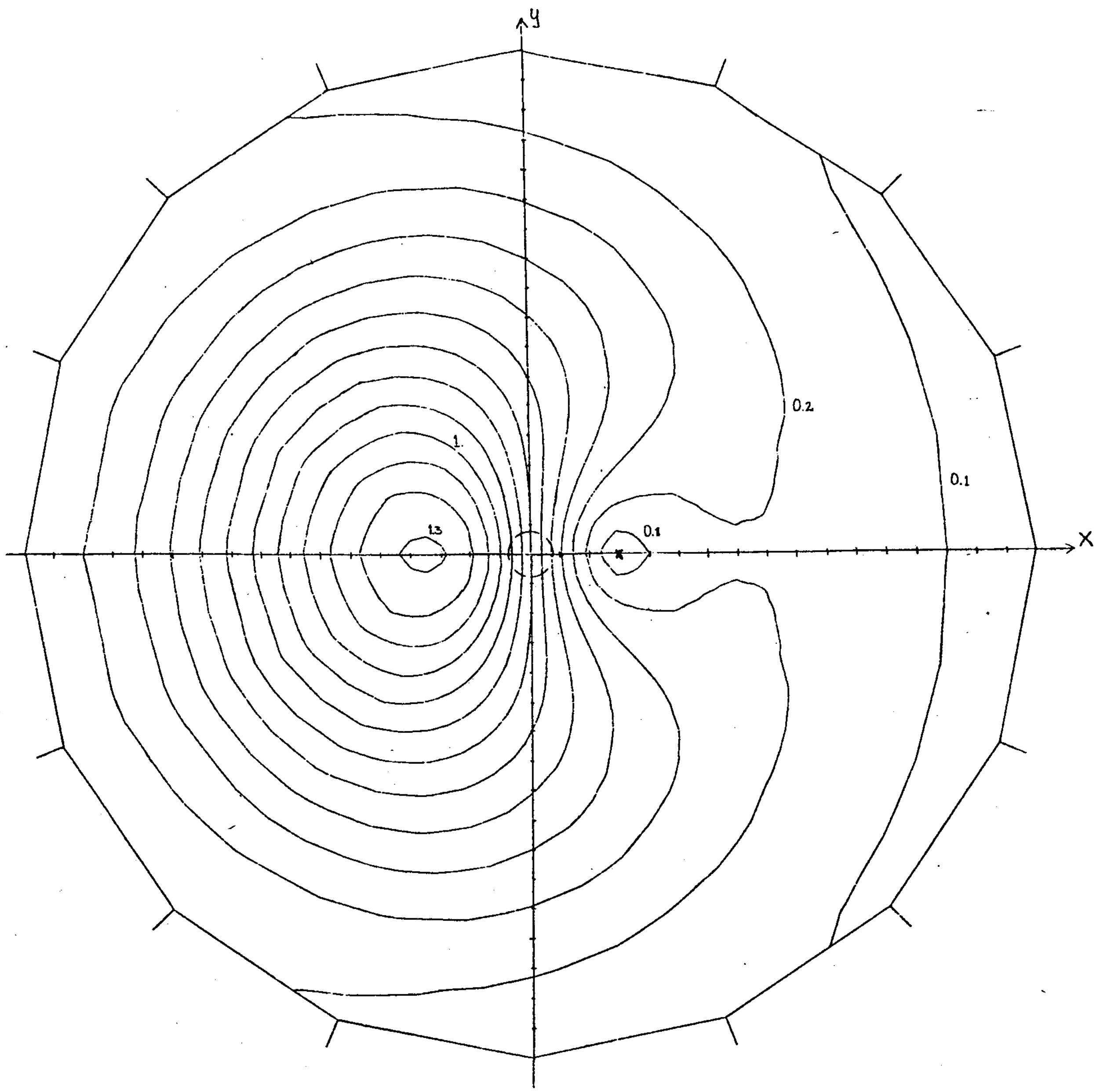


FIG. 7. The ratio $|\psi_0^{sc,anom}/\psi^{inc}|$ for the configuration in Fig. 6, but $k_1a=0.5$. (The scale is given in percent.)

is taken to be $k_1z_0=2$, and three different source positions are considered, namely $k_0\zeta_p=0$, 3, and 15. $\psi_0^{sc,anom}$ is evaluated on $S_0$ and for an array of points obtained by varying the cylindrical coordinates $\zeta$ and $\varphi$. This array varies slightly between the different examples, depending on the variation of the scattered field.

The simplest scattering situation is that of a single buried sphere. Some numerical plots concerning this case are shown in Figs. 6–8. Figures 6 and 7 show surfaces of constant value of the absolute value of the anomalous scattered field and the anomalous scattered field divided by the incoming field, respectively. There are some irregularities of the curves, which obviously are due to the linear interpolation in the computer routine producing these plots. The original calculated fields do not show these irregularities. The absolute value of the anomalous scattered field is given in an arbitrarily chosen scale, which is the same for all graphs showing $|\psi_0^{sc,anom}|$. The absolute scale can be obtained from the graphs showing $|\psi_0^{sc,anom}/\psi_0^{inc}|$. Figure 8 shows a series of graphs of the variation of $|\psi_0^{sc,anom}|$ along the $x$ axis for a selection of radii of the sphere.

In general, the graphs of $|\psi_0^{sc,anom}|$ show a maximum on the far side of the sphere. For larger values of the radius there is a minimum indicating that destructive interference come into play. (These gross features are roughly consistent with what one would expect from qualitative arguments based on phase differences of representative rays; however, any such arguments should of course be used with extreme caution at these long wavelengths.)

Figures 10–16 show the results from computations with the inhomogeneity consisting of two spheres. The radii of the spheres are denoted $a_1$ and $a_2$ and the total $T(1)$ matrix for the configuration of the two spheres

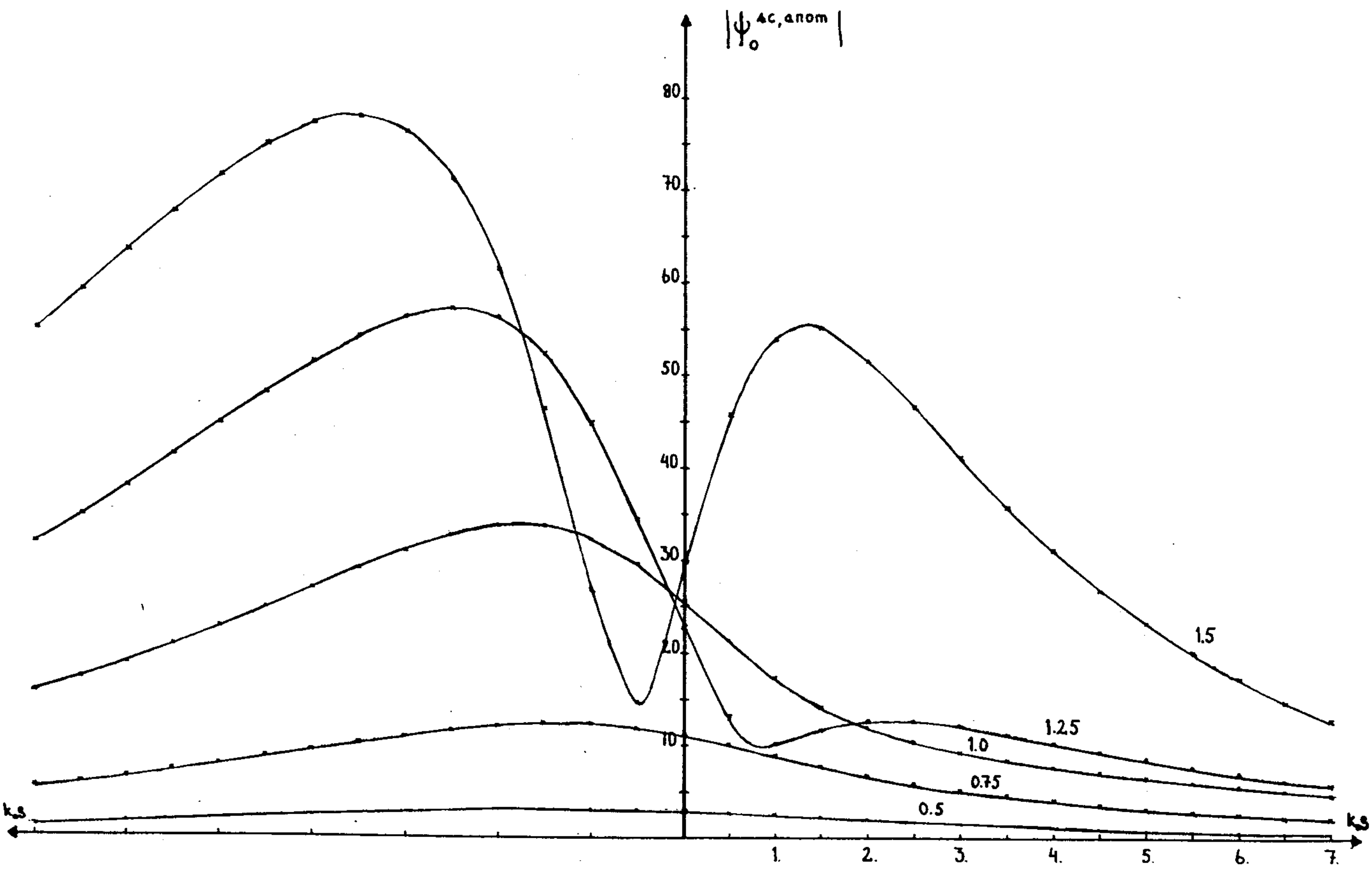


FIG. 8. The variation in $|\psi_0^{sc,anom}|$ along the $x$ axis but on the surface $S_0(k_1 z=2)$ for various radii of the buried sphere. In all curves $k_0\zeta_p=3$, $k_2/k_1=2$, $\zeta_2/\zeta_1=2$.

refers to spherical waves from the coordinate origin, which lies halfway between the centers of the spheres. The distance between the centers is denoted $d$. The orientation of the two spheres is given by the spherical angles $(\vartheta, \chi)$ of the axis from the origin to the center of the larger sphere (cf. Fig. 9). Computer programs for the calculation of the total $T$ matrix for a configuration of two spheres oriented along the $z$ axis are given in Ref. 37. For a different orientation this $T$ matrix is then similarity transformed by means of a (block diagonal in $n$) rotation matrix (i.e., each $n$ block consists of a $(2n+1)$-dimensional irreducible rotation matrix, cf. Refs. 31 and 42). In Figs. 10 and 11 the volume of the bigger sphere is eight times that of the smaller.

At these comparatively long wavelengths the relative influence of the two spheres could be expected to be roughly proportional to their volumes and the presence of the smaller sphere is seen to produce only a slight distortion of the field corresponding to an inhomogeneity consisting only of the bigger sphere. In Figs. 12–14 the spheres are bigger and of more equal size. In Fig. 14, which shows most structure, the $k$ values of the two spheres are different.

The results shown here are only some representative examples of the type of calculations which can now be performed in a straightforward manner. It is certainly of interest to perform more extensive calculations concerning the dependence on all the various parameters which are involved and also to present the results in terms of $|\psi_0^{sc,anom}/\psi_0^{sc,dir}|$, a suitable measure on the influence of the inhomogeneity. Conversely, one should also compare $|\psi_1^{sc,anom}|$ (i.e., calculated just below $S_0$) with what would be obtained in the absence of $S_0$ (i.e., for $k_0=k_1$, $\zeta_0=\zeta_1$). This would give a measure of the influence of the interface $S_0$. Work in these directions is in progress and will be reported elsewhere.

## IV. CONCLUDING REMARKS

It has already been remarked that the above results can be expected to apply, with appropriate modifications[2–5,7] to the electromagnetic and elastic cases as well and work in this direction is in progress.

So far we have only treated in any detail the solution for the case of a plane surface $S_0$, and the construction of an explicit solution for a case when $S_0$ is not a plane is obviously a problem of great interest. In particular, if $S_0$ is plane surface everywhere except in a finite region, the present formalism can be used in a perturbative treatment of the deviation from the plane surface solution even if the integral transforms cannot be inverted analytically. If the finite "hill" in addition has rotation symmetry, the integral transforms involved is only one dimensional. We get a one-dimensional integral transform also when we have a ridge which is infinite in extent in one direction, e.g., the $x$ axis, and

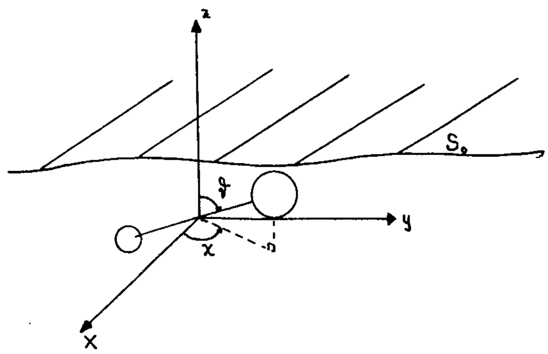


FIG. 9. Spherical angles ($\vartheta$, $\chi$) for the orientation of two spheres.

which is independent of the coordinate $x$ and of finite extent in the $y$ direction. (The inhomogeneity could then still be taken as three dimensional.)

In the present article we have concentrated on the direct scattering problem for the configuration of Fig. 1. It is obvious that the inverse problem for this configuration is of greatest importance and the present results concerning the direct problem could be exploited in an attack on the inverse one. It is in this context interesting to note that results based on the surface integral relations used in the $T$ matrix formalism have recently proved to be fruitful in attacking inverse scattering problems.[43]

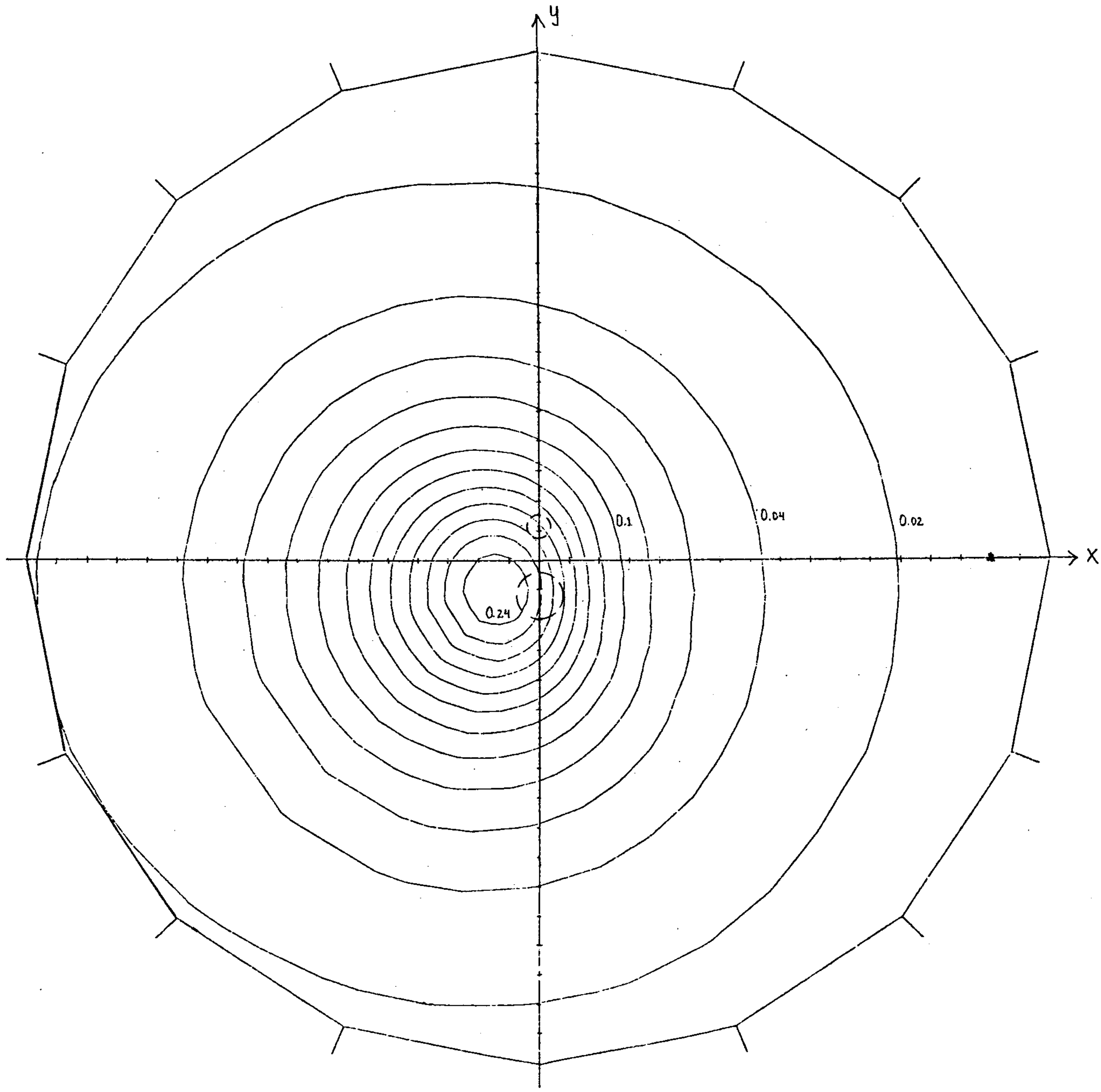


FIG. 10. The amplitude $|\psi_0^{sc,anom}|$ on the surface $S_0(k_1z=2)$ for two buried spheres of radii $k_1a_1=0.5$, $k_1a_2=0.25$, $\zeta_2/\zeta_1=\zeta_3/\zeta_1=2$, $k_2/k_1=k_3/k_1=2$. The separation distance is $k_1d=1.5$ and the symmetry axis has the polar angles $\vartheta=\pi/2$, $\chi=3\pi/2$. The source is located at $k_0\zeta_0=15$.

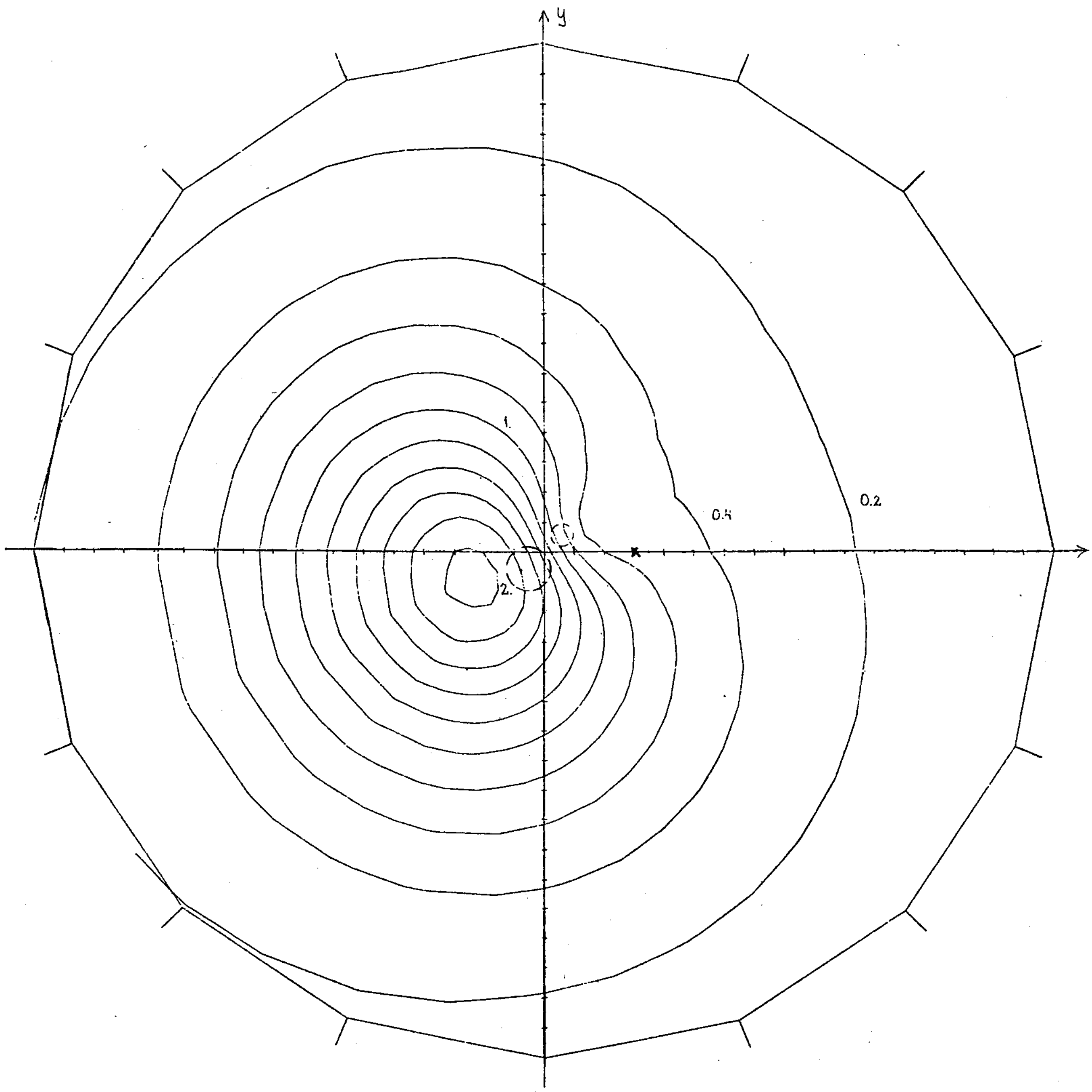


FIG. 11. The same as Fig. 10, but $\vartheta = 3\pi/4$, $\chi = 5\pi/4$, and $k_0\zeta_p = 3$.

## ACKNOWLEDGMENT

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## APPENDIX A: ON THE COMPLETENESS OF SPHERICAL AND PLANE WAVE SYSTEMS

In this appendix we show how the treatment of the completeness properties of certain systems of spherical and plane waves given in Ref. 11 (cf. also earlier references given there) can be extended and modified so as to cover the present case. Reference 11 treats a single closed or open and periodic surface, and we shall show how the systems considered there can be enlarged so as to be complete on $S_0$ and $S_1$ together.

Consider first two closed surfaces such as $S_0^{(2)}$ and $S_1^{(2)}$ in Fig. 3. In this case the system $\{\mathrm{Re}\psi_n, \psi_n\}$ is a complete system in $L^2(S_0^{(2)}+S_1^{(2)})$, the space of square integrable functions on $S_0^{(2)}+S_1^{(2)}$. (Here and in the following the integration is always with respect to the surface element.) That is, for any element $f(\mathbf{r}')$ in $L^2(S_0^{(2)}+S_1^{(2)})$ which is orthogonal to an arbitrary element in this set (i.e., an arbitrary linear combination of $\mathrm{Re}\psi_n$ and $\psi_n$), it follows that $f(\mathbf{r}')=0$. In order to show the completeness of $\{\mathrm{Re}\psi_n, \psi_n\}$ we make use of the fact that if $f(\mathbf{r}')$ is orthogonal to an arbitrary element in this set, it is in particular orthogonal to $\{\mathrm{Re}\psi_n\}$ and

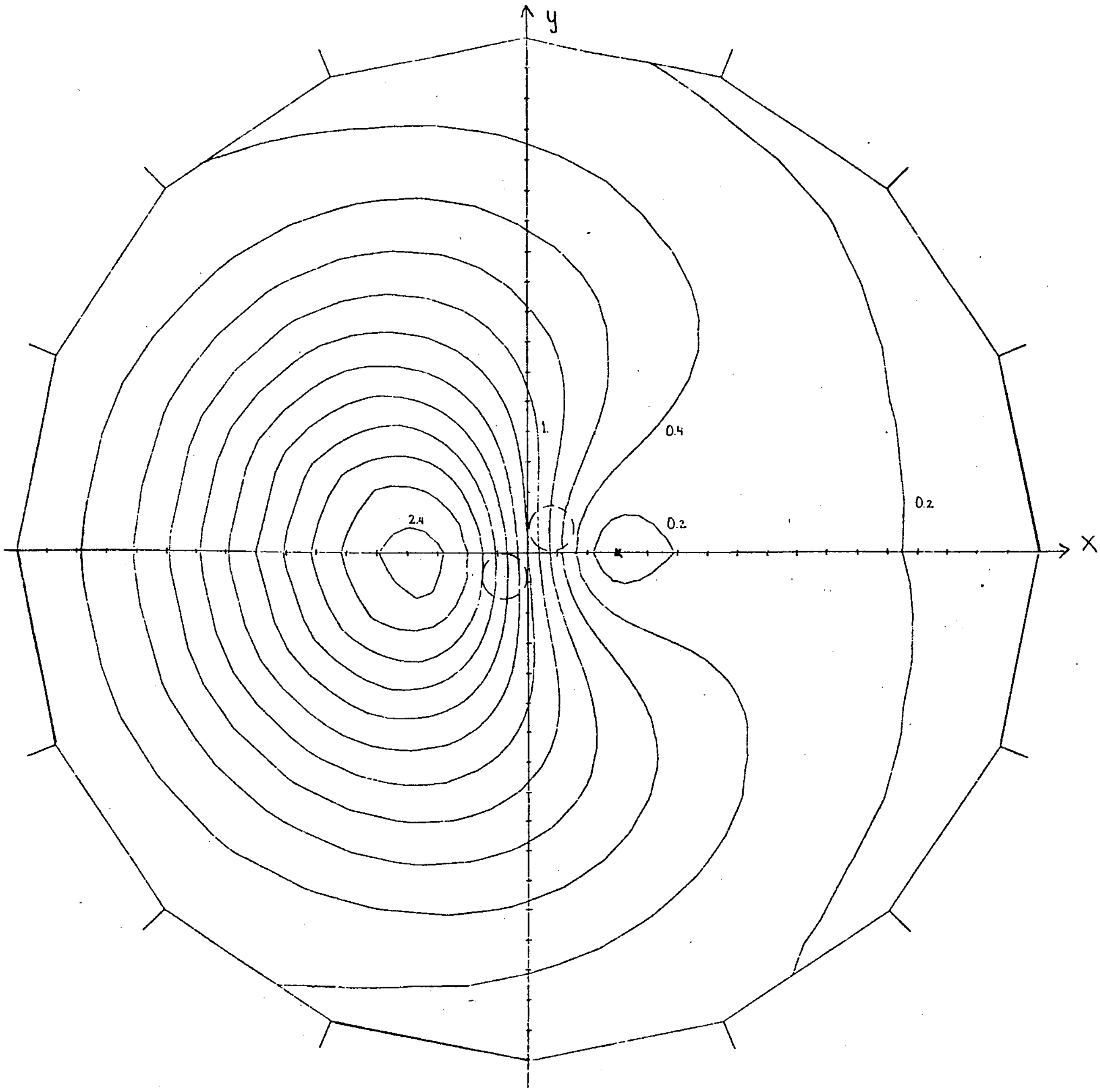


FIG. 12. The ratio $|\psi_0^{sc,anom}/\psi^{inc}|$ on the surface $S_0$ ($k_1z=2$) for two buried spheres. $k_1a_1=k_1a_2=0.5$, $\vartheta=\pi/2$, $\chi=\pi/4$, $k_0\zeta_p=3$. The remaining parameters are as in Fig. 10.

$\{\psi_n\}$, and we consider these two sets of conditions separately.

Thus we consider first the conditions

$$\int_{S_0^{(2)}+S_1^{(2)}} f(\mathbf{r}')\,\psi_n(k\mathbf{r}')\,dS'=0 \text{ (all } n) . \tag{A1}$$

Choose an $\mathbf{r}$ inside the inscribed sphere of $S_1^{(2)}$, multiply (A1) by $\mathrm{Re}\psi_n(k\mathbf{r})$, and sum over $n$. According to (13), we then have

$$\int_{S_0^{(2)}+S_1^{(2)}} f(\mathbf{r}')\,G(\mathbf{r},\mathbf{r}';k)\,dS'=0 .$$

Thus the function $\phi^i(\mathbf{r})$ defined by

$$\phi^i(\mathbf{r})\equiv \int_{S_0^{(2)}+S_1^{(2)}} f(\mathbf{r}')\,G(\mathbf{r},\mathbf{r}';k)\,dS'$$

is equal to zero inside the inscribed sphere of $S_1^{(2)}$. It is a solution to Helmholtz equation which can be continued up to $S_1^{(2)}$ and it is then zero everywhere inside $S_1^{(2)}$. It is sufficient to consider all $f(\mathbf{r}')$ in a dense set in $L^2(S_0^{(2)}+S_1^{(2)})$ and we may therefore furthermore assume that $f(\mathbf{r}')$ is continuous on $S_0^{(2)}+S_1^{(2)}$. Then $\phi^i(\mathbf{r})$ is continuous across $S_1^{(2)}$, i.e., we have $\phi^i(\mathbf{r})=0$ for $\mathbf{r}$ inside or on $S_1^{(2)}$.

Consider similarly the condition

$$\int_{S_0^{(2)}+S_1^{(2)}} f(\mathbf{r}')\,\mathrm{Re}\psi_n(k\mathbf{r}')\,dS'=0 . \tag{A2}$$

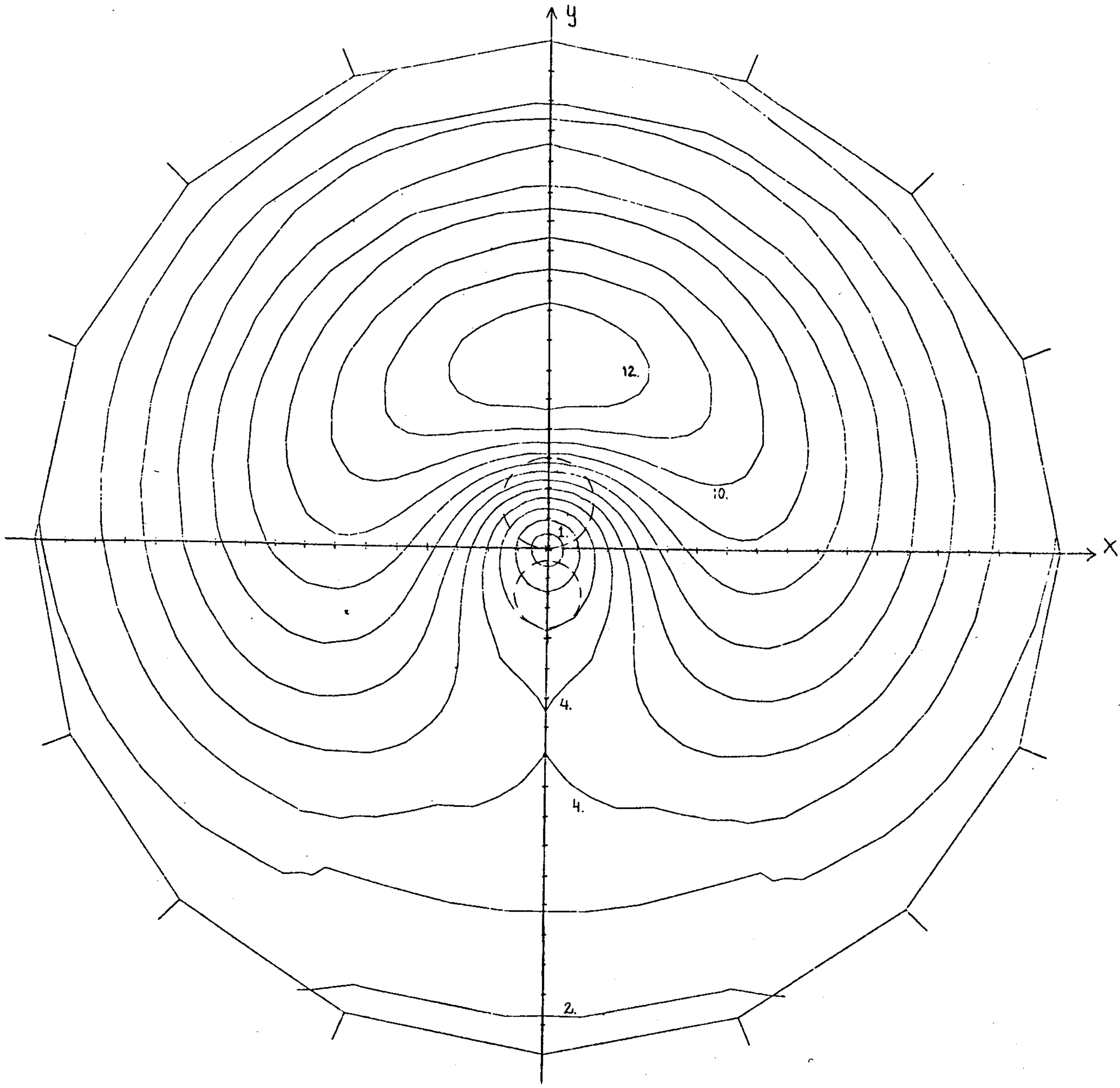


FIG. 13. The same as Fig. 12, but $k_1a_1=1$, $k_1a_2=0.75$, $k_1d=2$, $\vartheta=\chi=\pi/2$, $k_0\zeta_p=0$.

In this case we choose an $\mathbf{r}$ outside the circumscribed sphere of $S_0^{(2)}$, multiply (A2) by $\psi_n(k\mathbf{r})$ and sum over $n$ to get

$$\int_{S_0^{(2)}+S_1^{(2)}} f(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}';k)\, dS'=0.$$

Thus the function

$$\phi^e(\mathbf{r})\equiv \int_{S_0^{(2)}+S_1^{(2)}} f(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}';k)\, dS'$$

is zero outside the circumscribed sphere of $S_0^{(2)}$ and this solution of Helmholtz equation is then by continuation zero everywhere outside $S_0^{(2)}$ and for a continuous $f(\mathbf{r}')$ $\phi^e(\mathbf{r})$ is continuous across $S_0^{(2)}$. Thus we have that

$$\phi(\mathbf{r})\equiv \int_{S_0^{(2)}+S_1^{(2)}} f(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}';k)\, dS',$$

which is a solution to Helmholtz equation everywhere except on $S_0^{(2)}$ and $S_1^{(2)}$, is different from zero at most in the shell between $S_0^{(2)}$ and $S_1^{(2)}$. A nonzero solution of this homogeneous Dirichlet boundary value problem for the bounded region between $S_0^{(2)}$ and $S_1^{(2)}$ can exist for at most a discrete set of $k$ values.[44] We therefore assume that $k$ does not coincide with any of these discrete values and in this case we have $\phi(\mathbf{r})=0$ everywhere. But in general $f(\mathbf{r}')$ corresponds to the surface source density on $S_0^{(2)}$ and $S_1^{(2)}$ of $\phi(\mathbf{r})$, i.e., $f(\mathbf{r}')$ is proportional to the discontinuity of $\hat{n}\cdot\nabla\phi$ across these surfaces. Since $\phi(\mathbf{r})$ is equal to zero everywhere, $\hat{n}\cdot\nabla\phi_+=0=\hat{n}\cdot\nabla\phi_-$ on $S_0^{(2)}$ and $S_1^{(2)}$, and consequently

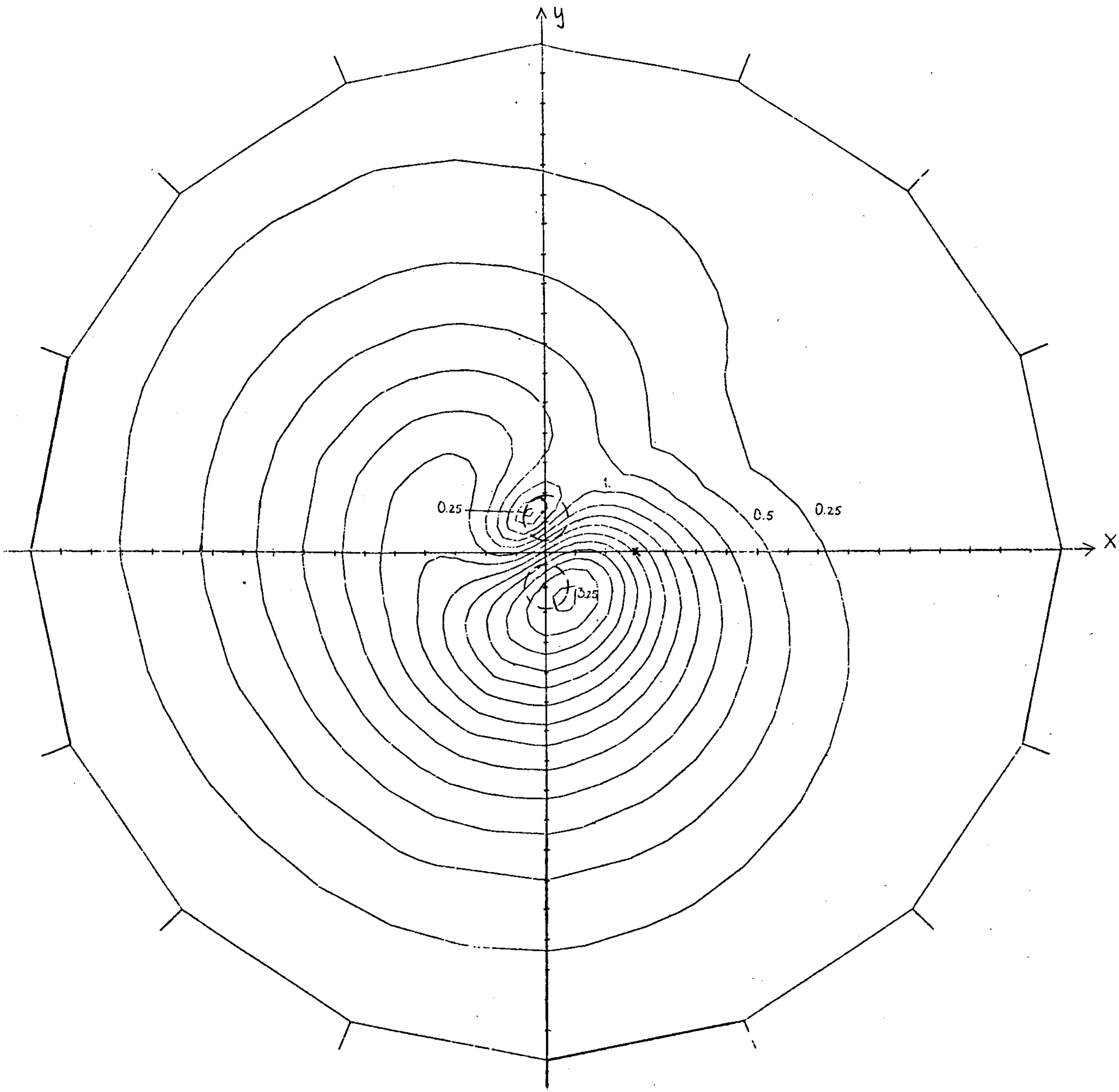


FIG. 14. The amplitude $|\psi_0^{sc,anom}|$ on the surface $S_0$ ($k_1 z=2$) for two buried spheres of radii $k_1a_1=k_1a_2=0.5$, $k_1d=1.5$, $\vartheta=\chi=\pi/2$, $k_2/k_1=1$, $k_3/k_1=2$, $\zeta_2/\zeta_1=\zeta_3/\zeta_1=2$, and $k_0\zeta_p=3$. Index 2 refers to the sphere on the positive $y$ axis and index 3 to the sphere on the negative $y$ axis.

$f(\mathbf{r}')=0$. Thus the two conditions (A1) and (A2) imply $f(\mathbf{r}')=0$ and the system $\{\mathrm{Re}\psi_n, \psi_n\}$ is complete on $S_0^{(2)}$ and $S_1^{(2)}$ [for $k$ not coinciding with any of the (discrete) resonance values of the region between $S_0^{(2)}$ and $S_1^{(2)}$].

Next we consider the configuration of Fig. 1. One technical complication is introduced in this case inasmuch as for the infinite surface $S_0$, the plane and spherical waves are not square integrable over $S_0$ and thus, for example, the integrals

$$\int_{S_0} f(\mathbf{r}')\,e^{i\mathbf{k}\cdot\mathbf{r}'}\,dS_0', \quad \int_{S_0} f(\mathbf{r}')\,\psi_n(k\mathbf{r}')\,dS_0'$$

do not exist for all $f(\mathbf{r}')$ in $L^2(S_0)$. However, the plane and spherical waves can be treated as generalized eigenfunctions and for these we have a corresponding completeness concept. (See, e.g., Ref. 45 for more details.) Thus we consider a suitably restricted subspace of $L^2(S_0+S_1)$ namely the set of infinitely differentiable functions on $S_0$ which are rapidly decreasing at infinity.[45]

Corresponding to the fact that we have one open and infinite surface and one closed and bounded, we consider the following set of functions

$$\{e^{i\mathbf{k}\cdot\mathbf{r}'},\ \psi_n\} \text{ for } \hat{k}\in C_- \text{ and all } n, \tag{A3}$$

and we show that for every infinitely differentiable rapidly decreasing function $f(\mathbf{r}')$ on $S_0$ and $S_1$, the conditions

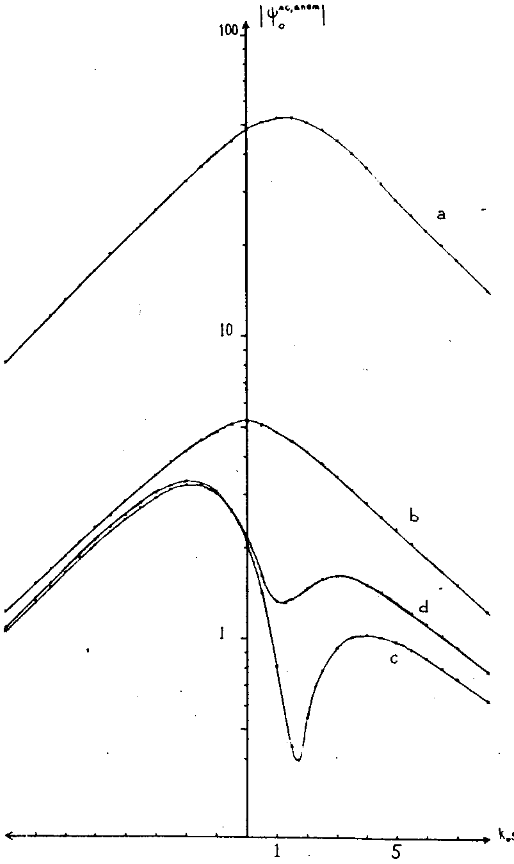


FIG. 15. The variation in $|\psi_0^{sc,anom}|$ along the $y$ axis but on the surface $S_0$ ($k_1z=2$) for two buried spheres $k_1a_1=k_1a_2=0.5$, $k_1d=1.5$, $\vartheta=\chi=\pi/2$, $k_3/k_1=2$, $\zeta_2/\zeta_1=\zeta_3/\zeta_1=2$, $k_0\zeta_p=3$, but with varied ratios of $k_2/k_1$ (again index 2 refers to the sphere on the positive $y$ axis, etc.). (a) $k_2/k_1=5$, (b) $k_2/k_1=2$, (c) $k_2/k_1=1$, (d) $k_2/k_1=0.5$.

$$\int_{S_0+S_1} f(\mathbf{r}')\, e^{i\mathbf{k}\cdot\mathbf{r}'}\, dS'=0, \quad \hat{k}\in C_- , \tag{A4}$$

and

$$\int_{S_0+S_1} f(\mathbf{r}')\, \psi_n(k\mathbf{r}')\, dS'=0 \quad (\text{all } n) \tag{A5}$$

imply $f(\mathbf{r}')=0$. The set (A3) is then complete.[45]

Consider first (A5). In order to exploit this condition we proceed as before: Choose an $\mathbf{r}$ inside the inscribed sphere of $S_1$, multiply by $\mathrm{Re}\,\psi_n(k\mathbf{r})$, and sum over $n$. In this way we obtain, as before,

$$\phi(\mathbf{r})\equiv \int_{S_0+S_1} f(\mathbf{r}')\, G(\mathbf{r},\mathbf{r}';k)\, dS' \tag{A6}$$

is zero everywhere inside and on $S_1$.

Similarly we multiply (A4) by $\exp(-i\mathbf{k}\cdot\mathbf{r})$ where $\mathbf{r}$ is chosen above $z=z_>$ and integrate over $C_-$ to obtain, by means of Eq. (5), the result that the expression $\phi(\mathbf{r})$ given by (A6) is zero everywhere above and on $S_0$. These properties of $\phi(\mathbf{r})$, together with the explicit expression (A6) suffice to guarantee that $\phi(\mathbf{r})$ is equal to zero everywhere[46] and as before we get $f(\mathbf{r}')=0$ and the set (A3) is complete on $S_0+S_1$.

However, the functions $\psi_n(k\mathbf{r}')$ are not always a suitable set to use on $S_0$. On $S_0$ we may express them in terms of plane waves. If the coordinate origin inside $S_1$ lies below $z=z_<$, $\psi_n(k\mathbf{r}')$ can be expanded in terms of $\{e^{i\mathbf{k}\cdot\mathbf{r}'}\}$ with $\hat{k}\in C_+$ only, whereas if the origin lies above $z=z_<$, both $\hat{k}\in C_-$ and $\hat{k}\in C_+$ appear [cf. Eq. (9)]. However, since all of $S_1$ is assumed to lie below $S_0$, the origin will lie below $S_0$ and therefore contributions corresponding to $\hat{k}\in C_+$ will always appear in the expansion of $\psi_n(k\mathbf{r}')$ for $\mathbf{r}'$ on $S_0$. Thus if a function on $S_0$ is expanded in terms of the set $\{e^{i\mathbf{k}\cdot\mathbf{r}'},\ \hat{k}\in C_-;\ \psi_n(k\mathbf{r}')\}$ and if $\psi_n(k\mathbf{r}')$ is transformed into plane waves, we obtain an expansion of this function in terms of plane waves $\exp(i\mathbf{k}\cdot\mathbf{r}')$ with $\hat{k}\in C_-$ and $\hat{k}\in C_+$, as in Eq. (17).

## APPENDIX B: EVALUATION OF THE INTEGRAL $I(\mathbf{k},\mathbf{r})$

In this appendix we evaluate the singular integral (21), i.e.,

$$I(\mathbf{k},\mathbf{r})\equiv \int_{S_0} [e^{i\mathbf{k}\cdot\mathbf{r}'}\nabla' G(\mathbf{r},\mathbf{r}';k) - (\nabla' e^{i\mathbf{k}\cdot\mathbf{r}'})\, G(\mathbf{r},\mathbf{r}',k)]\cdot \hat{n}_0\, dS_0' .$$

The easiest way is to use the limiting absorbtion procedure, i.e., to give $k$ a small imaginary part; $k=k'+i\epsilon$ with $\epsilon>0$. Using the new integration variable $\mathbf{r}''\equiv\mathbf{r}'-\mathbf{r}$, we have

$$I(\mathbf{k},\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} \int_{S_0}\left[e^{i\mathbf{k}\cdot\mathbf{r}''}\nabla''\left(\frac{e^{ikr''}}{4\pi r''}\right)-\frac{e^{ikr''}}{4\pi r''}\nabla''(e^{i\mathbf{k}\cdot\mathbf{r}''})\right]\cdot\hat{n}_0\, dS_0'' .$$

First we choose $\mathbf{r}$ in $V_0$ and close $S_0$ by adding a half sphere $\Sigma_R$ with radius $r''=R$. $S_0$ is closed in the upper or lower half space depending on whether $\hat{k}\in C_+$ or $\hat{k}\in C_-$. By Gauss' theorem the $1/r''$ singularity gives a contribution $+1$ if $\hat{k}\in C_+$ but zero when $\hat{k}\in C_-$. In the limit of $R\to\infty$, the remaining integral over $\Sigma_R$ gives no contribution since for $\hat{k}\in C_+$ we have

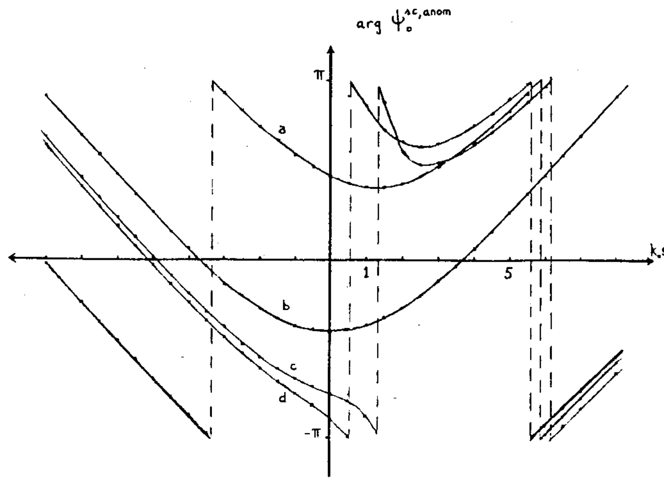


FIG. 16. The phase variation of $\psi_0^{sc,anom}$ along the $y$ axis. Data as in Fig. 15.

$$\left|\int_{\Gamma_R}\left[e^{i\mathbf{k}\cdot\mathbf{r}''}\nabla''\left(\frac{e^{ikr''}}{4\pi r''}\right)-\frac{e^{ikr''}}{4\pi r''}\nabla''(e^{i\mathbf{k}\cdot\mathbf{r}''})\right]\cdot\hat{n}\,dS''\right|$$
$$\leq\int_0^{2\pi}d\varphi\int_0^{\arccos(z_</R)}\exp[-Im(kR\cos\gamma)-\epsilon R]$$
$$\times\,|ik-1/R-ik\cos\gamma|\frac{R}{4\pi}\sin\vartheta\,d\vartheta$$
$$\leq e^{-\epsilon R}R\int_{z_</R}^{1}\exp(-tk'RIm\,\cos\alpha)K_1'(\mathbf{k})\,dt$$
$$\leq e^{-\epsilon R}K_1(\mathbf{k})\exp(-k^1z_<Im\,\cos\alpha)\rightarrow 0,\quad R\rightarrow\infty.$$

Here $\cos\gamma\equiv\hat{k}\cdot\hat{r}''$, $\hat{k}\equiv(\sin\alpha\cos\beta,\ \sin\alpha\sin\beta,\ \cos\alpha)$ and $K_1'(\mathbf{k})$, $K_1(\mathbf{k}')$ are constants, independent of $R$. For $\hat{k}\in C_-$ one has similarly

$$\left|\int_{\Gamma_R}\left[e^{i\mathbf{k}\cdot\mathbf{r}''}\nabla''\left(\frac{e^{ikr''}}{4\pi r''}\right)-\frac{e^{ikr''}}{4\pi r''}\nabla''(e^{i\mathbf{k}\cdot\mathbf{r}''})\right]\cdot\hat{n}\,dS''\right|$$
$$\leq e^{-\epsilon R}R\int_{-1}^{z_>/R}\exp(-tk'RIm\,\cos\alpha)K_2'(\mathbf{k})dt$$
$$\leq e^{-\epsilon R}K_2(\mathbf{k})\exp(-k'z_>Im\,\cos\alpha)\rightarrow 0,\quad R\rightarrow\infty.$$

Thus if $\mathbf{r}$ lies in $V_0$ we have

$$I(\mathbf{k},\mathbf{r})=\begin{Bmatrix}e^{i\mathbf{k}\cdot\mathbf{r}}\\0\end{Bmatrix}\text{ for }\begin{Bmatrix}\hat{k}\in C_+\\\hat{k}\in C_-\end{Bmatrix}.$$

In the second situation, i.e., when $\mathbf{r}$ lies in $V_1$, we can proceed as above. We close $S_0$ in the upper or lower half spaces, but this time the contribution comes from $\hat{k}\in C_-$. One finds

$$I(\mathbf{k},\mathbf{r})=\begin{Bmatrix}0\\-e^{i\mathbf{k}\cdot\mathbf{r}}\end{Bmatrix}\text{ for }\begin{Bmatrix}\hat{k}\in C_+\\\hat{k}\in C_-\end{Bmatrix}.$$

The minus sign appearing in this last expression comes from the fixed convention regarding the direction of the normal vector $\hat{n}_0$ on $S_0$.

## APPENDIX C: CONDITIONS FOR THE EXISTENCE OF A SOLUTION TO (49)

In this appendix we discuss briefly some sufficient conditions for the existence of a (bounded and also unique) solution to Eq. (49). An equation of the type (49) is said to be *regular* if $\sum_{n'}A_{nn'}<1$ for all $n$. If Eq. (49) is regular and if the right-hand side $\mathbf{d}$ is bounded by $|d_n|\leq K\zeta_n$, where $K$ is a constant and $\zeta_n=1-\sum_n|A_{nn'}|>0$, then a bounded solution $\mathbf{c}$ with $|c_n|<K$ exists and can be found by "successive approximation" or by "the method of reduction."[34] If, furthermore, the system (49) is *fully regular*, i.e., if $\zeta_n\geq\vartheta>0$ for all $n$, this solution is the only bounded solution to Eq. (49). By successive approximation we mean the use of the iteration scheme

$$c_n^{(i+1)}=d_n-\sum_{n'}A_{nn'}c_{n'}^{(i)},\quad i=0,1,2,\ldots,\ c_n^{(0)}=0.$$

The method of reduction refers to taking the limit $N\rightarrow\infty$ in the truncated system of equations

$$c_n=d_n-\sum_{n'=0}^{N}A_{nn'}c_{n'},\quad n=0,1,\ldots,N.$$

The above results are proved in Chaps. 1 and 2 of Ref. 34, where more mathematical details can be found.

We apply these results to Eq. (49) (or rather to a similarity transformed version of it: cf. below) with the explicit form (60) for $A_{nn'}$, in a simple case. Consider a spherical inhomogeneity and a source situated symmetrically above it. For a sphere Eq. (60) reads

$$A_{nn'}=\delta_{mm'}\delta_{\sigma\sigma'}T_{n'}(1)(-1)^{n'+m'}4\pi\gamma_n\gamma_{n'}$$
$$\times\int_{i\infty}^{1}P_n^m(x)P_{n'}^m(x)R[k_1(1-x^2)^{1/2}]\exp(2iz_0k_1x)\,dx,\tag{C1}$$

where

$$T_n(1)=-\frac{k_1C_{12}j_n'(k_1a)j_n(k_2a)-k_2j_n(k_1a)j_n'(k_2a)}{k_1C_{12}h_n^{(1)\prime}(k_1a)j_n(k_2a)-k_2h_n^{(1)}(k_1a)j_n'(k_2a)}\tag{C2}$$

(the prime denotes differentiation with respect to the whole argument) and the normalization factor $\gamma_n$ is given by[42]

$$\gamma_n=(-1)^m\left[\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}\right]^{1/2}.$$

The matrix $A$ in (C1) has different growth properties along rows and columns. We shall therefore study a similarity-transformed version of $A$ which has symmetrical growth properties on both sides of the diagonal. Consider

$$\tilde{A}=SAS^{-1},$$

where the (diagonal) matrix $S$ is given by

$$S_{nn'}=\delta_{nn'}\gamma_n^{-1}j_n(k_1a).\tag{C3}$$

This matrix is invertible if $k_1a$ does not coincide with any of the zeros of $j_n(x)$. Thus it is invertible if for instance $0<k_1a<3$. The transformed Eq. (49) will be written

$$(1+\tilde{A})\tilde{c}=\tilde{d},\tag{C4}$$

and according to Eqs. (C1), (C3), and (64) one has explicitly

$$\tilde{A}_{nn'}=\delta_{mm'}\delta_{\sigma\sigma'}T_{n'}(1)(-1)^{n'+m'}(2n'+1)(n'-m')!$$
$$\times[(n'+m')!]^{-1}j_n(k_1a)[j_{n'}(k_1a)]^{-1}\int_{i\infty}^{1}P_n^m(x)$$
$$\times P_{n'}^m(x)R[k_1(1-x^2)^{1/2}]\exp(2iz_0k_1x)\,dx,\tag{C5}$$

$$\tilde{d}_n=\sum_{n'}S_{nn'}d_{n'}=2k_1i^{m+1}(-1)^n j_n(k_1a)$$
$$\times\int_{i\infty}^{1}P_n^m(x)J_m[k_1\zeta_p(1-x^2)^{1/2}$$
$$\times\frac{\exp\{ik_1z_0x+i(z_p-z_0)[k_0^2-k_1^2(1-x^2)]^{1/2}\}}{C_{01}[k_0^2-k_1^2(1-x^2)]^{1/2}+k_1x}k_1x\,dx\begin{pmatrix}1\\0\end{pmatrix}$$
$$\text{for}\begin{Bmatrix}\sigma=e\\\sigma=o\end{Bmatrix}.\tag{C6}$$

The inhomogeneity is assumed to lie fully inside $V_1$ which requires $a<z_0$. We shall consider the long wavelength limit, i.e., an interval of the form $k_1a<C_1$, where $C_1$ is some constant. In the case of a symmetri-

cally situated source only terms corresponding to $m=0$ contribute [cf. (C6)]; the general case will demand a more elaborate analysis but is straightforward). Using suitable estimates for $|P_n|$ one obtains the required bound on $\sum_n |\tilde{A}_{nn'}|$. Thus for this special situation it can be shown that for a suitable choice of the constant $C_1$ the system (C4) is fully regular so that a unique bounded solution to this equation exists[47] (and thus also to Eq. (49) for every invertible matrix $S$). However, a corresponding analysis could be applied to more complicated situations. The conditions obtained are sufficient conditions and unique bounded solutions can be expected to exist for a much wider range of parameters and geometries than are indicated by these sufficient conditions. All the numerical examples which we have considered in the present article showed very fast convergence for this equation.


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# Paper II

78-42

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# Electromagnetic Scattering from Buried Inhomogeneities - a General Three-dimensional Formalism.[†]

by

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Abstract

We will in the present paper derive a general three-dimensional formalism for electro-magnetic scattering from buried inhomogeneities. We will exploit the transition matrix formalism - originally given by Waterman - to electromagnetic scattering in the presence of an infinite surface and a buried bounded inhomogeneity. The analysis explicitly assumes that the sources are located above the ground, but this restriction can easily be relaxed and a parallel derivation can be made for sources located in the ground or inside the buried obstacle. No explicit symmetry assumptions are made for the bounded inhomogeneity or the interface between the halfspaces, except that the interface be bounded by two parallel planes. The scattered field above the ground is calculated in terms of an expansion where the expansion coefficients are solutions of a matrix equation. The expression for the scattered field is separated into a directly scattered term - as if no scatterers were present - and the so called anomalous field, reflecting the presence of the inhomogeneity. We give some numerical examples for a flat interface and an inhomogeneity consisting of one or two buried spheres or a perfectly conducting spheroid.

## I Introduction

The transition matrix formalism has, since it was first formulated [1], successfully been extended to more complicated configurations and scatteres [2]-[9] - several finite scatterers, multilayered or not, or a periodic infinite surface [10]. The analysis has shown great similarities between the acoustic, electromagnetic and the elastic cases, giving a systematic formulation for scattering solutions of Helmholtz' equation in both the scalar and vector case. The effect of an infinite surface in the presence of finite scatterer has recently been treated for acoustic fields [11], and in this paper we give the corresponding extension to electromagnetic waves.

The two essential tools in the T-matrix formalism - suitable expansions of the Green's function and of the surface fields - will in this context be generalized to hold for vector plane waves. The vector character of the problem introduces a dyadic notation for the Green's function, but also generalization of the transformations between plane and spherical waves found in the scalar case, and these aspects will be analysed. The presence of an infinite surface introduces a continuous variable, which is inconvenient for numerical applications. We will in this paper show how the equations, at least for a flat surface, can be transformed into a matrix equation more suitable for numerical calculations. This property is a consequence of finiteness of the buried obstacle but also of the use of the spherical wave basis.

Electromagnetic scattering, from e.g. a dipole and an infinite plane surface is a canonical problem, and a long list of papers have analysed various aspects of this topic. Many of the results are found in the monograph by Baños [12], and references given there. The scattering problem from a buried inhomogeneity is a problem of greater complexity, and the results found in the literature are mainly concentrated on obstacles with certain symmetries, which simplifies the analysis to some extent, e.g. spheres, cylinders [13] - [23]. Very few results with a general bounded obstacle are found, see, however, e.g. [24]. In this paper we will develop an analysis for truly three-dimensional electromagnetic prospecting situations, making fairly weak assumptions on the geometry of the obstacle and structure of the source. Several extensions of the results given below is being pursued at presence and will appear elsewhere. The T-matrix method has been developed for the elastic wave case in Refs. [9] and [10]. It is expected that the results of [11] and the present paper can be extended to the elastic case as well and work in this direction is in progress.

The plan of the paper is the following. In section II we will make the necessary assumptions and definitions. We also derive the basic equations of the formalism, and analyse the differences from the corresponding scalar case. In section III the basic equations will be further developed, and the scattered field above the ground will be written down explicitly, both in the general case and the special case of a flat interface. Furthermore it will be shown that the results can be given a multiple scattering interpretation. In section IV we analyse and discuss various numerical aspects, and some numerical results are presented. In the final section V we discuss applications and future extensions of the theory.

## II T-matrix formulation of electromagnetic scattering in the presence of an infinite surface

In this section we will derive the necessary equations for electromagnetic scattering from a scattering geometry as depicted in Figure 1. The surface $S_o$ separates the two halfspaces $V_o$ and $V_1$. The interface $S_o$ need not be a plane but is assumed to be sufficiently regular for an application of Green's theorem and bounded by two parallel planes $z=z_>$ and $z=z_<$. The normal to these planes defines the z-axis and the axis is directed into $V_o$. The upper halfspace $V_o$ is assumed to be homogeneous, and the halfspace $V_1$ is also assumed homogeneous except for a finite region $V_2$. The propagation constants in each volume are indicated in the figure. (We here explicitly take the volume $V_2$ to be homogeneous but this restriction can be relaxed [7].)

The sources of the wave field will explicitly be assumed to be situated in the upper halfspace $V_o$, at the point P in Figure 1. This choice is not essential for the formalism and a parallel derivation can be made when the sources are in $V_1$ or $V_2$ (cf. Ref. [11]). This aspect will be developed further in a future paper.

In each volume respectively the electric field $\vec{E}_i(\vec{r})$ will satisfy (a time factor exp(-iωt) is suppressed)

$$\nabla\times\nabla\times\vec{E}_i(\vec{r}) - k_i^2\vec{E}_i(\vec{r}) = \vec{0} \qquad i=0,1,2 \qquad (1)$$

(Here and below we will derive all the equations in terms of the electric field $\vec{E}(\vec{r})$, but these equations hold equally well for the magnetic field $\vec{H}$, all that needs to be done is to replace the electric field $\vec{E}(\vec{r})$ by the magnetic field $\vec{H}$ [2], see also Eq. (18) and below.)

The incident field $\vec{E}_o^{inc}(\vec{r})$ is assumed to be prescribed and generated at a point P, but we will here make no further assumptions.

The method or scheme, which we will adopt here to solve the scattering problem, is a T-matrix formalism [1] - [2]. The starting point in this formalism is the following integral representation of the electric field $\vec{E}$, in terms of its tangential boundary values, see e.g. [25]

$$\left.\begin{matrix}\vec{E}(\vec{r})\\ \vec{0}\end{matrix}\right\} = \vec{E}^{inc}(\vec{r}) + \nabla\times\iint_S \hat{n}\times\vec{E}^+(\vec{r}')G(\vec{r},\vec{r}';k)\,dS' + k^{-2}\,\nabla\times\left\{\nabla\times\iint_S \hat{n}\times(\nabla'\times\vec{E}^+(\vec{r}'))G(\vec{r},\vec{r}';k)\,dS'\right\} \quad \left\{\begin{matrix}\vec{r}\text{ outside } S\\ \vec{r}\text{ inside } S\end{matrix}\right. \tag{2}$$

Here S is a closed surface and the sources are located outside S, and $\vec{E}^+$ and $\nabla\times\vec{E}^+$ indicate the field values taken from the outside. The free space Green's function $G(\vec{r},\vec{r}',k)$ satisfies the scalar Helmholtz' equation with a delta function as a source term:

$$\left(\nabla^2+k^2\right)\,G(\vec{r},\vec{r}';k) = -\,\delta(\vec{r}-\vec{r}') \tag{3}$$

To satisfy the requirement that we have an outgoing wave for large arguments we chose

$$G(\vec{r},\vec{r}';k) = \exp\{ik|\vec{r}-\vec{r}'|\}\big/4\pi|\vec{r}-\vec{r}'|$$

The integral representation Eq. (2), commonly known as the Poincaré-Huygens principle or the extinction theorem, will first be applied to a surface consisting of a finite part of $S_o$ and a lower half sphere. Let the radius of the sphere go to infinity and assume the integrals over $S_o$ exist, and furthermore that the integrals over the lower halfspace vanish (radiation conditions). We get

$$\left.\begin{matrix}\vec{E}_0(\vec{r})\\ \vec{0}\end{matrix}\right\} = \vec{E}_0^{inc.}(\vec{r}) + \nabla\times\iint_{S_0}\hat{n}_0\times\vec{E}_0^{+}(\vec{r}')\,G(\vec{r},\vec{r}';k_0)\,dS' + $$
$$+ k_0^{-2}\nabla\times\left\{\nabla\times\iint_{S_0}\hat{n}_0\times(\nabla'\times\vec{E}_0^{+}(\vec{r}'))G(\vec{r},\vec{r}';k_0)\,dS'\right\} \qquad \begin{cases}\vec{r}\text{ in }V_0\\ \vec{r}\text{ outside }V_0\end{cases} \tag{4}$$

It is here convenient to introduce the dyadic notation for the Green's function $G(\vec{r},\vec{r}';k)$ and since the T-matrix formalism is based on suitable expansions of the Green's function or dyadic we will here discuss the expansion of the Green's dyadic in some detail. Since the scattering geometry, as depicted in Figure 1, consists of an infinite surface $S_0$ bounded by two parallel planes an expansion in plane waves is suitable. The aim of this expansion is to use the same expansion of the Green's dyadic over the whole surface $S_0$ in the integrals of Eq. (4). The plane wave expansion of the Green's function which we shall find most suitable is, (see e.g. [26] or [11])

$$G(\vec{r},\vec{r}';k) = \frac{ik}{8\pi^2}\int_0^{2\pi}d\beta\int_{C_\pm}e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\sin\alpha\,d\alpha \tag{5}$$

Here $\vec{k}=k(\sin\alpha\cos\beta,\ \sin\alpha\sin\beta,\ \cos\alpha)$ and $C_\pm$ are contours in the complex $\alpha$-plane (see Figure 2). The $C_+$ contour is relevant if $z>z'$ and $C_-$ if $z<z'$. This expansion can be interpreted as an expansion in harmonic waves (corresponding to the real part of the contours $C_\pm$) and evanescent waves (the complex contributions in the contours $C_\pm$). This type of expansion has been extensively analysed in the literature and for further details see Baños [12] and references given there.

The unit dyadic $\overleftrightarrow{I}$ can be written as $\sum_{j=1}^{3}\hat{a}_j\hat{a}_j$ where

$\hat{a}_j$; j=1,2,3 is any righthanded orthogonal triplet of unit vectors. We will here explicitly take the unit vectors $\hat{a}_j$ to be the spherical unit vectors of $\hat{k}\equiv\vec{k}/k$. We shall use the following notations

$$\begin{cases} \hat{a}_1 \equiv \hat{\alpha} \\ \hat{a}_2 \equiv \hat{\beta} \\ \hat{a}_3 \equiv \hat{k} \end{cases} \tag{6}$$

and we can write

$$\overleftrightarrow{\mathbf{1}}G(\vec{r},\vec{r}\,';k) = \sum_{j=1}^{3} \frac{ik}{8\pi^2} \int_0^{2\pi} d\beta \int_{C_\pm} \hat{a}_j\hat{a}_j \, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} \sin\alpha \, d\alpha \tag{7}$$

The main reason for using spherical unit vectors in the dyadic is to get accordance with the spherical vector waves introduced below, but also since this choice separates the longitudinal part of the Green's dyadic in a simple fashion (the longitudinal part corresponds in our notation to j=3).

The spherical vector waves will be used to describe the scattering from the inhomogeneity, and consequently the corresponding expansion of the Green's dyadic will be needed. The spherical vector waves $\vec{\psi}_{\tau\sigma ml}(k\vec{r})$ and $\mathrm{Re}\vec{\psi}_{\tau\sigma ml}(k\vec{r})$ are defined as

$$\vec{\psi}_{\tau\sigma ml}(k\vec{r}) \equiv \frac{\gamma_{lm}}{\sqrt{l(l+1)}} \left(k^{-1}\nabla\times\right)^{\tau} \left[ k\vec{r}\, h_l^{(1)}(kr)\, P_l^m(\cos\vartheta) \begin{pmatrix} \cos m\varphi \\ \sin m\varphi \end{pmatrix} \right] \tag{8}$$

$$l=1,2,\ldots;\; m=0,1,\ldots,l;\; \sigma=e,o;\; \tau=1,2$$

$$\gamma_{lm} \equiv (-1)^m \sqrt{\varepsilon_m \frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \quad ; \quad \varepsilon_m \equiv 2-\delta_{m,0}$$

The vector waves $\mathrm{Re}\vec{\psi}_{\tau\sigma ml}(k\vec{r})$ is defined analogously with $h_l^{(1)}(kr)$ everywhere replaced by $j_l(kr)$.

These vector waves can alternatively be written in terms of the vector spherical harmonics $\vec{A}_{\tau\sigma ml}(\hat{r})$ as:

$$\vec{\psi}_{1\sigma ml}(k\vec{r}) = \vec{A}_{1\sigma ml}(\hat{r})\, h_l^{(1)}(kr)$$
$$\vec{\psi}_{2\sigma ml}(k\vec{r}) = \vec{A}_{2\sigma ml}(\hat{r})\, \frac{d}{d(kr)}\left[kr\, h_l^{(1)}(kr)\right]/kr + \sqrt{l(l+1)}\, \vec{A}_{3\sigma ml}(\hat{r})\, h_l^{(1)}(kr)/kr$$

where

$$\vec{A}_{1\sigma ml}(\hat{r}) \equiv \frac{1}{\sqrt{l(l+1)}}\, \nabla \times \left(\vec{r}\, \gamma_{lm} P_l^m(\cos\vartheta) \begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix}\right) = -\hat{r} \times \vec{A}_{2\sigma ml}(\hat{r})$$

$$\vec{A}_{2\sigma ml}(\hat{r}) \equiv \frac{1}{\sqrt{l(l+1)}}\, r\nabla \left(\gamma_{lm} P_l^m(\cos\vartheta) \begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix}\right) = \hat{r} \times \vec{A}_{1\sigma ml}(\hat{r}) \qquad (9)$$

$$\vec{A}_{3\sigma ml}(\hat{r}) \equiv \hat{r}\, \gamma_{lm} P_l^m(\cos\vartheta) \begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix}$$

We will below, when convenient, abbreviate the indices as follows:

$$\vec{\psi}_n \equiv \vec{\psi}_{\tau n} \equiv \vec{\psi}_{\tau\sigma ml} \qquad \vec{A}_n \equiv \vec{A}_{\tau n} \equiv \vec{A}_{\tau\sigma ml}$$

The expansion of the Green's dyadic in terms of spherical vector waves is found in the literature e.g. [27]

$$\overleftrightarrow{\mathrm{I}}G(\vec{r},\vec{r}\,';k) = ik \sum_{\substack{lm\sigma \\ \tau=1,2}} \vec{\psi}_n(k\vec{r}_>)\, \mathrm{Re}\vec{\psi}_n(k\vec{r}_<) + \overleftrightarrow{\mathrm{I}}_{irr.} \qquad (10)$$

Here $\vec{r}_<$ and $\vec{r}_>$ are to be identified with the argument $\vec{r}$ or $\vec{r}\,'$ according to $|\vec{r}_<| = \min(r,r')$ ; $|\vec{r}_>| = \max(r,r')$. The dyadic $\overleftrightarrow{\mathrm{I}}_{irr.}$ is an irrotational dyadic.

Consider an $\vec{r}$ in $V_o$ with $z>z_>$ in Eq. (4). Insert the plane wave expansion of the Green's dyadic Eq. (7), and we get, after a change of order of integration, a representation of the scattered electric field in the region $V_0$ with $z>z_>$.

$$\vec{E}_0^{sc}(\vec{r}) = \int_0^{2\pi} d\beta_0 \int_{C_+} \vec{f}(\vec{k}_0)\, e^{i\vec{k}_0\cdot\vec{r}} \sin\alpha_0\, d\alpha_0 \qquad z>z_> \tag{11}$$

where $\vec{f}(\vec{k}_0) = \sum_{j=1}^{2} f_j(\vec{k}_0)\,\hat{a}_j$ and

$$f_j(\vec{k}_0) \equiv \frac{ik_0}{8\pi^2} \iint_{S_0} \left\{ (\hat{n}_0\times\vec{E}_0^+)\cdot(-i\vec{k}_0\times\hat{a}_j) + [\hat{n}_0\times(\nabla'\times\vec{E}_0^+)]\cdot\hat{a}_j \right\} e^{-i\vec{k}_0\cdot\vec{r}'}\, dS' \qquad \vec{k}_0\in C_+ \tag{12}$$

Note that only the transverse part of the expansion of the Green's dyadic is used - the longitudinal part is superfluous, since the electric field itself is transverse. Here and below the summation over j (or $\tau$), if not indicated, will be just over 1 and 2.

For an $\vec{r}$ not in $V_0$ and furthermore $z<z_<$ we can again use Eq. (7) in Eq. (4) and this gives (after a change of order of integration) a plane wave expansion of the incoming field.

$$\vec{E}_0^{inc.}(\vec{r}) = \int_0^{2\pi} d\beta_0 \int_{C_-} \vec{a}(\vec{k}_0)\, e^{i\vec{k}_0\cdot\vec{r}} \sin\alpha_0\, d\alpha_0 \qquad z<z_< \tag{13}$$

Here $\vec{a}(\vec{k}_0) = \sum_{j=1}^{2} a_j(\vec{k}_0)\,\hat{a}_j$ and

$$a_j(\vec{k}_0) \equiv \frac{-ik_0}{8\pi^2} \iint_{S_0} \left\{ (\hat{n}_0\times\vec{E}_0^+)\cdot(-i\vec{k}_0\times\hat{a}_j) + [\hat{n}_0\times(\nabla'\times\vec{E}_0^+)]\cdot\hat{a}_j \right\} e^{-i\vec{k}_0\cdot\vec{r}'}\, dS' \qquad \vec{k}_0\in C_- \tag{14}$$

The incoming field $\vec{E}_0^{inc}$ is as usual a prescribed field, whose sources are assumed situated in $V_0$. This means that in any case the sources are above the fictitious plane $z=z_<$ and an expansion of the incoming field $\vec{E}_0^{inc}$ as in Eq. (13) can be found for a wide class of sources (for examples see [11] and below). The similarity between $f_j(\vec{k}_0)$ and $a_j(\vec{k}_0)$ should be noted - the essential difference is just the domain of $\vec{k}_o$ as indicated in Eqs. (12) and (14).

The problem can now be stated more explicitly. For a given $\vec{a}(\vec{k}_o)$, i.e. incoming field, find the vector amplitude $\vec{f}(\vec{k}_o)$, which in Eq. (11) gives the scattered field for $z>z_>$. This relation between $\vec{a}(\vec{k}_o)$ and $\vec{f}(\vec{k}_o)$ - a quantity called the total T-operator, since it is a generalization of the concept of the T-matrix for a finite scatterer - will be found by elimination of the surface fields on $S_o$. These surface fields are influenced by the lower half-space and the inhomogeneity by the boundary conditions on $S_o$ and $S_1$.

The boundary conditions will be discussed in detail below, but first we will use the Poincaré-Huygens principle once more to find an additional relation of the surface fields. Thus apply Eq. (2) when S consists of $S_1$, a finite part of $S_o$ and a upper halfspace. Once again let the radius go to infinity and assume necessary existence of integrals and radiation conditions to hold. We get

$$\left.\begin{matrix}\vec{E}_1(\vec{r})\\ \vec{0}\end{matrix}\right\} = -\nabla\times\iint_{S_o}\hat{n}_o\times\vec{E}_1^-(\vec{r}')G(\vec{r},\vec{r}';k_1)dS' - k_1^{-2}\nabla\times\left\{\nabla\times\iint_{S_o}\hat{n}_o\times(\nabla'\times\vec{E}_1^-(\vec{r}'))G(\vec{r},\vec{r}';k_1)dS'\right\}+$$
$$+\nabla\times\iint_{S_1}\hat{n}_1\times\vec{E}_1^+(\vec{r}')G(\vec{r},\vec{r}';k_1)dS' + k_1^{-2}\nabla\times\left\{\nabla\times\iint_{S_1}\hat{n}_1\times(\nabla'\times\vec{E}_1^+(\vec{r}'))G(\vec{r},\vec{r}';k_1)dS'\right\}$$
$$\left\{\begin{matrix}\vec{r}\text{ in }V_1\\ \vec{r}\text{ outside }V_1\end{matrix}\right. \qquad (15)$$

For an $\vec{r}$ in $V_1$ this equation can be used to find an expansion of the electric field inside $V_1$, but this property will not be exploited any further here. We will instead use the Eq. (15) for $\vec{r}\notin V_1$, and first we consider an $\vec{r}\in V_o$ and furthermore outside the circumscribed sphere of the inhomogeneity $S_1$. Since $\vec{r}$ is outside the circumscribed sphere of $S_1$, we can identify $\vec{r}_>$ and $\vec{r}_<$ in Eq. (10) with $\vec{r}$ and $\vec{r}'$ respectively - valid over the whole surface $S_1$. Thus we get after a change of summation and integration (we have here explicitly taken the origin inside the volume $V_2$).

$$\nabla\times\iint_{S_0}\hat{n}_0\times\vec{E}_1^-(\vec{r}')G(\vec{r},\vec{r}';k_1)dS' + k_1^{-2}\nabla\times\left\{\nabla\times\iint_{S_0}\hat{n}_0\times(\nabla'\times\vec{E}_1^-(\vec{r}'))G(\vec{r},\vec{r}';k_1)dS'\right\} =$$

$$= ik_1\sum_n\vec{\Psi}_n(k_1\vec{r})\iint_{S_1}\left\{(\hat{n}_1\times\vec{E}_1^+(\vec{r}'))\cdot(\nabla'\times\mathrm{Re}\vec{\Psi}_n(k_1\vec{r}')) + [\hat{n}_1\times(\nabla'\times\vec{E}_1^+(\vec{r}'))]\cdot\mathrm{Re}\vec{\Psi}_n(k_1\vec{r}')\right\}dS' \quad (16)$$

The next step will be to consider an $\vec{r}$ inside the inscribed sphere of $S_1$. We then again can identify $\vec{r}_>$ and $\vec{r}_<$ - this time with $\vec{r}'$ and $\vec{r}$ respectively - over the whole surface of $S_1$, and using Eq. (10) in (15) we get

$$\nabla\times\iint_{S_0}\hat{n}_0\times\vec{E}_1^-(\vec{r}')G(\vec{r},\vec{r}';k_1)dS' + k_1^{-2}\nabla\times\left\{\nabla\times\iint_{S_0}\hat{n}_0\times(\nabla'\times\vec{E}_1^-(\vec{r}'))G(\vec{r},\vec{r}';k_1)dS'\right\} =$$

$$= ik_1\sum_n\mathrm{Re}\vec{\Psi}_n(k_1\vec{r})\iint_{S_1}\left\{(\hat{n}_1\times\vec{E}_1^+(\vec{r}'))\cdot(\nabla'\times\vec{\Psi}_n(k_1\vec{r}')) + [\hat{n}_1\times(\nabla'\times\vec{E}_1^+(\vec{r}'))]\cdot\vec{\Psi}_n(k_1\vec{r}')\right\}dS' \quad (17)$$

Again only the transverse part of $\overleftrightarrow{I}G(\vec{r},\vec{r}';k)$ contributes.

The surface fields can all be expressed in terms of just $\hat{n}_0\times\vec{E}_1^-$ and $\hat{n}_1\times\vec{E}_2^-$ and the corresponding tangential components of the magnetic field by the boundary conditions, i.e. the continuity of the tangential electric and magnetic fields:

$$\begin{aligned}
&\hat{n}_0\times\vec{E}_0^+(\vec{r}') = \hat{n}_0\times\vec{E}_1^-(\vec{r}')\\
&\hat{n}_0\times(\nabla'\times\vec{E}_0^+(\vec{r}')) = c_{01}\,\hat{n}_0\times(\nabla'\times\vec{E}_1^-(\vec{r}'))\\
&\hat{n}_1\times\vec{E}_1^+(\vec{r}') = \hat{n}_1\times\vec{E}_2^-(\vec{r}')\\
&\hat{n}_1\times(\nabla'\times\vec{E}_1^+(\vec{r}')) = c_{12}\,\hat{n}_1\times(\nabla'\times\vec{E}_2^-(\vec{r}'))
\end{aligned} \quad (18)$$

where $c_{01}\equiv\mu_{0r}/\mu_{1r}$ and $c_{12}\equiv\mu_{1r}/\mu_{2r}$ . (If $\vec{E}$ is replaced by $\vec{H}$, let $c_{01}\equiv\varepsilon_{0r}/\varepsilon_{1r}$ and $c_{12}\equiv\varepsilon_{1r}/\varepsilon_{2r}$ .)

Next we expand the surface fields $\hat{n}_0\times\vec{E}_1^-$ and $\hat{n}_1\times\vec{E}_2^-$ in suitable complete sets of functions. By means of these expansions, an algorithm for the elimination of the surface fields can be constructed. Here we have a variety of complete systems available. The standard expansion system for the surface field inside a finite scatterer is the regular spherical vector waves, see e.g. Ref. [2]. This system will also be used here, since we are aiming at an identification of the T-matrix of the inhomogeneity in terms of spherical waves. For the surface field $\hat{n}_1 x \vec{E}_2^-$ we thus assume an expansion

$$\hat{n}_1 \times \vec{E}_2^-(\vec{r}') = \sum_n \alpha_n \hat{n}_1 \times \mathrm{Re}\vec{\psi}_n(k_2\vec{r}') \qquad (19)$$

This sum is symbolical, since the convergence and completeness of such an expansion is in a mean square sense. The expansion coefficients $\alpha_n$ will in general depend on the truncation order and preassigned accuracy required, but this is not explicitly indicated here to avoid a heavy formalism. For more details on completeness and convergence of this system see e.g. [28] - [29] and appendix. We also note that an expansion such as in Eq. (19) does not rely on the Rayleigh hypothesis [28].

Since the surface $S_o$ is infinite the spherical vector wave functions are not convenient for the expansion of the surface field $\hat{n}_o x \vec{E}_1^-$. The completeness properties on a configuration consisting of an infinite and a finite surface is discussed in the appendix. As shown in Ref. [11] for the scalar case and in the appendix for the vector case, both the up- and down-going plane waves are needed when we expand the surface field $\hat{n}_o x \vec{E}_1^-$.

$$\hat{n}_o \times \vec{E}_1^-(\vec{r}') = \int_0^{2\pi} d\beta_1\, \hat{n}_o \times \left\{ \int_{C_-} \vec{\alpha}(\vec{k}_1) + \int_{C_+} \vec{\beta}(\vec{k}_1) \right\} e^{i\vec{k}_1\cdot\vec{r}'} \sin\alpha_1\, d\alpha_1 \qquad (20)$$

We note that the vector amplitudes $\vec{\alpha}(\vec{k}_1)$ and $\vec{\beta}(\vec{k}_1)$ in the expansion above, only have components along $\hat{a}_1$ and $\hat{a}_2$, because the transverse character of the solution. Again this type of expansion is independent of any Rayleigh hypothesis.

These expansions can now be inserted into Eqs. (16) and (17) together with the boundary conditions Eq. (18), and we get

$$\sum_{j=1}^{2}\int_0^{2\pi} d\beta_1 \left\{ \int_{C_-} \alpha_j(\vec{k}_1) + \int_{C_+} \beta_j(\vec{k}_1) \right\} \vec{I}_j(\vec{k}_1,\vec{r}) \sin\alpha_1 d\alpha_1 = -i \sum_{nn'} \vec{\psi}_n(k_1\vec{r}') \, Q_{nn'}(Re,Re)\, \alpha_{n'} \quad (21)$$

$$\sum_{j=1}^{2}\int_0^{2\pi} d\beta_1 \left\{ \int_{C_-} \alpha_j(\vec{k}_1) + \int_{C_+} \beta_j(\vec{k}_1) \right\} \vec{I}_j(\vec{k}_1,\vec{r}) \sin\alpha_1 d\alpha_1 = - i \sum_{nn'} Re\vec{\psi}_n(k_1\vec{r}) \, Q_{nn'}(Out,Re)\, \alpha_{n'} \quad (22)$$

Eq. (21) holds for all $\vec{r}\in V_0$ and outside the circumscribing sphere of $S_1$, while Eq. (22) holds for all $\vec{r}$ inside the inscribed sphere of $S_1$. We also have introduced the notation [2].

$$Q_{nn'}(Out,Re) \equiv k_1 \iint_{S_1} \hat{n}_1 \cdot \left\{ (\nabla' \times \vec{\psi}_n(k_1\vec{r}')) \times Re\vec{\psi}_{n'}(k_2\vec{r}') + c_{12}\, \vec{\psi}_n(k_1\vec{r}') \times (\nabla' \times Re\vec{\psi}_{n'}(k_2\vec{r}')) \right\} dS' \quad (23)$$

The matrix $Q_{nn'}(Re,Re)$ is defined analogously, now with the regular spherical vector waves in all places. The conventions concerning signs, normalizations etc. of the Q-matrices used in the present paper are identical to those of Refs. [3] and [5]. Furthermore we have introduced the symbol

$$\vec{I}_j(\vec{k}_1,\vec{r}) \equiv \nabla \times \iint_{S_0} \hat{n}_0 \times \hat{a}_j \, G(\vec{r},\vec{r}';k_1)\, e^{i\vec{k}_1\cdot\vec{r}'} dS' + k_1^{-2} \nabla \times \left\{ \nabla \times \iint_{S_0} \hat{n}_0 \times (i\vec{k}_1 \times \hat{a}_j)\, G(\vec{r},\vec{r}';k_1)\, e^{i\vec{k}_1\cdot\vec{r}'} dS' \right\} \quad (24)$$

$\alpha_j(\vec{k}_1)$ and $\beta_j(\vec{k}_1)$ are the $\hat{a}_j$-component of the vector amplitudes $\vec{\alpha}(\vec{k}_1)$ and $\vec{\beta}(\vec{k}_1)$ in Eq. (20). Eq. (24) can be simplified by application of Green's theorem to the volume above or below $S_o$ depending whether $\vec{k}_1$ belongs to $C_+$ or $C_-$. We get (see also [11] app. B)

$$\vec{I}_j(\vec{k}_1;\vec{r}) = \left\{ \begin{matrix} \hat{a}_j e^{i\vec{k}_1\cdot\vec{r}} \\ \vec{0} \end{matrix} \right\} \qquad \vec{k}_1 \in C_{\pm} \qquad \vec{r} \text{ above } S_o \tag{25}$$

$$\vec{I}_j(\vec{k}_1;\vec{r}) = \left\{ \begin{matrix} \vec{0} \\ -\hat{a}_j e^{i\vec{k}_1\cdot\vec{r}} \end{matrix} \right\} \qquad \vec{k}_1 \in C_{\pm} \qquad \vec{r} \text{ below } S_o \tag{26}$$

This property of the vector $\vec{I}_j(\vec{k}_1;\ \vec{r})$ simplify Eqs. (21) and (22)

$$\int_0^{2\pi} d\beta_1 \int_{C_+} \vec{\beta}(\vec{k}_1) e^{i\vec{k}_1\cdot\vec{r}} \sin\alpha_1 \, d\alpha_1 = -i \sum_{nn'} \vec{\psi}_n(k,\vec{r})\, Q_{nn'}(Re,Re)\,\alpha_{n'} \tag{27}$$

$$\int_0^{2\pi} d\beta_1 \int_{C_-} \vec{\alpha}(\vec{k}_1) e^{i\vec{k}_1\cdot\vec{r}} \sin\alpha_1 \, d\alpha_1 = i \sum_{nn'} Re\vec{\psi}_n(k,\vec{r})\, Q_{nn'}(Out,Re)\,\alpha_{n'} \tag{28}$$

As before the first equation holds for $\vec{r} \in V_o$ and outside the circumscribed sphere of $S_1$, while the second holds for $\vec{r}$ inside the inscribed sphere of $S_1$. To proceed we introduce the transformation between plane and outgoing spherical vector waves. The formulas linking these two types of waves are very similar to the formulas given in the scalar case, see Eq. (9) in [11], but the vector character introduces more complex transformation functions. This transformation, which can be found in e.g. [26], is written as

$$\vec{\psi}_n(k\vec{r}) = \frac{i}{2\pi} \int_0^{2\pi} d\beta \int_{C_{\pm}} \vec{B}_n(\hat{k}) e^{i\vec{k}\cdot\vec{r}} \sin\alpha \, d\alpha \qquad z \gtrless 0 \tag{29}$$

Here $C_{\pm}$ are the same contours as for the expansions of the Green's dyadic Eq. (5), see Figure 2, and furthermore the transformation vectors $\vec{B}_n(\hat{k})$ are defined as

$$\vec{B}_{1n}(\hat{k}) \equiv i^{-l-1}\vec{A}_{1n}(\hat{k})$$
$$\vec{B}_{2n}(\hat{k}) \equiv i^{-l}\vec{A}_{2n}(\hat{k})$$
$$\text{or}\quad \vec{B}_n(\hat{k}) \equiv (-i)^{l+2-\tau}\vec{A}_n(\hat{k}) \qquad \tau=1,2 \tag{30}$$

Devaney & Wolf [26] also gives the formula

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \vec{B}_n(\hat{r})e^{i\vec{k}\cdot\vec{r}}\sin\vartheta\, d\vartheta = 4\pi i^{-1}\,\mathrm{Re}\,\vec{\psi}_n(r\vec{k}) \tag{31}$$

We observe that this is the regular analogue of Eq. (29) - a transformation between regular spherical vector waves and plane waves - but here the integration contour is purely real. This transformation can be proven to hold for real values of $\alpha$, but is valid for complex propagation angles $\alpha$ as well - in the common domain of analyticity of the right and left hand sides.

We introduce Eq. (29) into Eq. (27) and we get

$$\vec{\beta}(\vec{k}_1) = \frac{1}{2\pi}\sum_{nn'}\vec{B}_n(\hat{k}_1)Q_{nn'}(\mathrm{Re},\mathrm{Re})\alpha_{n'} \qquad \vec{k}_1 \in C_+ \tag{32}$$

(The comparison of the integrands on a plane z= constant is valid, since the plane wave expansion is essentially a two-dimensional Fourier integral [12].)

We take the scalar product of Eq. (28) and the vector spherical harmonics $\vec{A}_n(\hat{r})$ and integrate over the unit sphere (this can be done since Eq. (28) is valid inside the inscribed sphere of $S_1$). By orthogonality and Eq. (31) we get

$$\int_0^{2\pi} d\beta_1 \int_{C_-} \vec{\alpha}(\vec{k}_1)\cdot\vec{B}_n^{\dagger}(\hat{k}_1)\sin\alpha_1\, d\alpha_1 = -\frac{1}{4\pi}\sum_{n'} Q_{nn'}(\mathrm{Out},\mathrm{Re})\alpha_{n'} \tag{33}$$

Here we have introduced the vectors $\vec{B}_n^\dagger(\hat{k})$, which are analogous to $\vec{B}_n(\hat{k})$ (-i is exchanged with +i)

$$\vec{B}_n^\dagger(\hat{k}) \equiv i^{l+2-\tau}\vec{A}_n(\hat{k}) \tag{34}$$

The plane wave analogue to the Q-matrices, Eq. (23), will now be introduced to eliminate the surface fields inEqs. (12) and (14). We define:

$$Q_{jj'}(\vec{k}_0,\vec{k}_1) \equiv -\frac{k_0}{8\pi^2}\iint_{S_0}\left\{(\hat{n}_0\times\hat{a}_j)\cdot(-i\vec{k}_0\times\hat{a}_{j'}) + c_{01}[\hat{n}_0\times(i\vec{k}_1\times\hat{a}_{j'})]\cdot\hat{a}_j\right\} e^{i(\vec{k}_1-\vec{k}_0)\cdot\vec{r}'}dS' \tag{35}$$

We notice here that for a flat surface $S_o$, the $Q_{jj'}(\vec{k}_o,\vec{k}_1)$ will be diagonal, i.e. it will be proportional to $\delta(k_{ox}-k_{1x})\,\delta(k_{oy}-k_{1y})$, which is analogous to the diagonal matrix in Eq. (23) for a spherical surface $S_1$. This diagonal character of $Q_{jj'}(\vec{k}_o,\vec{k}_1)$ is essentially the Snell's law of refraction, which will be more appearent below.

By introduction of the boundary conditions Eq. (18) and the surface field expansion Eq. (20) in Eqs. (12) and (14) we get

$$f_j(k_0) = -i\sum_{j'=1}^{2}\int_0^{2\pi}d\beta_1\left\{\int_{C_-}\alpha_{j'}(\vec{k}_1)+\int_{C_+}\beta_{j'}(\vec{k}_1)\right\}Q_{jj'}(\vec{k}_0,\vec{k}_1)\sin\alpha_1 d\alpha_1 \qquad \vec{k}_0\in C_+ \tag{36}$$

$$a_j(\vec{k}_0) = i\sum_{j'=1}^{2}\int_0^{2\pi}d\beta_1\left\{\int_{C_-}\alpha_{j'}(\vec{k}_1)+\int_{C_+}\beta_{j'}(\vec{k}_1)\right\}Q_{jj'}(\vec{k}_0,\vec{k}_1)\sin\alpha_1 d\alpha_1 \qquad \vec{k}_0\in C_- \tag{37}$$

Thus we have derived for the five quantities $\vec{f}(\vec{k}_o)$, $\vec{a}(\vec{k}_o)$, $\vec{\alpha}(\vec{k}_1)$, $\vec{\beta}(\vec{k}_1)$ and $\alpha_n$ four equations (36), (37), (32) and (33). From these equations we can extract a relation between any pair of quantities but we will here just concentrate on our primary goal, i.e. to find the vector amplitude $\vec{f}(\vec{k}_o)$ expressed in terms of the vector amplitude $\vec{a}(\vec{k}_o)$ of the incoming field.

## III Calculation of the scattered field

We will in this section develop the four basic equations (36),(37), (32) and (33) further. As pointed out above, we will only consider calculations of the vector amplitude $\vec{f}(\vec{k}_o)$ in terms of the prescribed incoming amplitude $\vec{a}(\vec{k}_o)$. The derivation of other types of relations between the quantities $\vec{f}(\vec{k}_o)$, $\vec{a}(\vec{k}_o)$, $\vec{\alpha}(\vec{k}_1)$, $\vec{\beta}(\vec{k}_1)$ and $\alpha_n$ can be treated in an analogous fashion.

In the transition matrix formalism the incoming field is influenced by the scatterer in a way, which is completely described by the T-matrix of the obstacle [1]-[2]. This makes it desirable to extract the T-matrix for the inhomogeneity $S_1$ from the four basic equations. Formally we solve Eq. (33) for $\alpha_{n'}$ and insert it into Eq. (32). We get

$$\vec{\beta}(\vec{k}_1) = 2\sum_{nn'} \vec{B}_n(\hat{k}_1) T_{nn'} \int_0^{2\pi} d\beta_1' \int_{C_-} \vec{\alpha}(\vec{k}_1') \cdot \vec{B}_{n'}^{+}(\hat{k}_1') \sin\alpha_1' \, d\alpha_1' \qquad \vec{k}_1 \in C_+ \tag{38}$$

The T-matrix is defined formally as (see e.g. [3] and [5])

$$T_{nn'} \equiv -\sum_{n''} Q_{nn''}(Re, Re)\, Q^{-1}_{n''n'}(Out, Re) \tag{39}$$

This type of relation between the vector amplitudes $\vec{\alpha}(\vec{k}_1)$ and $\vec{\beta}(\vec{k}_1)$, as expressed in Eq. (38), was expected. The downgoing wave amplitude $\vec{\alpha}(\vec{k}_1)$, given in the plane wave basis, is transformed into the spherical basis by the transformation vectors $\vec{B}_n^{+}(\hat{k}_1')$. The scattering effects from the inhomogeneity is totally determined by the T-matrix of the obstacle. Thus the downgoing wave amplitude is transformed by the T-matrix into an outgoing wave amplitude, which once again is transformed - as the T-matrix is related to the spherical wave basis - to the plane wave basis by the vectors $\vec{B}_n(\hat{k}_1)$. The structure of Eq. (38) thus has a very explicit physical interpretation. It is here also convenient to note that our original assumption of a homogeneous scatterer $V_2$ can be relaxed, and any type of scatterer is applicable as soon as its T-matrix is determined. Algorithms for several homogeneous or layered scatterers or scatterers with continuously varying material parameters are found in [3], [5] and [7].

We now concentrate on the elimination of the surface field amplitudes $\vec{\alpha}(\vec{k}_1)$ and $\vec{\beta}(\vec{k}_1)$ from Eqs. (36), (37) and (38). Formally this can be performed by inverting the integral equations. This formal solution is in practise of limited use, but it offers some insight into the structure of the problem and in particular a multiple scattering interpretation can be extracted. We formally write

$$\vec{f} = -i\left(Q^{\uparrow\downarrow}\vec{\alpha} + Q^{\uparrow\uparrow}\vec{\beta}\right)$$
$$\vec{a} = i\left(Q^{\downarrow\downarrow}\vec{\alpha} + Q^{\downarrow\uparrow}\vec{\beta}\right)$$
$$\vec{\beta} = T\vec{\alpha}$$

with a solution

$$\vec{f} = \left(R - Q^{\uparrow\uparrow}TQ^{\downarrow\downarrow^{-1}}\right)\left(1 + Q^{\downarrow\uparrow}TQ^{\downarrow\downarrow^{-1}}\right)^{-1}\vec{a} \tag{40}$$

Here we have by arrows $\downarrow\uparrow$ symbolized the directions of the plane wave, up- and downgoing, in the range and domain of the operators Q respectively. With these symbols we have $R \equiv -Q^{\uparrow\downarrow}Q^{\downarrow\downarrow-1}$. (The reflection coefficient of the surface $S_o$.) The multiple scattering interpretation of the solution, Eq. (40), given be an expansion of the inverse in Eq. (40) in powers of $Q^{\downarrow\uparrow}TQ^{\downarrow\downarrow-1}$ is depicted in Fig. 3.

The usefulness of the formalism in numerical applications will to a large degree depend upon the possibility of discretizing the final equations in a suitable way. Equations with a continuous variable often lead to inversion of integral equations, which is a delicate problem for numerical calculations. Thus, we want to rewrite the basic equations (36), (37) and (38) - at least for some class of simple infinite surfaces $S_o$ - as a discrete set of equations. We define the projection of the plane wave amplitude $\vec{\alpha}(\vec{k}_1)$ on the spherical basis as

$$C_{n\tau} \equiv C_n \equiv \int_0^{2\pi} d\beta_1 \int_{C_-} \vec{\alpha}(\vec{k}_1) \cdot \vec{B}_n^{\dagger}(\hat{k}_1) \sin\alpha_1 d\alpha_1 \tag{41}$$

and the equations (38) and (37) now read

$$\vec{\beta}(\vec{k}_1) = 2 \sum_{nn'} \vec{B}_n(\hat{k}_1) T_{nn'} C_{n'} \qquad \vec{k}_1 \in C_+ \tag{42}$$

$$a_j(\vec{k}_o) = i \sum_{j'} \int_0^{2\pi} d\beta_1 \left\{ \int_{C_-} \alpha_{j'}(\vec{k}_1) + \sum_{nn'} 2 \int_{C_+} B_{nj'}(\hat{k}_1) T_{nn'} C_{n'} \right\} \times \\ \times\, Q_{jj'}(\vec{k}_o, \vec{k}_1) \sin\alpha_1 d\alpha_1 \qquad \vec{k}_o \in C_- \tag{43}$$

Here we have adopted the notation $B_{nj}(\hat{k}) \equiv \vec{B}_n(\hat{k}) \cdot \hat{a}_j$ for the $\hat{a}_j$-component of $\vec{B}_n(\hat{k})$, and similary we define $B_{nj}^{\dagger}(\hat{k}) \equiv \vec{B}_n^{\dagger}(\hat{k}) \cdot \hat{a}_j$.

The next step will be to invert Eq. (43) in order to isolate $\vec{\alpha}(\vec{k}_1)$. (This step is formal, since as already pointed out the $Q_{jj'}(\vec{k}_o, \vec{k}_1)$ functions are not ordinary functions, i.e the equations have to be interpreted in a distribution sense. Nevertheless, for a flat surface the Eqs. (36) and (37) will degenerate to just simple algebraic expressions and the inversion is trivial.)

$$\alpha_j(\vec{k}_1) = -i \sum_{j'} \int_0^{2\pi} d\beta_o \int_{C_-} a_{j'}(\vec{k}_o) Q_{jj'}^{-1}(\vec{k}_1, \vec{k}_o) \sin\alpha_o d\alpha_o \; + \\ + 2 \sum_{nn'j'} \int_0^{2\pi} d\beta_1' \int_{C_+} B_{nj'}(\hat{k}_1') T_{nn'} C_{n'} R_{jj'}(\vec{k}_1, \vec{k}_1') \sin\alpha_1' d\alpha_1' \qquad \vec{k}_1 \in C_- \tag{44}$$

Here we have introduced the reflection coefficient $\mathcal{R}_{jj'}(\vec{k}_1,\vec{k}_1')$ from below:

$$\mathcal{R}_{jj'}(\vec{k}_1,\vec{k}_1') \equiv -\sum_{j_1}\int_0^{2\pi} d\beta_0 \int_{C_-} Q^{-1}_{jj_1}(\vec{k}_1,\vec{k}_0)\, Q_{j_1 j'}(k_0,\vec{k}_1')\sin\alpha_0\, d\alpha_0$$

$$\vec{k}_1 \in C_-\,,\ \vec{k}_1' \in C_+ \tag{45}$$

We proceed by multiplying Eq. (44) by $B^{\dagger}_{nj}(\hat{k}_1)$ and sum over j and integrate $\vec{k}_1$ over a $C_-$ contour. The left hand side then can be identified as a $c_n$ quantity and we get

$$c_n = d_n - \sum_{n'} A_{nn'}c_{n'} \tag{46}$$

The vector $d_n$ and the matrix $A_{nn'}$ are defined as

$$d_n \equiv -i\sum_{jj'}\int_0^{2\pi} d\beta_1 \int_{C_-}\sin\alpha_1\, d\alpha_1 \int_0^{2\pi} d\beta_0 \int_{C_-}\sin\alpha_0\, d\alpha_0\ a_{j'}(\vec{k}_0)\, Q^{-1}_{jj'}(\vec{k}_1,\vec{k}_0)\, B^{\dagger}_{nj}(\hat{k}_1) \tag{47}$$

$$A_{nn'} \equiv -2\sum_{n''jj'}\int_0^{2\pi} d\beta_1 \int_{C_-}\sin\alpha_1\, d\alpha_1 \int_0^{2\pi} d\beta_1' \int_{C_+}\sin\alpha_1'\, d\alpha_1' \times$$

$$\times\, B_{n''j'}(\hat{k}_1')\, T_{n'n''}\, \mathcal{R}_{jj'}(\vec{k}_1,\vec{k}_1')\, B^{\dagger}_{nj}(\hat{k}_1) \tag{48}$$

We here note that the vector $d_n$ is a completely known vector, when the geometry and incoming field is given, and can be interpreted as the incoming field transmitted through the surface $S_o$. Also the matrix $A_{nn'}$ is known and does not contain any quantities depending on the incoming field - it depends just on geometry and the material parameters.

The infinite matrix equation (46) can be solved for the unknown $c_n$, and necessary conditions for a solution to exist can be analysed in the same way as in the scalar case, (see [11] app. C), and we do not repeat these results here.

The solution of Eq. (46) determines the vector amplitudes $\vec{\alpha}(\vec{k}_1)$ and $\vec{\beta}(\vec{k}_1)$ by the equations (42) and (44), and the final solution - the vector amplitude $f_j(\vec{k}_0)$, see Eq. (36), - can be written in terms of $c_n$ as:

$$f_j(\vec{k}_0) = \sum_{j'} \int_0^{2\pi} d\beta_0' \int_{C_-} R_{jj'}(\vec{k}_0,\vec{k}_0')\, a_{j'}(\vec{k}_0')\, \sin\alpha_0'\, d\alpha_0' - \\ -2i \sum_{nn'j'} \int_0^{2\pi} d\beta_1 \int_{C_+} \sin\alpha_1\, d\alpha_1 \Big\{ Q_{jj'}(\vec{k}_0,\vec{k}_1) + \sum_{j''} \int_0^{2\pi} d\beta_1' \int_{C_-} \sin\alpha_1'\, d\alpha_1' \times \\ \times Q_{jj''}(\vec{k}_0,\vec{k}_1')\, R_{j''j'}(\vec{k}_1',\vec{k}_1) \Big\}\, B_{nj'}(\hat{k}_1)\, T_{nn'}\, c_{n'} \qquad \vec{k}_0 \in C_+ \tag{49}$$

The reflection coefficient $R_{jj'}(\vec{k}_0,\vec{k}_0')$ from above is the direct analogue to the T-matrix for finite scaterer, see Eq.(39) and defined as:

$$R_{jj'}(\vec{k}_0,\vec{k}_0') \equiv - \sum_{j''} \int_0^{2\pi} d\beta_1 \int_{C_-} Q_{jj''}(\vec{k}_0,\vec{k}_1)\, Q^{-1}_{j''j'}(\vec{k}_1,\vec{k}_0')\, \sin\alpha_1\, d\alpha_1 \\ \vec{k}_0 \in C_+ \,,\; \vec{k}_0' \in C_- \tag{50}$$

The scattered field $\vec{E}_0^{sc}(\vec{r})$ for $z>z_>$, given by Eq. (11), can now be written in terms of the incoming field amplitude $\vec{a}(\vec{k}_0)$ and the solution $c_n$ of the matrix Eq. (46). Insert Eq. (49) in Eq. (11) and we get

$$\vec{E}_0^{sc.}(\vec{r}) = \int_0^{2\pi} d\beta_0 \int_{C_+} \sum_{jj'} \int_0^{2\pi} d\beta_0' \int_{C_-} R_{jj'}(\vec{k}_0,\vec{k}_0')\, a_{j'}(\vec{k}_0')\, \sin\alpha_0'\, d\alpha_0'\, \hat{a}_j e^{i\vec{k}_0\cdot\vec{r}} \sin\alpha_0\, d\alpha_0 - \\ -2i \int_0^{2\pi} d\beta_0 \int_{C_+} \sum_{\substack{nn'\\ jj'}} \Big\{ \int_0^{2\pi} d\beta_1 \int_{C_+} \sin\alpha_1\, d\alpha_1 \Big[ Q_{jj'}(\vec{k}_0,\vec{k}_1) + \\ + \sum_{j''} \int_0^{2\pi} d\beta_1' \int_{C_-} \sin\alpha_1'\, d\alpha_1'\, Q_{jj''}(\vec{k}_0,\vec{k}_1')\, R_{j''j'}(\vec{k}_1',\vec{k}_1) \Big] B_{nj'}(\hat{k}_1)\, T_{nn'}\, c_{n'} \Big\} \times \\ \times \hat{a}_j\, e^{i\vec{k}_0\cdot\vec{r}} \sin\alpha_0\, d\alpha_0 = \vec{E}_0^{sc,dir.}(\vec{r}) + \vec{E}_0^{sc,anom.}(\vec{r}) \tag{51}$$

The first term in Eq. (51) can be identified as the directly reflected term, i.e. the total scattered field if no scatterer were present. All information about the scatterer is contained in the second term, called the anomalous scattered field, both directly via the T-matrix and indirectly via $c_n$, as a solution to Eq. (46). This second term contains all the multiple scattering effects and mutual interaction between $S_o$ and $S_1$, as can be seen by an expansion of the matrix $(1+A)^{-1}$ in powers of the reflection coefficient R and the T-matrix. An interpretation analogous to the one of Eq. (40) can be made, by interpreting R as a reflection at $S_o$ from below and T as scattering (reflection) from $S_1$. The details of this are left to the reader.

So far the main assumption concerning the infinite surface $S_o$ has been its confinement between the two parallel planes $z=z_>$ and $z=z_<$. The final Eq. (51) is rather complicated and it is illustrative to examine the simplifications which occur when $S_o$ is a flat surface. Let the surface $S_o$ be $z=z_o$=constant. The equations (45), (50), (47), (48) and (51) will then simplify, due to the diagonal character of Eq. (35).

$$\mathcal{R}_{jj'}(\vec{k}_1,\vec{k}_1') = -\delta_{jj'} R_j(\lambda_1)\, e^{2iz_o(k_1^2-\lambda_1^2)^{1/2}} \frac{\delta(\alpha_1-\pi+\alpha_1')\,\delta(\beta_1-\beta_1')}{\sin\alpha_1} \qquad \vec{k}_1\in C_-,\ \vec{k}_1'\in C_+ \tag{52}$$

$$R_{jj'}(\vec{k}_o,\vec{k}_o') = \delta_{jj'} R_j(\lambda_o)\, e^{-2iz_o(k_o^2-\lambda_o^2)^{1/2}} \frac{\delta(\alpha_o-\pi+\alpha_o')\,\delta(\beta_o-\beta_o')}{\sin\alpha_o} \qquad \vec{k}_o\in C_+,\ \vec{k}_o'\in C_- \tag{53}$$

$$d_n = \sum_j \int_0^{2\pi} d\beta_o \int_{C_-} a_j(\vec{k}_o)\, B^{\dagger}_{nj}(\hat{k}_o^*)\, e^{iz_o((k_1^2-\lambda_o^2)^{1/2}-(k_o^2-\lambda_o^2)^{1/2})} \frac{2(k_o^2-\lambda_o^2)^{1/2}}{D_j(\lambda_o)} \sin\alpha_o\, d\alpha_o \tag{54}$$

$$A_{nn'} = 2\sum_{n''j} \int_0^{2\pi} d\beta_1 \int_{C_-} B^{\dagger}_{nj}(\hat{k}_1)\, B_{n''j}(\tilde{\hat{k}}_1)\, T_{n''n'}\, R_j(\lambda_1)\, e^{2iz_o(k_1^2-\lambda_1^2)^{1/2}} \sin\alpha_1\, d\alpha_1 \tag{55}$$

$$\vec{E}_o^{sc.}(\vec{r}) = \vec{E}_o^{sc.,dir.}(\vec{r}) + \vec{E}_o^{sc.,anom}(\vec{r}) =$$

$$= \sum_j \int_0^{2\pi} d\beta_o \int_{C_+} a_j(\tilde{\vec{k}}_o) R_j(\lambda_o) e^{-2iz_o(k_o^2-\lambda_o^2)^{1/2} + i\vec{k}_o\cdot\vec{r}} \hat{a}_j \sin\alpha_o d\alpha_o + \tag{56}$$

$$+ \frac{4c_{o1}k_o}{k_1} \sum_{nn'j} \int_0^{2\pi} d\beta_o \int_{C_+} B_{nj}(\hat{k}_o^{\dagger}) T_{nn'} c_{n'} e^{iz_o((k_1^2-\lambda_o^2)^{1/2} - (k_o^2-\lambda_o^2)^{1/2}) + i\vec{k}_o\cdot\vec{r}} \times$$

$$\times \hat{a}_j \frac{(k_o^2-\lambda_o^2)^{1/2}}{D_j(\lambda_o)} \sin\alpha_o d\alpha_o$$

Here we have introduced the wellknown Fresnel reflection coefficients [30].

$$R_1(\lambda) \equiv \frac{N_1(\lambda)}{D_1(\lambda)} \equiv \frac{c_{o1}k_1(1-(\lambda/k_o)^2)^{1/2} - k_o(1-(\lambda/k_1)^2)^{1/2}}{c_{o1}k_1(1-(\lambda/k_o)^2)^{1/2} + k_o(1-(\lambda/k_1)^2)^{1/2}} \tag{57}$$

$$R_2(\lambda) \equiv \frac{N_2(\lambda)}{D_2(\lambda)} \equiv \frac{(k_o^2-\lambda^2)^{1/2} - c_{o1}(k_1^2-\lambda^2)^{1/2}}{(k_o^2-\lambda^2)^{1/2} + c_{o1}(k_1^2-\lambda^2)^{1/2}}$$

$R_1$ is the reflection coefficient for the electric field in the plane of incidence (note $\hat{a}_1=\hat{\alpha}$ in our convention) and $R_2$ is the corresponding quantity for the electric field perpendicular to the plane of incident ($\hat{a}_2=\hat{\beta}$), $D_j(\lambda)$ appearing in Eqs. (54) and (56) are the denominator of $R_j(\lambda)$ respectively, and $\lambda_i=k_i\sin\alpha_i$; i=0,1. The square root is defined such that $\text{Im}(k_i^2-\lambda_i^2)^{1/2} \geqslant 0$. Furthermore we have introduced the notation

$$\tilde{\hat{k}}_1 \equiv (\sin\alpha_1\cos\beta_1, \sin\alpha_1\sin\beta_1, -\cos\alpha_1)$$

$$\hat{k}_o^* \equiv \frac{k_o}{k_1}(\sin\alpha_o\cos\beta_o, \sin\alpha_o\sin\beta_o, -((k_1/k_o)^2 - \sin^2\alpha_o)^{1/2})$$

$$\tilde{\vec{k}}_o \equiv k_o(\sin\alpha_o\cos\beta_o, \sin\alpha_o\sin\beta_o, -\cos\alpha_o)$$

$$\hat{k}_o^{\dagger} \equiv \frac{k_o}{k_1}(\sin\alpha_o\cos\beta_o, \sin\alpha_o\sin\beta_o, ((k_1/k_o)^2 - \sin^2\alpha_o)^{1/2})$$

## IV Numerical applications

In this section we will analyse the numerical applications of the theory. The inhomogeneity is completely described by its T-matrix, and the applicability of the formalism for a flat interface depends on the existence of suitable algorithms for computing the T-matrix of the obstacle. Hitherto a long list of computer programmes is available for computing the T-matrix of various scatterers [31] - [34], and in this section we will illustrate the use of some of these.

The T matrix for a single sphere is trivial, since it is diagonal and can easily be computed. Scattering from a single buried sphere will be examplified in a series of figures, displaying various physical quantities. More interesting applications are scattering from a non-spherical obstacle. Computer programmes for a general rotationally symmetric body or for two spheres are available [31]-[34]. By a rotation of the T-matrix a general orientation of the common axis can be treated, and a series of figures for two buried spheres or a perfectly conducting spheroid will illustrate these calculations.

In all numerical examples shown here, the incoming field is a vertically orientated electric dipole, but more complicated source distributions can easily be introduced, cf. Eq. (13). Let the source point P be $\vec{r}_t=(\rho_t,0,z_t)$, where $(\rho,\phi,z)$ are cylindrical coordinates, and we have

$$\vec{E}_0^{inc}(\vec{r})=i\sqrt{\frac{8\pi}{3}}\,\vec{\psi}_{2e01}(k_0(\vec{r}-\vec{r}_t))=\frac{1}{k_0^2}\nabla\times\left\{\nabla\times\left[\hat{z}\,\frac{e^{ik_0|\vec{r}-\vec{r}_t|}}{k_0|\vec{r}-\vec{r}_t|}\right]\right\} \tag{58}$$

From Eq. (29) and Eq. (13) we then get

$$\vec{a}(\vec{k}_o) = \frac{1}{2\pi i}\,\hat{a}_1 \sin\alpha_o\, e^{-i\vec{k}_o\cdot\vec{r}_t} \qquad \vec{k}_o \in C_- \tag{59}$$

Eq. (54) then simplifies into ($z_t = z_0$)

$$\begin{aligned} d_n = \frac{2k_1\gamma_n}{ik_o\sqrt{\ell(\ell+1)}} \int_{i\infty}^{1} \frac{(-i)^{\ell-m}k_o t\,dt}{k_o\left(1-\left(\frac{k_o}{k_1}\right)^2(1-t^2)\right)^{1/2}+C_{o1}k_1 t}\; e^{ik_1 z_o\left(1-\left(\frac{k_o}{k_1}\right)^2(1-t^2)\right)^{1/2}} \times \\ \times J_m\left(k_o s_t (1-t^2)^{1/2}\right)\Big\{ im\,\delta_{\sigma 0}\delta_{\tau 1} P_\ell^m\left(\left(1-\left(\frac{k_o}{k_1}\right)^2(1-t^2)\right)^{1/2}\right) - \\ - \frac{\delta_{\sigma e}\delta_{\tau 2}}{2\ell+1}\Big[\ell(\ell+1-m)P_{\ell+1}^m\left(\left(1-\left(\frac{k_o}{k_1}\right)^2(1-t^2)\right)^{1/2}\right) - \\ - (\ell+1)(\ell+m)P_{\ell-1}^m\left(\left(1-\left(\frac{k_o}{k_1}\right)^2(1-t^2)\right)^{1/2}\right)\Big]\Big\} \end{aligned} \tag{60}$$

where the $\gamma_n$ is defined as in Eq. (8).

We also rewrite the anomalous scattered field $\vec{E}_o^{sc,anom}(\vec{r})$ as

$$\vec{E}_o^{sc,anom}(\vec{r}) = \sum_{nn'} \vec{F}_n(\vec{r})\, T_{nn'}\, c_{n'} \tag{61}$$

where $c_n$ is the solution of Eq. (46) and where

$$\begin{aligned} \vec{F}_n(\vec{r}) \equiv \frac{4c_{o1}k_o}{k_1}\sum_j \int_0^{2\pi} d\beta_o \int_{C_+} e^{i\vec{k}_o\cdot\vec{r}+iz_o\left((k_1^2-k_o^2\sin\alpha_o)^{1/2}-k_o\cos\alpha_o\right)} \times \\ \times \frac{k_o\cos\alpha_o}{D_j(\lambda_o)}\;\hat{a}_j\, B_{nj}(\hat{k}_o^{\dagger})\,\sin\alpha_o\, d\alpha_o \end{aligned} \tag{62}$$

The numerical procedure will be:

1) For a given source-position $\vec{r}_t$ compute the $d_n$-vector, Eq. (60)
2) Compute the $T_{nn'}$-matrix of the inhomogeneity
3) Generate the $A_{nn'}$-matrix, Eq. (55)
4) Solve the Eq. (46) for the $c_n$-vector
5) Generate for an array of measuring-points $\vec{r}$ the $\vec{F}_n(\vec{r})$-vector, Eq. (62)
6) Combine the quantities in Eq. (61).

We note that the integrals appearing in the vector $\vec{F}_n(\vec{r})$ and the $A_{nn'}$-matrix are independent of the inhomogeneity, and can consequently with advantage be generated and stored for a given contrast in the physical parameters of the volumes $V_o$ and $V_1$. This observation makes some of the steps in the scheme above superfluous, if only the properties of the inhomogeneity are varied in consecutive field computations.

The numerical calculations include a large number of numerical integrations along the $C_+$ and $C_-$ contours, or along a transformed, equivalent contour. In all these numerical integrations we will use a fast, improved quadrature procedure, which explicitly makes use of the previously computed functional values, when the subdivision of the integral interval increases. The integrals, which are exponentially decaying for $z_o>0$, are numerically stable, and show fast convergence for the contrasts used here.

We will in this paper present calculations of the anomalous scattered electric field $\vec{E}_o^{sc,anom}(\vec{r})$. The remaining part of the scattered field - $E_o^{sc,dir}(\vec{r})$ - is thoroughly analysed in e.g. Ref. [12], and will for a given air-earth (i.e. $V_o$-$V_1$) contrast be the same for all buried obstacles. Computer programmes for this directly scattered field are also available, see e.g. [35], and it is for this reasons that we here omit that part of the total field computation.

The numerical results can be represented in a number of ways, but we will here mainly give the results as the absolute value of various components of $\vec{E}_o^{sc,anom}(\vec{r})$ and its phase. Furthermore a number of graphs will show the absolute value of the ratio of the vertical component of the anomalous scattered field to the corresponding component of the incoming field. In the numerical examples shown below we assume all space to be lossless, more explicitly we take; $k_1/k_o=2.5$, $C_{o1}=1$, $k_1 z_o=2$. The source is a vertical electric dipole on the surface $z_t=z_o$ and the $d_n$-vector is computed for a series of source positions $k_o\rho_t=0,3,8$.

As mentioned above the simplest obstacle, which the theory can be applied to is a single buried sphere, since its T-matrix is diagonal. In Figs. 4-5 we show some examples of numerical calculations for this type of situation. Figure 4 shows the z- and x-component of the anomalous scattered field. This plot shows surfaces of constant value of the absolute value. Due to the interpolation routines in the computer these plots exhibit some irregularities, which do not appear in the originally calculated field. In Figure 5 we show the ratio of the absolute value of the z-component of $\vec{E}_o^{sc,anom}(\vec{r})$ to the vertical incoming field $\vec{E}_o^{inc}(\vec{r})$. The plot illustrates the variation of this quantity for two different rays through the z-axis: $\phi=225^o$ and $\phi=45^o$ and $\phi=\pm 90^o$ respectively. In general the z-component of the field shows a much more complex interference pattern than the x- and y-components of the field, presumably due to the orientation of the source and to the boundary conditions (continuity of the horizontal components). More detailed analysis of the graphs is not possible at this stage, but additional computations must be made.

To illustrate the applicability of the theory to non-spherical buried configurations we also calculate the anomalous scattered field from two buried spheres. The radii of the spheres are $a_1$ and $a_2$ and the separation between their centres is d. The orientation of the common axis (from the smaller to the bigger sphere) is given in spherical angles $\theta$, $\chi$ as depicted in Fig. 6. The T-matrix for two spheres is computed for an orientation along the z-axis, see Ref. [34]. For a different orientation of the spheres, the T-matrix is similarity-transformed by the rotation matrices (these matrices are block-diagonal $(2l+1) \times (2l+1)$ matrices) cf. Ref. [36]. In Figs. 7-9 we show computer plots of surfaces of constant values for various components and source positions of the anomalous scattered electric field. Again we have the complex interference pattern of the vertical component of the field, while the horizontal components show a less complex scattering picture. Fig. 10 shows the variation in the vertical component of the anomalous scattered field to the corresponding component of the incoming field along two rays for data as in Fig. 9. We have in Fig. 11 plotted the effect of a variation in the k value of one of the spheres, while all other parameters are fixed.

In Fig. 12 the scattering picture from one buried sphere at variable depth is presented. Note that these results can be obtained by letting one of the spheres in the computer programme for two spheres vanish, and by then considering a variation of the separation between the remaining and the fictitious sphere. In this way a repeated calculation of the integrals in Eqs. (55), (60) and (62) can be avoided.

Scattering from a single buried non-spherical obstacle is illustrated in Figs. 13-14. The T-matrix for an arbitrarily shaped dielectric or perfectly conducting rotationally symmetric body is computed for an orientation along the z-axis, see e.g. [31] - [33] . By an application of the rotational matrices mentioned above we then can obtain any orientation of the symmetry axis of the inhomogeneity. In this paper we have for simplicity taken the obstacle to be a perfectly conducting spheroid. In Fig. 13 we display the z-component of the anomalous scattered field for two various orientations of an oblate spheroid. Fig. 14 is an analogous plot for a prolate spheroid.

The results presented above are just intended to serve as examples of the kind of numerical computations which can be performed within the present formalism. Similar computations can now be done straightforwardly for various other configurations and parameter contrasts. In this section we have only illustrated a few quantities of physical interest. Other quantities of interest in e.g. electromagnetic prospecting situations would be the total scattered field - i.e. the direct plus the anomalous scattered field - and various ratios, for example the wave tilt i.e. the ratio of the horizontal to the vertical component of the total scattered electric field or the surface impedence. Another extention of the numerical calculations would be computations with complex calues of k. All these aspects will be considered in future papers.

## V Concluding remarks

We have in this paper presented a transition matrix formalism for electromagnetic scattering from buried inhomogeneities. This formalism is an extension of the corresponding scalar results [11]. The extension to elastic waves is believed not to cause any difficulties and work is in progress that analyses this case.

In this paper the formalism is developed for a general interface $S_o$, but only results for a flat surface $S_o$ has been illustrated numerically. Compared to actual surfaces encountered in real prospecting situations, this is a strong restriction. A more detailed treatment of scattering from configurations that involve a nonflat surface $S_o$ is therefore of great interest and results in this direction would substantially increase the usefulness of the formalism. If the deviation from a plane is restricted to a finite region one can avoid integral equations over infinite intervals. By adding and subtracting in a suitable way one obtains an integral equation over a finite intervall and a corresponding contribution from an infinite plane surface, which can be treated analytically. The proper expansion systems for the surface field for such a configuration are of fundamental importance, and work in these directions is in progress and will be reported elsewhere.

All media in the analysis above were assumed to be lossless, but this restriction can be relaxed and an analogous formulation for media with losses can be made. We also expect that an extension to a layered structure instead of the single infinite surface $S_o$ will give a fairly straightforward modification of the results given above. The reflection from the surface $S_0$ is completely determined by its reflection coefficient and if a stratification is introduced this reflection coefficient for the single surface will be replaced by its total reflection coefficient for the layered structure.

We have in all numerical examples illustrated various components of the anomalous scattered field and its quotient with the vertical incoming field. Of interest are various other ratios, for example a comparison with the total electric field or the directly scattered field. The source has in all examples been a vertical electric dipole. An interesting comparison would be computations with sources of horizontal polarizations or having a more complex structure. These extensions are straightforward and not reported here. Furthermore an equivalent theory can be formulated for the source position in $V_1$ or $V_2$. This will lead to modifications in the basic equations and can be analysed analogously (see [11] for the acoustic case).

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## Appendix

We will in this appendix analyse the completeness properties of the tangential spherical and plane vector wave systems. The completeness properties of the spherical waves on a finite surface will be studied first. The plane wave system is then shown to be complete on an infinite surface and these results are finally extended to hold for a combination of finite and infinite surfaces as depicted in Fig. 1. In the first part of this appendix we will make a short review of the properties of the layer distributions at the boundary of a smooth surface in the electromagnetic case. For a review on the corresponding results in the scalar case, see [37].

Consider the following vector

$$\vec{\phi}(\vec{r}) = \iint_S \vec{f}(\vec{r}')\, G(\vec{r},\vec{r}';k)\, dS' \qquad \text{(A.1)}$$

where S is a closed smooth bounded surface and $\vec{f}(\vec{r})$ is a surface field, i.e. a continuous vector field defined on S such that $\hat{n}\cdot\vec{f}(\vec{r}')=0$. $\vec{\phi}(\vec{r})$ is defined for every $\vec{r}$ not on S, but it can furthermore be shown that $\vec{\phi}(\vec{r})$ is continuous everywhere, see e.g. Lemma 73 in [38].

Taking the curl of (A.1), we get

$$\nabla\times\vec{\phi}(\vec{r}) = \iint_S \vec{f}(\vec{r}')\times\nabla' G(\vec{r},\vec{r}';k)\, dS' \qquad \text{(A.2)}$$

This operation is valid for $\vec{r}$ not on S, but for an $\vec{r}$ approaching S from the positive or negative side (with respect to $\hat{n}$) we have a definite limit and furthermore there is a jump in the tangential component of (A.2) when crossing the surface S. Theorem 46 in [38] gives

$$\hat{n}(\vec{r}_0)\times\left\{\nabla\times\vec{\phi}(\vec{r}_0)\big|_+ - \nabla\times\vec{\phi}(\vec{r}_0)\big|_-\right\} = \vec{f}(\vec{r}_0) \qquad \vec{r}_0\in S \tag{A.3}$$

If we operate once more with the curl operator on (A.1) we get for an $\vec{r}$ not on S

$$\nabla\times\nabla\times\vec{\phi}(\vec{r}) = -\iint_S \vec{f}(\vec{r}\,')\,\nabla'^2 G(\vec{r},\vec{r}\,';k)\,dS' - \nabla\iint_S \vec{f}(\vec{r}\,')\cdot\nabla' G(\vec{r},\vec{r}\,';k)\,dS'$$
$$= \iint_S k^2\vec{f}(\vec{r}\,')\,G(\vec{r},\vec{r}\,';k)\,dS' - \nabla\iint_S \vec{f}(\vec{r}\,')\cdot\nabla' G(\vec{r},\vec{r}\,';k)\,dS' \tag{A.4}$$

As before the first term on the right hand side is continuous everywhere, and furthermore the second term can be shown to have a continuous tangential component, if we assume that $\vec{f}(\vec{r}\,')$ is a continuously differentiable surface field.

We define the surface gradiant

$$\nabla_0' G(\vec{r},\vec{r}\,';k) \equiv \nabla' G(\vec{r},\vec{r}\,';k) - \hat{n}\frac{\partial}{\partial n'} G(\vec{r},\vec{r}\,';k) \tag{A.5}$$

and we get since $\vec{f}(\vec{r}\,')\cdot\hat{n}=0$

$$\iint_S \vec{f}(\vec{r}\,')\cdot\nabla' G(\vec{r},\vec{r}\,';k)\,dS' = \iint_S \vec{f}(\vec{r}\,')\cdot\nabla_0' G(\vec{r},\vec{r}\,';k)\,dS' =$$
$$= -\int_S \nabla_0'\vec{f}(\vec{r}\,')\,G(\vec{r},\vec{r}\,';k)\,dS' \tag{A.6}$$

The last equality follows from an application of the corresponding Gauss' theorem for the surface gradient $\nabla_0$, see Lemma 58 in [38].

Thus we get

$$\nabla \iint_S \vec{f}(\vec{r}') \cdot \nabla' G(\vec{r},\vec{r}';k)\, dS' = \iint_S \nabla_0' \cdot \vec{f}(\vec{r}')\, \nabla' G(\vec{r},\vec{r}';k)\, dS' \tag{A.7}$$

Eq. (A.7) has a jump in the normal component when one crosses the surface. The discontinuity is, see Theorem 45 in [38]

$$\nabla \iint_S \vec{f}(\vec{r}') \cdot \nabla' G(\vec{r}_0,\vec{r}';k)\, dS' \Big|_+ - \nabla \iint_S \vec{f}(\vec{r}') \cdot \nabla' G(\vec{r}_0,\vec{r}';k)\, dS' \Big|_- =$$

$$= \hat{n}(\vec{r}_0)\, \nabla_0' \cdot \vec{f}(\vec{r}_0) \qquad \vec{r}_0 \in S \tag{A.8}$$

Thus the left hand side of (A.4) has a continuous tangential component when crossing the surface.

We will now discuss the completeness of the tangential components of the regular spherical vector waves on a smooth closed bounded surface S. The results obtained for $\hat{n}\times \mathrm{Re}\vec{\psi}_n$ are analogous to those of [29] for $\hat{n}\times\vec{\psi}_n$. However, we shall prefer to use a somewhat more explicit notation and make the derivation in a way which is more closely analogous to the treatment of the scalar case given in [28].

Let $L^2(S,dS)$ denote the class of complex-valued square-integrable surface fields and define a scalarproduct and norm as

$$\langle \vec{\phi}_1, \vec{\phi}_2 \rangle \equiv \iint_S \vec{\phi}_1^{\,*} \cdot \vec{\phi}_2\, dS \qquad \vec{\phi}_1, \vec{\phi}_2 \in L^2(S,dS)$$

$$\|\phi\| \equiv \sqrt{\langle \vec{\phi}, \vec{\phi} \rangle} \qquad \vec{\phi} \in L^2(S,dS) \tag{A.9}$$

We observe that the spherical vector waves $\{\hat{n}\times \mathrm{Re}\vec{\psi}_n(k\vec{r}\,)\}$ belong to $L^2(S,dS)$.

We will here adopt the following definition of completeness and closure. A sequence $\{\vec{\phi}_i\}$ is closed if,

$$\langle \vec{\phi}_i, \vec{\phi} \rangle = 0 \qquad \text{for all i implies } \vec{\phi}=\vec{0}$$

A sequence $\{\vec{\phi}_i\}$ is complete if all elements in $L^2(S,dS)$ can be approximated arbitrarily closely by a finite linear combination of elements in $\{\vec{\phi}_i\}$, i.e. for given $\vec{\phi}\in L^2(S,dS)$ and $\varepsilon>0$ find $\alpha_1(N),\ldots,\alpha_N(N)$ such that

$$\|\vec{\phi} - \sum_i^N \alpha_i(N)\vec{\phi}_i\| < \varepsilon$$

These two concepts are equivalent for the space $L^2(S,dS)$, i.e. a system that is complete if and only if it is closed, see [39]. It should be noted that [39] has a reverse definition of a closed and a complete sequence. In this appendix we will show the completeness of a system by showing that it is closed.

Thus assume that

$$\langle \vec{f}, \hat{n}\times \mathrm{Re}\vec{\psi}_n \rangle = 0 \qquad \text{for all n} \tag{A.10}$$

Let here $\vec{f}$ belong to a suitable dense set in $L^2(S,dS)$. Take for simplicity $\vec{f}$ to be continuously differentiable on S. Take an $\vec{r}$ outside the circumscribing sphere of S, i.e. $|\vec{r}|>\max_{\vec{r}'\in S}|\vec{r}'|$ and multiply Eq. (A.10) by $\vec{\psi}_n(k\vec{r})$ and sum over n ($\tau=1,2$). Make use of Eq. (10) and we get

$$\vec{0} = \iint_S (\overleftrightarrow{\mathbf{I}}G(\vec{r},\vec{r}';k)\times\vec{f}^*(\vec{r}'))\cdot\hat{n}\,dS' + \vec{\psi}_{irr}(\vec{r}) \tag{A.11}$$

Here $\vec{\psi}_{irr}(\vec{r})$ is a irrotational vector. We define

$$\vec{v}(\vec{r}) \equiv -\nabla \times \iint_S \left( \overleftrightarrow{I} G(\vec{r},\vec{r}\,';k) \times \vec{f}^{\,*}(\vec{r}\,') \right) \cdot \hat{n}\, dS' =$$

$$= \iint_S (\hat{n} \times \vec{f}^{\,*}(\vec{r}\,')) \times \nabla' G(\vec{r},\vec{r}\,';k)\, dS' \qquad \text{(A.12)}$$

This vector is zero outside the circumscribing sphere of S, and furthermore since it is a solenoidal solution of the vector Helmholtz' equation it can be continued up to S and thus zero for all $\vec{r}$ outside S.

Now take the curl of Eq. (A.12) For $\vec{r} \notin S$ we then have

$$\nabla \times \vec{v}(\vec{r}) =$$

$$= k^2 \iint_S \hat{n} \times \vec{f}^{\,*}(\vec{r}\,')\, G(\vec{r},\vec{r}\,';k)\, dS' - \nabla \iint_S (\hat{n} \times \vec{f}^{\,*}(\vec{r}\,')) \cdot \nabla' G(\vec{r},\vec{r}\,';k)\, dS' \qquad \text{(A.13)}$$

and we have $\nabla \times \vec{v}(\vec{r}) = \vec{0}$ for an $\vec{r}$ outside S. Furthermore this quantity has a continuous tangential component across the surface S as reviewed above, see Eqs. (A.1) and (A.8).

We will now study Eq. (A.13) for an $\vec{r}$ inside S. The right hand side is a solenoidal solution to the vector Helmholtz' equation with zero boundary values for the tangential component. A non-zero solution to this problem can only exist for a discrete set of k values [38] and if we exclude these k values we have that $\nabla \times \vec{v}(\vec{r})$ is zero everywhere. An application of Stokes' theorem to a narrow loop across the surface S will show the continuity of the tangential components of $\vec{v}(\vec{r})$ and we thus have, see Eqs. (A.2) - (A.3)

$$\vec{0} = \hat{n} \times \left( \vec{v}_+(\vec{r}_0) - \vec{v}_-(\vec{r}_0) \right) = \hat{n}(\vec{r}_0) \times \vec{f}^{\,*}(\vec{r}_0) \qquad \vec{r}_0 \in S \qquad \text{(A.14)}$$

or since $\vec{f}\cdot\hat{n}=0$

$$\vec{f} = -\hat{n}\times(\hat{n}\times\vec{f}) = \vec{0} \tag{A.15}$$

But, as discussed in [28], it is sufficient to consider a dense subset of $L^2(S,dS)$ and make an approximation of $\vec{f}\in L^2(S,dS)$ by a sequence of functions in this dense sets. This sequence will tend to zero [28], and we conclude:

$$\langle\vec{f},\hat{n}\times\mathrm{Re}\vec{\psi}_n\rangle = 0 \quad \text{for all } n,\ \vec{f}\in L^2(S,dS) \Rightarrow \vec{f}=\vec{0}$$

Now let S be an infinite surface bounded above by a plane $z=z_>$. We consider the tangential plane vector waves defined as

$$\{\hat{n}\times\hat{a}_j e^{i\vec{k}\cdot\vec{r}}\} \qquad j=1,2;\ \vec{k}\in C_- \tag{A.16}$$

The tangential plane vector waves do not belong to $L^2(S,dS)$ but they can be treated as generalized eigenfunctions and for these one has the corresponding completeness relation, see e.g. [40].

In analogy with Eq. (A.10) we now start with

$$\iint \vec{f}^*(\vec{r}\,')\cdot(\hat{n}\times\hat{a}_j e^{i\vec{k}\cdot\vec{r}\,'})\,dS' = 0 \qquad j=1,2;\ \vec{k}\in C_- \tag{A.17}$$

Let here $\vec{f}$ belong to a suitable dense subset of $L^2(S,dS)$ e.g. let $\vec{f}$ be a infinitely differentiable function on S which is rapidly decreasing at infinity.

Take an $\vec{r}$ for which we have $z>z_>$ and multiply by $\hat{a}_j e^{-i\vec{k}\cdot\vec{r}}$ and integrate $\vec{k}$ over a $C_-$ contour and sum over j. We get as before (see Eq. (17)),

$$\vec{0} = \iint_S (\overleftrightarrow{J} G(\vec{r},\vec{r}\,';k) \times \vec{f}^{\,*}(\vec{r}\,')) \cdot \hat{n}\, dS' + \vec{\psi}_{irr}(\vec{r}) \tag{A.18}$$

Define, as above

$$\vec{v}(\vec{r}) \equiv \iint_S (\hat{n} \times \vec{f}^{\,*}(\vec{r}\,')) \times \nabla' G(\vec{r},\vec{r}\,';k)\, dS' \tag{A.19}$$

We conclude as above that this vector is zero everywhere above S, since it is zero for $z>z_>$, and furthermore can be continued to every $\vec{r}$ above S. Again we observe that the curl of (A.19) has a continuous tangential part when crossing S. The curl of Eq. (A.19) for an $\vec{r}$ below S defines a solenoidal solution of the vector Helmholtz' equation, which satisfies the radiation condition and has a zero tangential component on the boundary S. We assume this problem to be unique, so we have as above that $\nabla \times \vec{v} \equiv 0$ everywhere. (The uniqueness of the analogous situation in the scalar case is certain, at least for a large class of surfaces S [41] - [44].) The surface field $\vec{f}$ is proportional to the jump in the tangential component of $\vec{v}$ across S and as before we have $\vec{f} \equiv \vec{0}$. Thus we have

$$\langle \vec{f}, \hat{n} \times \hat{a}_j e^{i\vec{k}\cdot\vec{r}} \rangle = 0 \qquad j=1,2\,;\ k \in C_- \quad \vec{f} \in L^2(S,dS) \Rightarrow \vec{f} = \vec{0}$$

and the tangential plane vector waves (see Eq. (A.16)) are complete on a surface S as defined above.

Finally we will consider the geometry depicted in Fig. 1 and we will show the completeness of the following system in $L^2(S_0+S_1,dS)$

$$\left\{ \hat{n}\times\hat{a}_j e^{i\vec{k}\cdot\vec{r}},\ \hat{n}\times\vec{\psi}_n(k\vec{r}) \right\} \qquad j=1,2\ ;\ \vec{k}\in C_- \tag{A.20}$$

We assume

$$\iint_{S_0+S_1} \vec{f}^{\,*}(\vec{r}\,')\cdot(\hat{n}\times\hat{a}_j e^{i\vec{k}\cdot\vec{r}\,'})\, dS' = 0 \qquad j=1,2\ ;\ \vec{k}\in C_- \tag{A.21}$$

$$\iint_{S_0+S_1} \vec{f}(\vec{r}\,')\cdot(\hat{n}\times\vec{\psi}_n(k\vec{r}\,'))\, dS' = 0 \qquad \text{for all } n \tag{A.22}$$

Here it is assumed that $\vec{f}$ is an infinitely differentiable function on $S_0$ and $S_1$ and is rapidly decreasing at infinity.

Choose an $\vec{r}$ inside the inscribed sphere of $S_1$ and multiply (A.22) by $\mathrm{Re}\vec{\psi}_n(k\vec{r})$ and sum over n. Furthermore take an $\vec{r}$ with $z>z_>$ and multiply (A.21) by $\hat{a}_j e^{-i\vec{k}\cdot\vec{r}}$ and sum over j and integrate over a $C_-$ contour. We have that

$$\vec{v}(\vec{r}) \equiv \iint_{S_0+S_1} (\hat{n}\times\vec{f}^{\,*}(\vec{r}\,'))\times\nabla' G(\vec{r},\vec{r}\,';k)\, dS' \tag{A.23}$$

is zero for all $\vec{r}$ above $S_o$ and $\vec{r}$ inside $S_1$. As before, the curl of (A.23) is a solenoidal solution of vector Helmholtz' equation in $V_1$ with zero tangential components on $S_o$ and $S_1$, and this solution is assumed unique. In the scalar case this is true at least for a restricted class of surfaces $S_o$ such as a plane surface $S_o$ or a plane surface with a hill of finite extent, see [42] - [44]. Thus we can, in the same way as before, conclude that

$$\left.\begin{array}{ll} \langle \vec{f}, \hat{n}\times\hat{a}_j e^{i\vec{k}\cdot\vec{r}} \rangle = 0 & j=1,2\ ;\ \vec{k}\in C_- \\ \langle \vec{f}, \hat{n}\times\vec{\psi}_n \rangle = 0 & \text{for all } n \end{array}\right\} \Rightarrow \vec{f}=\vec{0} \tag{A.24}$$

and the system in Eq. (A.20) is complete in $L^2(S_o+S_1,dS)$.

In some applications it would be more convenient to work only with the tangential plane vector waves, i.e.

$$\{\hat{n}\times\hat{a}_j e^{i\vec{k}\cdot\vec{r}}\} \qquad \vec{k}\in C_\pm\ ;\ j=1,2 \tag{A.25}$$

This is always possible, since $\hat{n}\times\vec{\psi}_n(k\vec{r})$ can be expressed in plane waves, cf. Eq. (29). If the origin lies below $z=z_<$ only $\vec{k}\in C_+$ will contribute, but since we assume $S_1$ to be below $S_o$ there always will be a contribution from $\vec{k}\in C_+$. This result was used when we introduced Eq. (20) for the surface field $\hat{n}\times\vec{E}_1^-$.

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Figure captions

Fig. 1 Geometry and notations of the scattering problem.

Fig. 2 The integration contours $C_+$ and $C_-$.

Fig. 3 Multiple-scattering interpretation of Eq. (40).

Fig. 4 Upper part: The amplitude of the z-component of the anomalous scattered field $|\vec{E}_o^{sc,anom}\cdot\hat{z}|$ on the surface $S_o(k_1 z_o=2)$ for a buried sphere of radius $k_1 a=0.5$, $k_1/k_2=2$, $C_{12}=1$. The source is located at $k_o\rho_t=3$ and the scale factor is $10^{-4}$.

Lower part: The corresponding x-component of the anomalous field. (The scale on x- and y-axes are in units of $k_o$. For additional information about parameters see the text.)

Fig. 5 The variation in $|\vec{E}_o^{sc,anom}\cdot z|/|\vec{E}_o^{inc}\cdot\hat{z}|$ along two rays through the z-axis. The obstacle is a buried sphere with data as in Fig. 4. The rays have azimutal angles a) $\phi=225^o$ and $45^o$; b) $\phi=-90^o$ and $\phi=+90^o$. (The scale is given in percent.)

Fig. 6 Spherical angles $(\theta,\chi)$ for the orientation of two spheres.

Fig. 7 The amplitude of the y-component of the anomalous scattered field $|\vec{E}_o^{sc,anom}\cdot\hat{y}|$ on the surface $S_o$ $(k_1 z_o=2)$ for two buried spheres of radii $k_1 a_1=0.5$, $k_1 a_2=0.25$, $k_1/k_2=2.$, $C_{12}=1.$, $k_1/k_3=2.$, $C_{13}=1$. The separation distance is $k_1 d=1.$ and the orientation of the symmetry axis is $\theta=\chi=\pi/2$. The source is located at $k_o\rho_t=0.$ and the scale factor is $10^{-4}$. (The index 3 refers to the smaller sphere and $C_{13}\equiv\mu_{1r}/\mu_{3r}$.)

Fig. 8 The amplitude of the z-component of the anomalous scattered field $|\vec{E}_o^{sc,anom}\cdot\hat{z}|$ on the surface $(k_1 z_o=2)$ for two buried spheres of radii $k_1a_1=k_1a_2=0.5$ $k_1/k_2=k_1/k_3=2.$, $C_{12}=C_{13}=1.$, $k_1d=1.5$, $\theta=\pi/2$, $\chi=\pi/4$. The source position $k_o\rho_t=0.$ and the scale factor is $10^{-4}$.

Fig. 9 The amplitude of the x-component of the anomalous scattered field $|\vec{E}_o^{sc,anom}\cdot\hat{x}|$ on the surface $S_o(k_1z_o=2.)$ for two buried spheres of radii $k_1a_1=0.75$, $k_1a_2=0.5$, $k_1/k_2=2$, $C_{12}=1$, $k_1/k_3=2.$, $C_{13}=1.$, $k_1d=1.5$, $\theta=\pi/2$, $\chi=\pi/4$. The source position $k_o\rho_t=3$ and the scale factor is $10^{-5}$.

Fig. 10 The variation in $|\vec{E}_o^{sc,anom}\cdot\hat{z}|/|\vec{E}_o^{inc}\cdot\hat{z}|$ for data as in Fig. 9 along two rays a) $\phi=270^o$ and $\phi=90^o$ b) $\phi=225^o$ and $\phi=45^o$. The scale is in percent.

Fig. 11 The amplitude and phase variation of $\vec{E}_o^{sc,anom}\cdot\hat{x}$ along a ray $\phi=225^o$ and $\phi=45^o$ on the surface $S_o(k_1z_o=2)$ for two buried spheres $k_1a_1=k_1a_2=0.5$, $k_1d=1.5$, $\theta=\chi=\pi/2$ $k_1/k_3=2.$, $C_{12}=C_{13}=1.$, $k_o\rho_t=3$. (Here the index 2 refers to the sphere on the positive y-axis and index 3 to the sphere on the negative axis) a) $k_1/k_2=2$, b) $k_1/k_2=0.5$. The scale factor is $10^{-3}$.

Fig. 12 The amplitude and phase variation of $\vec{E}_o^{sc,anom}\cdot\hat{z}$ along a ray $\phi=180^o$ and $\phi=0^o$ on the surface $S_o$ for a single buried sphere $k_1a_1=0.5, k_1/k_2=2$, $C_{12}=1$, $k_o\rho_t=3$ as a variation of the distance h between the surface $S_0$ and the centre of the sphere. a) $k_1h=2.5$, b) $k_1h=2$, c) $k_1h=1.5$.

Fig. 13 The amplitude of the z-component of the anomalous scattered field $|\vec{E}_o^{sc,anom} \cdot \hat{z}|$ on the surface $S_o$ ($k_1 z_o=2$) for a single buried perfectly conducting spheroid. The semi axis in the direction of rotational symmetry is $k_1 a=0.2$ and the other semi axis is $k_1 b=0.6$. The source position is $k_o \rho_t=3$ and the scale factor $10^{-4}$. Upper part: orientation of the common axis $\theta=\pi/2$, $\chi=0$. Lower part: orientation of the common axis $\theta=0$, $\chi=0$.

Fig. 14 The same as in Fig. 13 but $k_1 a=0.6$ and $k_1 b=0.2$.

Figure 1

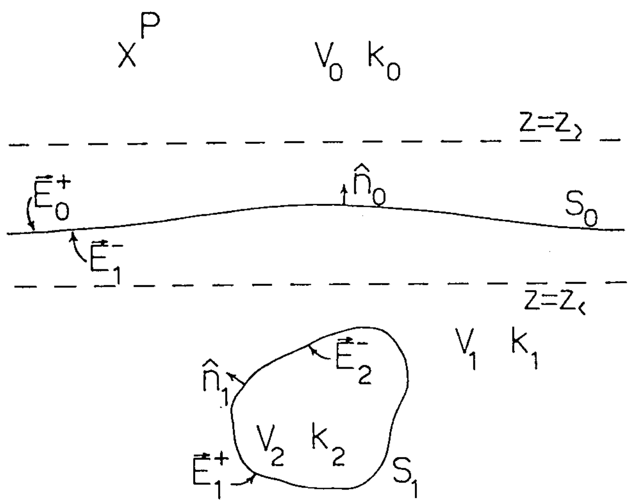



Figure 2

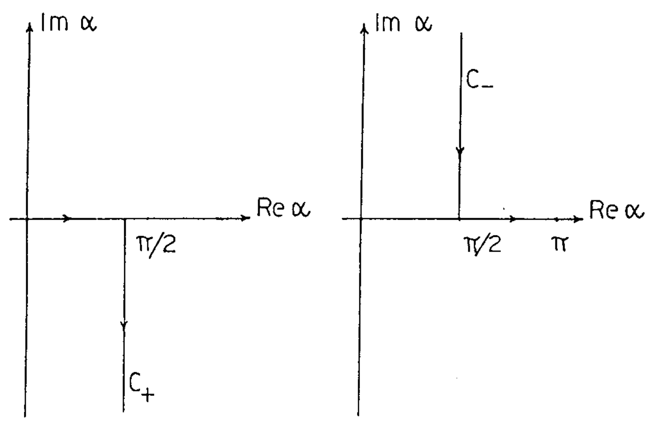


Figure 3

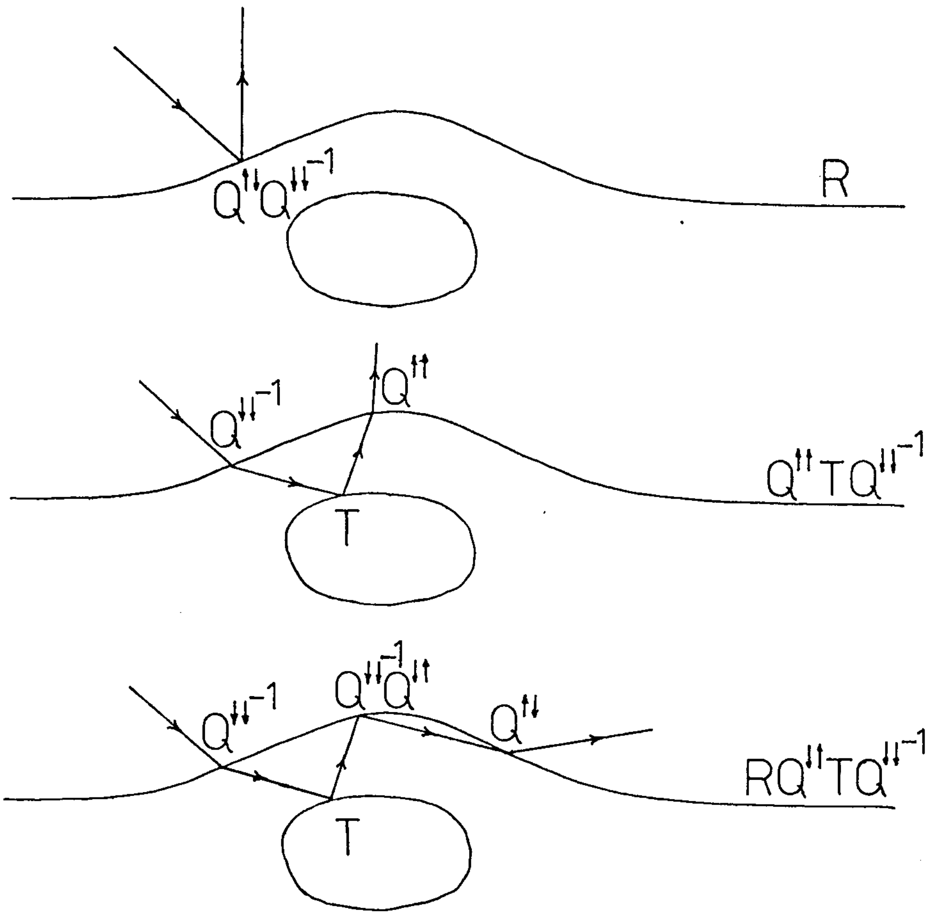

$Q^{\uparrow\downarrow}Q^{\downarrow\downarrow^{-1}}$
R
$Q^{\downarrow\downarrow^{-1}}$
$Q^{\uparrow\uparrow}$
$Q^{\uparrow\uparrow}TQ^{\downarrow\downarrow^{-1}}$
T
$Q^{\downarrow\downarrow^{-1}}$
$Q^{\downarrow\downarrow^{-1}}Q^{\downarrow\uparrow}$
$Q^{\uparrow\downarrow}$
$RQ^{\downarrow\uparrow}TQ^{\downarrow\downarrow^{-1}}$
T

Figure 4

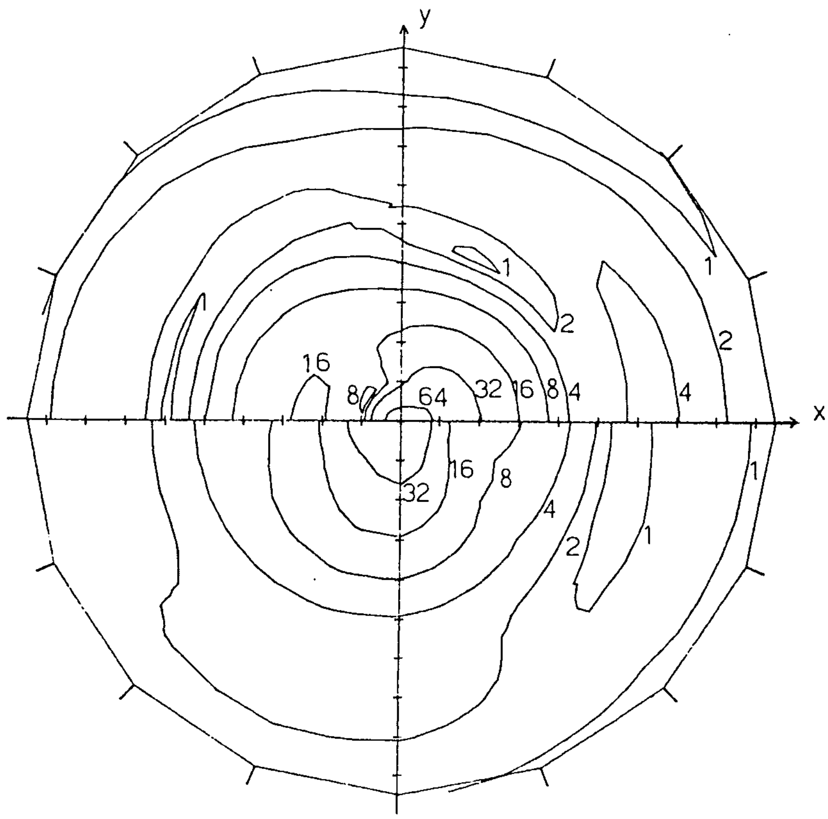



Figure 5

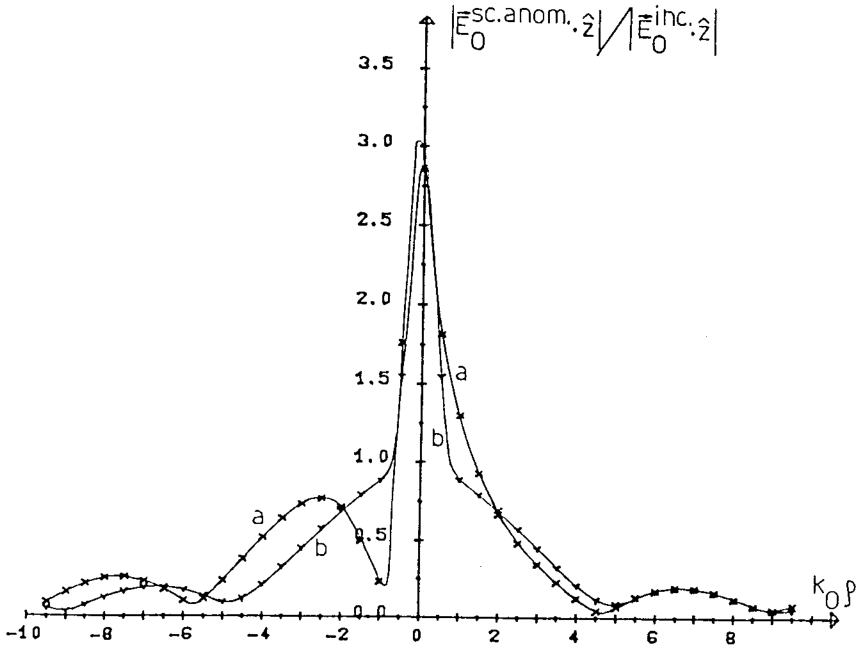


Figure 6

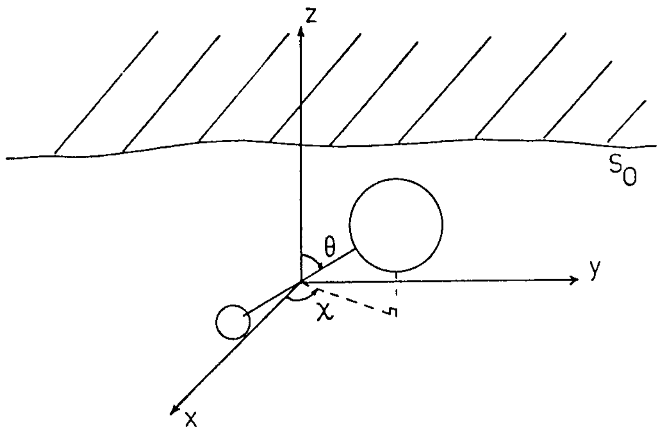



Figure 7

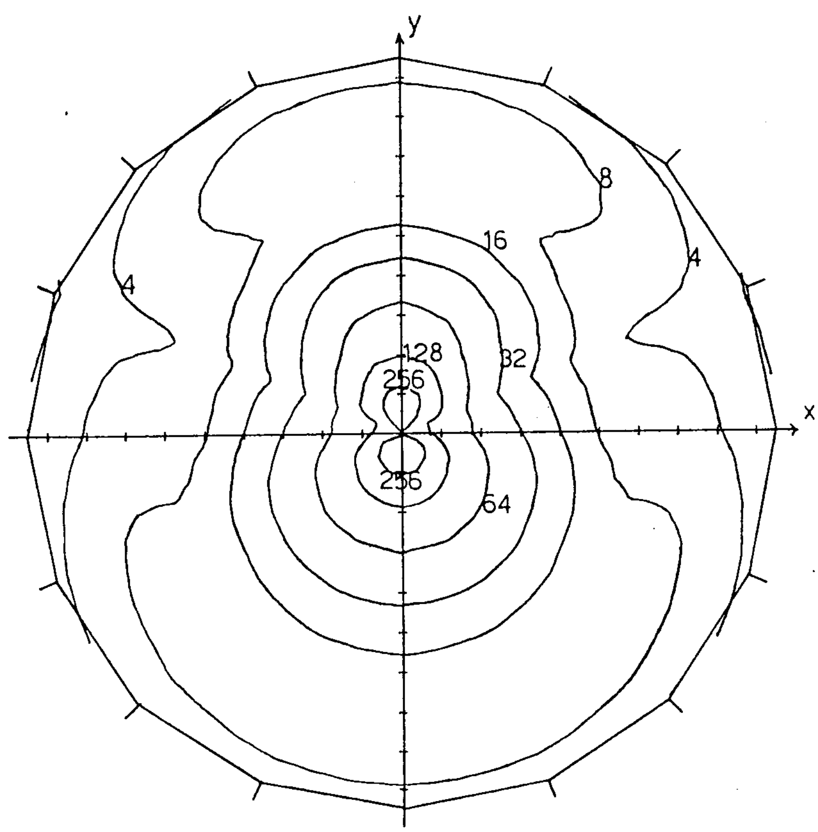


Figure 8

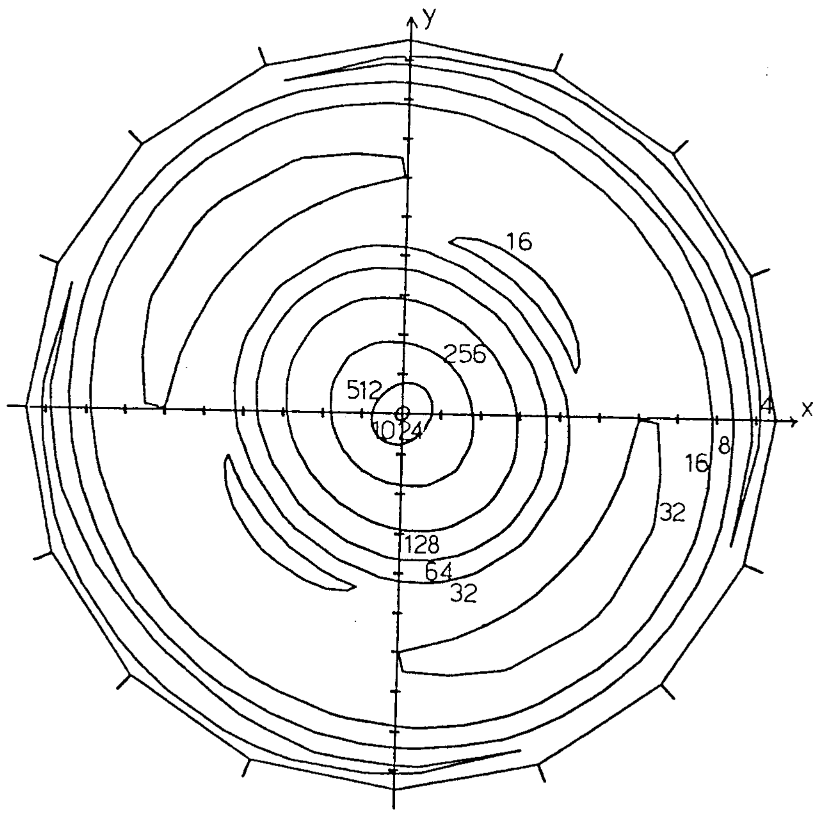



Figure 9

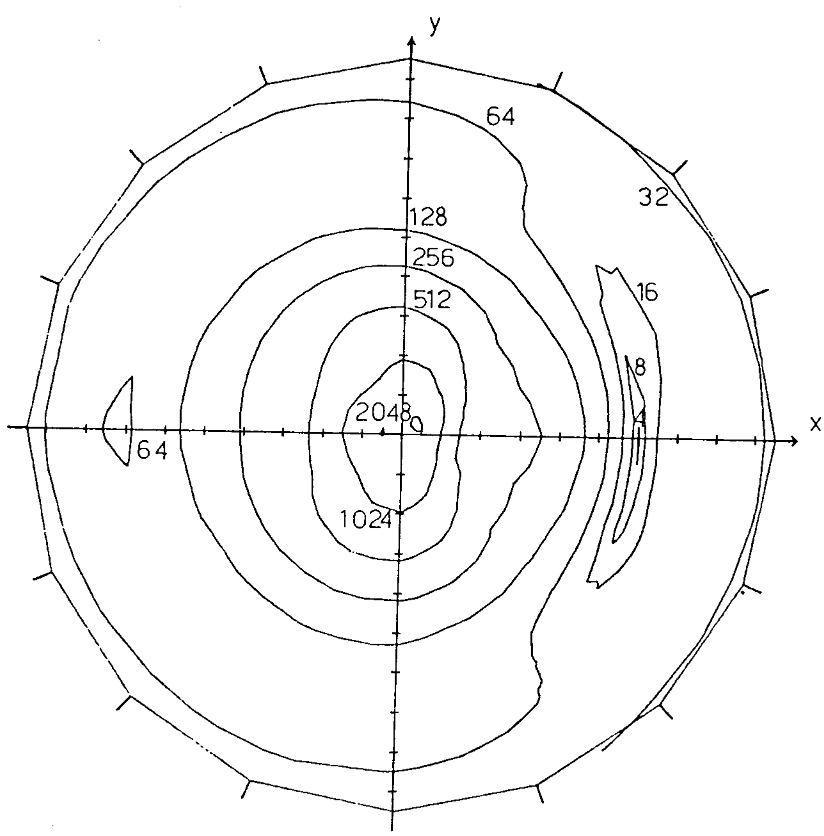
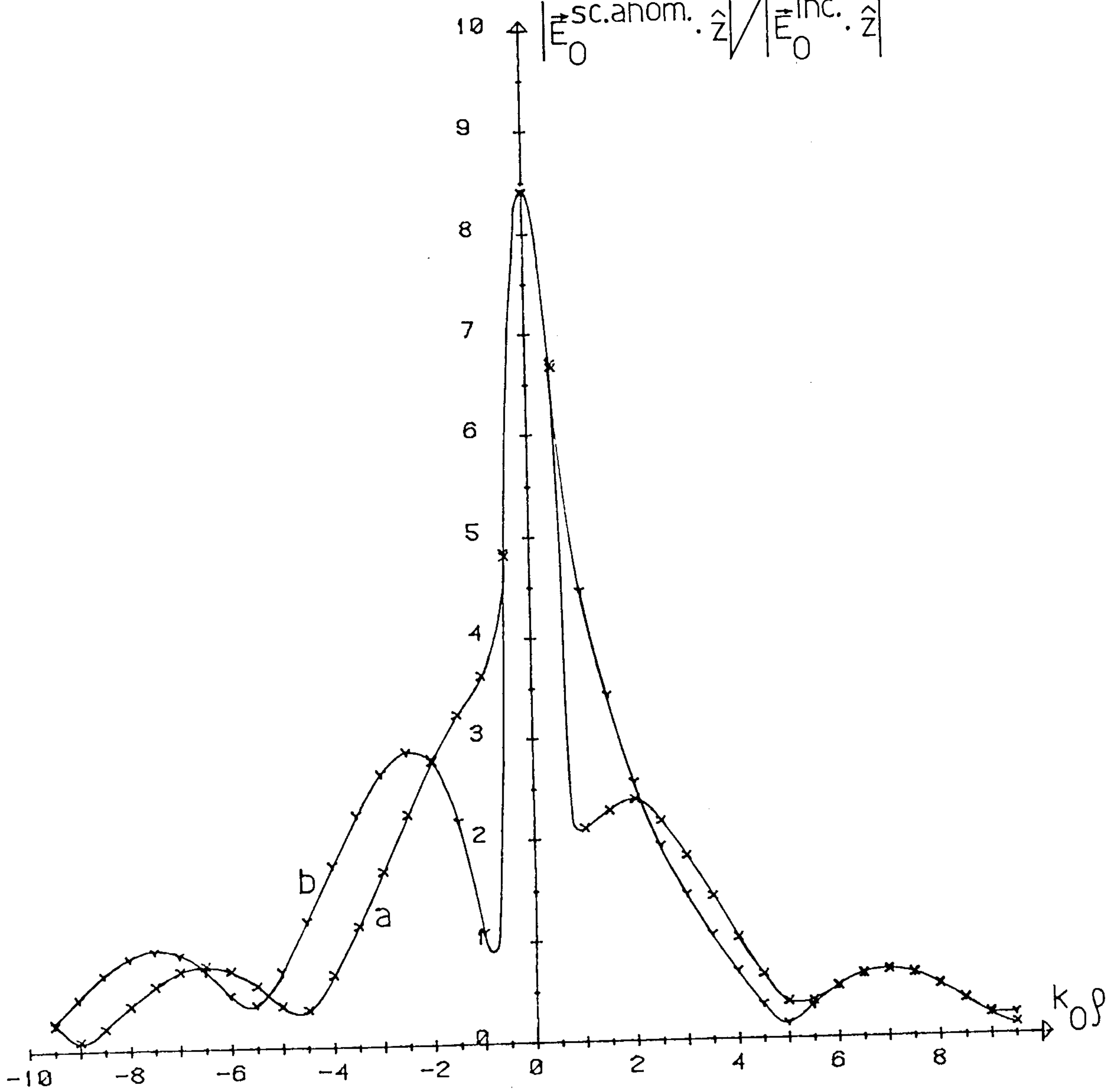
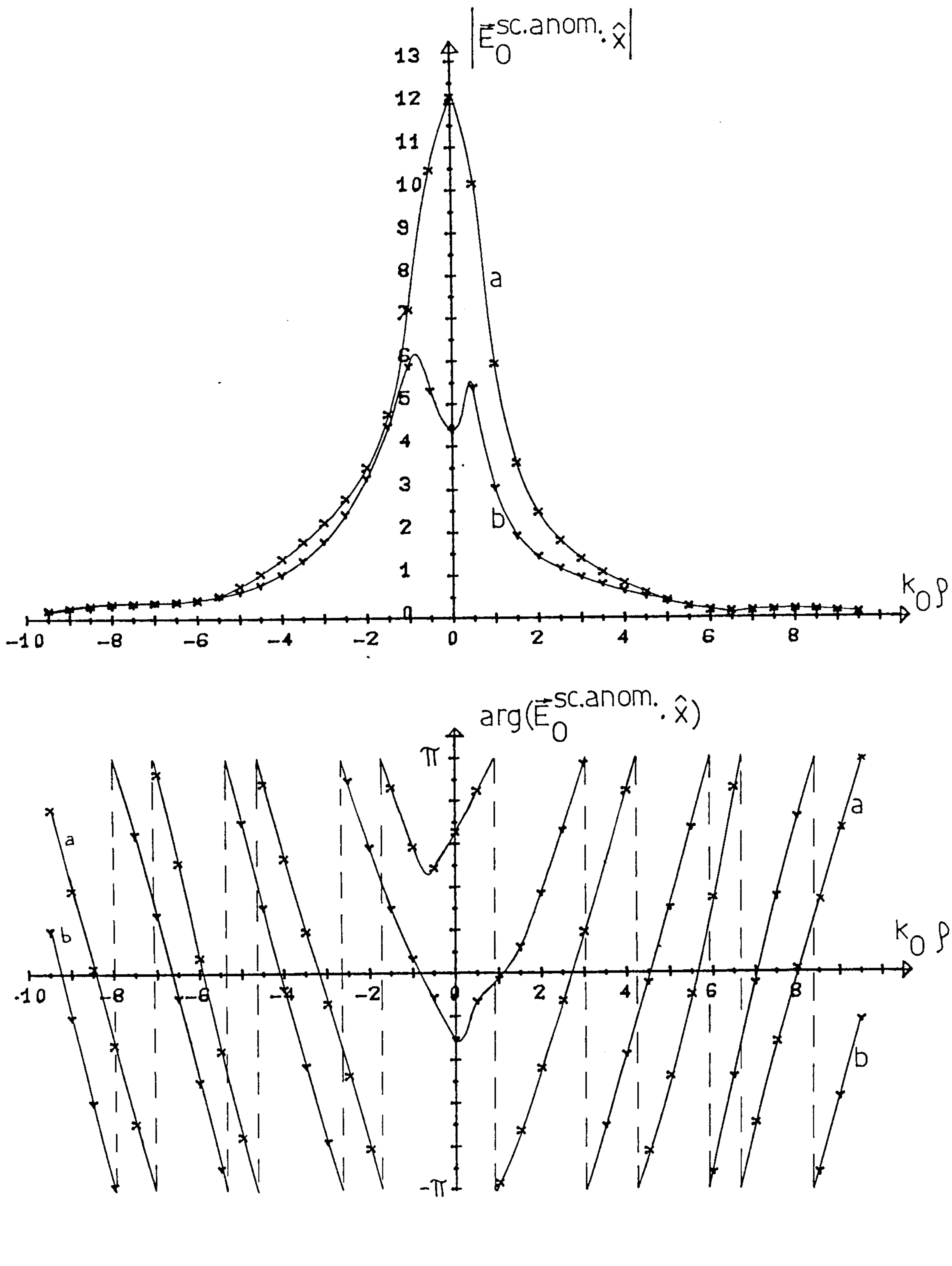


Figure 10



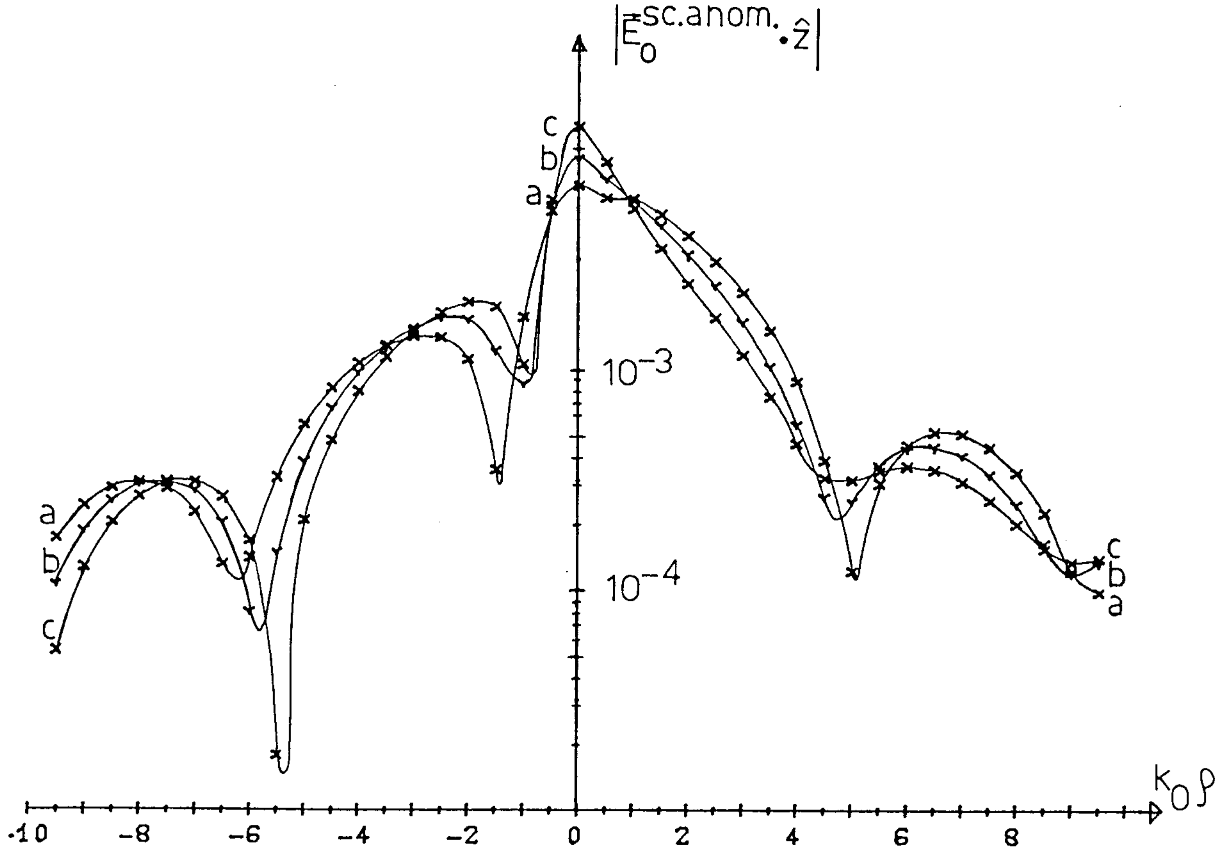
$\left|\vec{E}_0^{sc.anom.} \cdot \hat{z}\right| / \left|\vec{E}_0^{inc.} \cdot \hat{z}\right|$
10
9
8
7
6
5
4
3
2
1
0
b
a
-10
-8
-6
-4
-2
0
2
4
6
8
$k_0\rho$

Figure 11

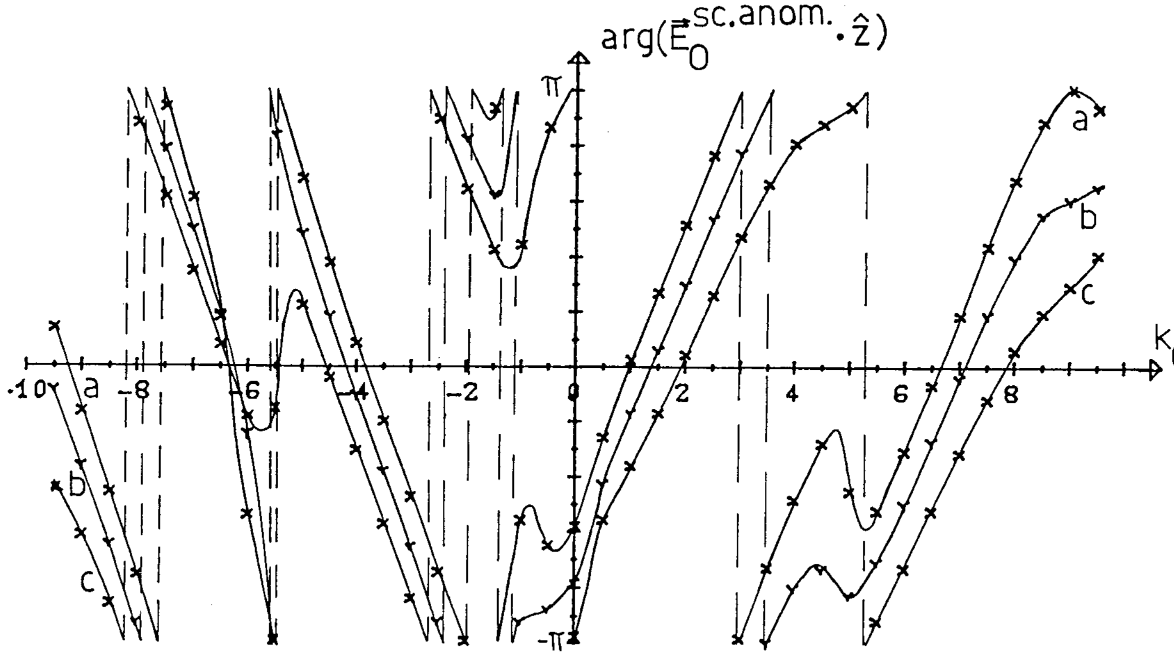


$|\vec{E}_0^{sc.anom.} \cdot \hat{x}|$
13
12
11
10
9
8
7
6
5
4
3
2
1
0
a
b
$k_0 \rho$
-10
-8
-6
-4
-2
0
2
4
6
8
$\arg(\vec{E}_0^{sc.anom.} \cdot \hat{x})$
π
-π
a
b
$k_0 \rho$

Figure 12



|E0 sc.anom. · ẑ|
10^-3
10^-4
a
b
c
kOρ
-10
-8
-6
-4
-2
0
2
4
6
8

arg(E0 sc.anom. · ẑ)
π
-π
a
b
c
kOρ
-10
-8
-6
-4
-2
0
2
4
6
8

Figure 13

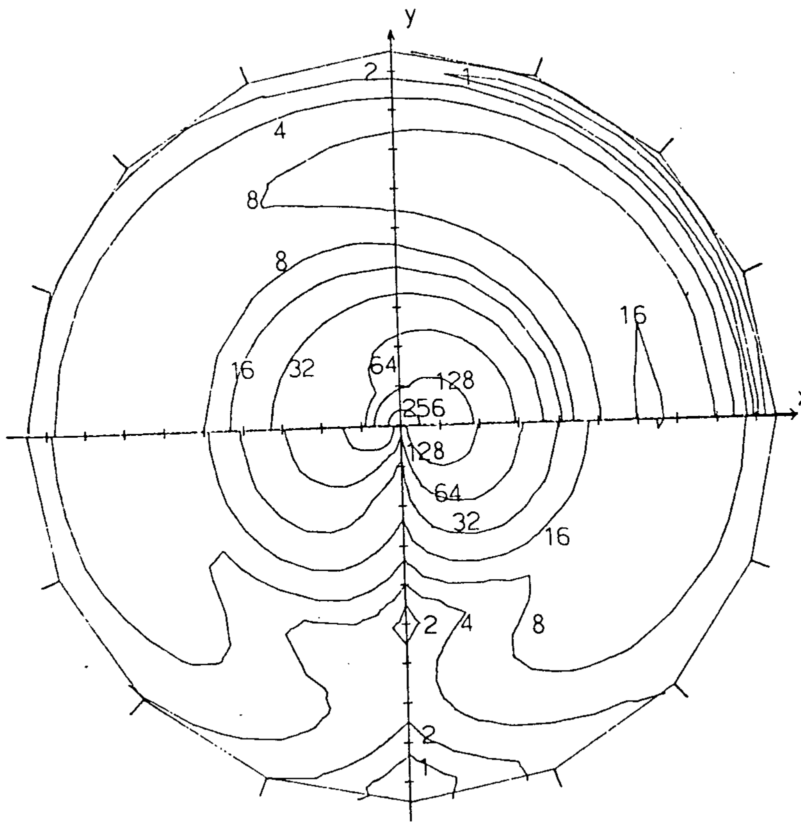
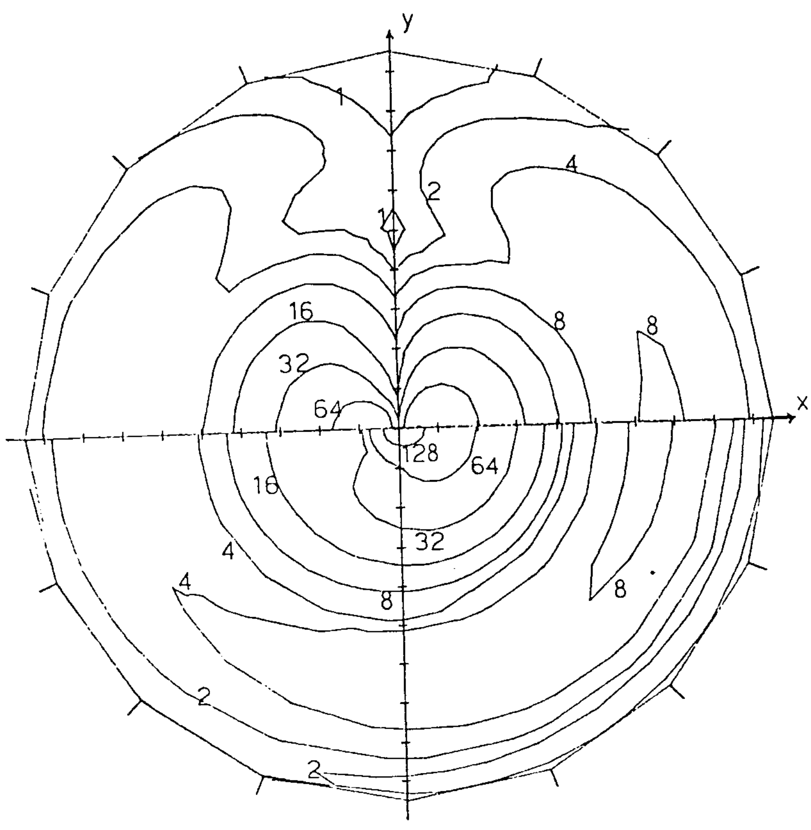



Figure 14



# Paper III

79-29

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# Electromagnetic scattering from a buried three-dimensional inhomogeneity in a lossy ground†

by

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## Abstract

The T matrix method (also called the "extended boundary condition method" or "null field approach") introduced by Waterman, has recently been generalized to interfaces of infinite extent (G. Kristensson and S. Ström, J. Acoust. Soc. Am. 64, 917-936 (1978) and G. Kristensson, "Electromagnetic Scattering from Buried Inhomogeneities - a General Threedimensional Formalism", Rep. 78-42, Inst. of Theoretical Physics, Göteborg (1978), to appear in J. Appl. Phys.). This paper extends the formalism to lossy materials. Here we explicitly assume that the ground and the inhomogeneity have losses, but the formalism also applies to a lossy medium above the ground with only minor changes. In developing the theory, we assume the source to be situated above the ground but it is otherwise arbitrary. A similar formalism can be constructed when the source position is located in the ground or in the inhomogeneity. The scattered field is calculated both above and below the ground. Above the ground the scattered field separates into two parts, which have direct physical interpretation; one field, here called the directly scattered field, which is the total scattered field when no buried obstacle is present, and a second field, the anomalous field, which reflects the presence of the inhomogeneity. We present some numerical computations of the field both above and below the ground for a flat earth and a buried perfectly conduction spheroid. The main theoretical developments are given in an appendix, where we study the transformation between plane and spherical vector waves for a complex wave number.

## I Introduction

Waterman [1] originally developed the T matrix method of scattering from a scatterer of finite extent (cf. also [2] for the elastic case). This approach (also called the "extended boundary condition method" or "null field approach") has been generalized to scattering from a buried inhomogeneity in the lossless case for both acoustic, electromagnetic and elastic waves [3] - [5]. The present paper will extend the formalism to the lossy case, which in many scattering problems is the situation of greatest interest. The scattering configuration will be truly three dimensional, and the formalism contains rather weak assumptions on the source distributions, the geometry of the scatterer etc. We will here explicitly develop the scattering formalism for electromagnetic waves with a source above the ground but the results are applicable with appropriate modifications to the acoustic and elastic cases and sources in other regions of interest.

The integral representation of the field is the basic element in the formalism. Suitable expansions of the Green's dyadic are central in the method and depending on the situation it is expanded in either plane or spherical vector waves. The transformation properties between these two elementary waves - plane and spherical - play an important part in the formalism and we analyse its properties in the lossy case in detail. The plane and spherical waves also enter in the expansions of the pertinent surface fields.

The inhomogeneity is completely described by its T matrix referring to spherical waves. The T matrix enters in the formalism as a building block in the construction of the solution, and in this context many results derived from scattering from obstacles of finite extent can be used. The interaction between the ground and the inhomogeneity is described by the solution of a matrix equation, where a power series expansion of the inverse of the matrix formally can be identified as multiple scattering contributions [3], [4]. We will in this paper study this matrix equation for the lossy case, and give explicit expressions in the flat interface case. Both the field above and below the ground are analysed and explicit expressions are given in the flat earth case.

The electromagnetic scattering from a source, say a dipole, in the presence of a homogeneous halfspace has been studied thoroughly and a long list of references are given in [4] addressing themselves to this problem, both with and without an inhomogeneity present in one of the halfspaces. Many of these treatments are purely numerical, while others pursue an analytic solution of the problem.

In section II the scattering problems to be considered are defined, the fundamental assumptions are introduced and the basic equations derived, while in section III the use of the formalism is illustrated in some numerical examples. In an appendix, we analyse the transformation properties between plane and spherical waves in the lossy case for both scalar and vector waves as needed for the theory developed in section II.

## II T matric formalism for a lossy ground

### Basic equations

In this section we will point out the essential differences and similarities between the lossless case [4], and the situation where losses are present in the ground and the inhomogeneity. Most equations are identical in structure to the corresponding lossless ones, and at some instances we will therefore be rather brief and we refer to [4] for more details.

Consider a scattering geometry as depicted in Fig. 1. The surface $S_o$ separates the halfspaces $V_o$ and $V_1$, which are assumed to be homogeneous except for a finite region $V_2$. This inhomogeneity is bounded by the surface $S_1$. Besides the implicit assumptions on the surfaces $S_o$ and $S_1$, namely that they fulfil the necessary regularity conditions for an application of the Green's theorem, we also assume $S_o$ to lie between the two parallel planes $z=z_>$ and $z=z_<$ (the z-axis is defined as perpendicular to these planes). We assume that the source and the inhomogeneity are located in separate halfspaces (the source location is marked with a P). A parallel formulation can be made when both source and inhomogeneity occupy the same halfspace [3] or the source is inside the obstacle. Furthermore we will in this context permit the different regions to have losses. In many practical applications one encounters a situation where the halfspace $V_o$ can be assumed to be lossless, and this is the explicit case we will consider here. The introduction of losses also in $V_o$ is straightforward and the details are left to the reader.

In each volume the electric field $\vec{E}_i$ satisfies (we assume the time factor $e^{-i\omega t}$ throughout this paper)

$$\nabla\times\nabla\times\vec{E}_i(\vec{r})-k_i^2\vec{E}_i(\vec{r})=\vec{0} \qquad \vec{r} \text{ in } V_i \;;\; i=0,1,2 \tag{1}$$

Here $k_i^2=\omega^2\mu_o\varepsilon_o\varepsilon_{i_r}\mu_{i_r}+i\omega\sigma_i\mu_{i_r}\mu_o$ i=0,1,2, where $\varepsilon_o$ and $\mu_o$ are the dielectric constant and permeability of free space and $\sigma_i$ the conductivity in $V_1$. As discussed above we will in this paper only consider the case $\sigma_o=0$. All equations in this paper will explicitly be written down for the electric field $\vec{E}_i$. The same analysis holds for the $\vec{H}_i$ field, and what is discussed below about the $\vec{E}_i$ field can equally well be applied to the magnetic field $\vec{H}_i$ if we interpret the source distributions as the corresponding magnetic ones, e.g. an electric dipole becomes a magnetic dipole, and the necessary substitutions are made, see Eq. (2) below.

The boundary conditions are continuity in the tangential magnetic and electric fields on the surface $S_o$ and $S_1$ (for notations see Fig. 1), i.e.

$$\begin{cases} \hat{n}_i\times\vec{E}_i^+(\vec{r}')=\hat{n}_i\times\vec{E}_{i+1}^-(\vec{r}') & \vec{r}'\in S_i \;;\; i=0,1 \\ \hat{n}_i\times[\nabla'\times\vec{E}_i^+(\vec{r}')]=c_i\hat{n}_i\times[\nabla'\times\vec{E}_{i+1}^-(\vec{r}')] & \\ c_i=\mu_{i_r}/\mu_{i+1_r} \qquad i=0,1 & \end{cases} \tag{2}$$

(If we replace $\vec{E}$ with $\vec{H}$, let $c_i=(\frac{k_i}{k_{i+1}})^2\,\mu_{i+1_r}\,\mu_{i_r}^{-1}$.)

The starting point in the T matrix formalism [1] is the following integral representation of the field $\vec{E}$ in terms of a surface integral over the tangential components of $\vec{E}$ and $\vec{H}$ on the bounding surface S.

$$\left.\begin{matrix}\vec{E}(\vec{r})\\ \vec{0}\end{matrix}\right\} = \vec{E}^{inc.}(\vec{r}) + \nabla\times\iint_S \hat{n}\times\vec{E}^{+}(\vec{r}')\,G(\vec{r},\vec{r}';k)\,dS' + $$

$$+ k^{-2}\nabla\times\left\{\nabla\times\iint_S \hat{n}\times(\nabla'\times\vec{E}^{+}(\vec{r}'))\,G(\vec{r},\vec{r}';k)\,dS'\right\} \qquad (3)$$

$$\left\{\begin{matrix}\vec{r} \text{ outside } S\\ \vec{r} \text{ inside } S\end{matrix}\right.$$

S is a bounded surface and $\vec{E}^{+}$ and $\nabla\times\vec{E}^{+}$ are the field values on the outside of S ($\hat{n}$ is directed outwards on S). The Green's function $G(\vec{r},\vec{r}';k)$ satisfies the Helmholtz' equation with a delta function source term.

$$(\nabla^2 + k^2)\,G(\vec{r},\vec{r}';k) = -\delta(\vec{r}-\vec{r}') \qquad (4)$$

The requirement of an outgoing wave at infinity gives us the solution to Eq. (4)

$$G(\vec{r},\vec{r}';k) = \exp(ik|\vec{r}-\vec{r}'|)/4\pi|\vec{r}-\vec{r}'| \qquad (5)$$

It should be noted here that k can be a complex number, which will be the case when we apply Eq. (3) to volumes with losses.

As discussed in [4] both plane and spherical vector waves are introduced as well as the transformation between these. In these quantities we must justify the analytic continuation of k-values into the complex plane. The definition of spherical waves are found in [4]. The extension to complex k values in these definitions introduces no problems; the complex quantities just appear in the radial dependence argument kr but leave the spherical vector harmonics $\vec{A}_n(\hat{r})$ unaffected. The transformation between spherical and plane vector waves are discussed in detail in the appendix and we have, (see Eq. (A.22)):

$$\vec{\Psi}_n(k\vec{r}) = \frac{i}{2\pi}\int_0^{2\pi} d\beta \int_{C_\pm} \vec{B}_n(\hat{k})\, e^{i\vec{k}\cdot\vec{r}} \sin\alpha\, d\alpha \qquad z \gtrless 0 \tag{6}$$

The complex contours $C_\pm$ are depicted in Fig. 2 (see the appendix for a definition of the contours). The contour $C_+$ - the upgoing waves - is used when $z>0$ and $C_-$ - the downgoing waves - when $z<0$. It should be noted that along the $C_\pm$ contours $k \sin\alpha$ is real and that Eq. (6) essentially is a two-dimensional Fourier integral in $k_x$, $k_y$ rewritten in the spherical angles $(\alpha,\beta)$ of $\vec{k}$, i.e. $\vec{k}=k(\sin\alpha \cos\beta, \sin\alpha \sin\beta, \cos\alpha)$. The definition of $\vec{B}_n(\vec{k})$ is found in [4]:

$$\vec{B}_{\tau n}(\hat{k}) \equiv i^{\tau-2-l}\, \vec{A}_{\tau n}(\hat{k}) \qquad \tau = 1,2 \tag{7}$$

Note that we here, and when convenient also below, abbreviate the indices as follows $n \equiv \tau n \equiv \tau\sigma ml$. As a special case ($l=0$) of Eq. (A.15) we also get the plane wave expansion of the Green's function in Eq. (5)

$$G(\vec{r},\vec{r}';k) = \frac{ik}{8\pi^2}\int_0^{2\pi} d\beta \int_{C_\pm} e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \sin\alpha\, d\alpha \qquad z \gtrless z'$$

The Green's dyadic $\overleftrightarrow{I}G(\vec{r},\vec{r}';k)$ can thus be written [4]:

$$\overleftrightarrow{I}G(\vec{r},\vec{r}';k) = \sum_{j=1}^{3} \frac{ik}{8\pi^2}\int_0^{2\pi} d\beta \int_{C_\pm} \hat{a}_j \hat{a}_j\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \sin\alpha\, d\alpha \qquad z \gtrless z' \tag{8}$$

Here $\overleftrightarrow{I}$ is the unit dyadic and $\hat{a}_j$, j=1,3 are the spherical unit vectors of $\hat{k}=\vec{k}/k$ and with the following convention:

$$\hat{a}_1 = \hat{\alpha}$$
$$\hat{a}_2 = \hat{\beta}$$
$$\hat{a}_3 = \hat{k}$$

The separation of the Green's dyadic in spherical vector waves is found in e.g. [6]:

$$\overleftrightarrow{I}G(\vec{r},\vec{r}';k) = ik\sum_{\substack{n \\ \tau=1,2}} \text{Re}\vec{\psi}_n(k\vec{r}_<)\,\vec{\psi}_n(k\vec{r}_>) + \overleftrightarrow{I}_{irr.} \tag{9}$$

The argument $\vec{r}_>$ and $\vec{r}_<$ are chosen to be $\vec{r}$ or $\vec{r}'$ according to $|\vec{r}_<|=\min(r,r')$, $|\vec{r}_>|=\max(r,r')$ and the dyadic $\overleftrightarrow{I}_{irr.}$ is an irrotational dyadic. Eq. (9) holds for complex values of k, at least for values of k in our domain of interest, i.e. $\arg k \in [0,\pi/4)$ [7]. In analogy with Eq. (6) we also need the transformation between the regular spherical vector waves and the plane waves. This is found in e.g. [8]:

$$\text{Re}\vec{\psi}_n(k\vec{r}) = \frac{i}{4\pi}\int_0^{2\pi} d\beta \int_0^{\pi} \vec{B}_n(\hat{k})\,e^{i\vec{k}\cdot\vec{r}} \sin\alpha \, d\alpha \tag{10}$$

The extension to complex values of k in this integral over a finite interval causes no problems and the formula holds in the common domain of analyticity of the two sides.

We now apply the integral representation Eq. (3) to a surface S consisting of a finite part of $S_o$ and lower half sphere. Assume that the fields encountered in the integral over the lower half-sphere satisfy the appropriate radiation condition. As the radius of the half sphere approaches infinity this integral then vanishes. By introducing the plane wave expansion of the Green's dyadic, see Eq. (8), in the surface integral over $S_o$ we obtain the following plane wave expansions of the scattered and incoming field, respectively (which are formally the same as in the loss-less case):

$$\vec{E}_0^{sc.}(\vec{r}) = \int_0^{2\pi} d\beta_0 \int_{C_+} \vec{f}(\vec{k}_0)\, e^{i\vec{k}_0\cdot\vec{r}} \sin\alpha_0\, d\alpha_0 \qquad z > z_> \tag{11}$$

$$\vec{E}_0^{inc.}(\vec{r}) = \int_0^{2\pi} d\beta_0 \int_{C_-} \vec{a}(\vec{k}_0)\, e^{i\vec{k}_0\cdot\vec{r}} \sin\alpha_0\, d\alpha_0 \qquad z < z_< \tag{12}$$

$$f_j(\vec{k}_0) \equiv \frac{ik_0}{8\pi^2} \iint_{S_0} \left\{ (\hat{n}_0\times\vec{E}_0^+)\cdot(\hat{a}_j\times i\vec{k}_0) + [\hat{n}_0\times(\nabla'\times\vec{E}_0^+)]\cdot\hat{a}_j \right\} e^{-i\vec{k}_0\cdot\vec{r}'} dS' \qquad \vec{k}_0 \in C_+ \tag{13}$$

$$a_j(\vec{k}_0) \equiv -\frac{ik_0}{8\pi^2} \iint_{S_0} \left\{ (\hat{n}_0\times\vec{E}_0^+)\cdot(\hat{a}_j\times i\vec{k}_0) + [\hat{n}_0\times(\nabla'\times\vec{E}_0^+)]\cdot\hat{a}_j \right\} e^{-i\vec{k}_0\cdot\vec{r}'} dS' \qquad \vec{k}_0 \in C_- \tag{14}$$

where $\vec{f}(\vec{k}_o) = \sum_{j=1}^{2} f_j(\vec{k}_o)\hat{a}_j$ and analogously for $a_j(\vec{k}_o)$.

The elimination of the surface fields will be done by another application of the integral representation Eq. (3). This time S consists of $S_1$ and a finite part of $S_o$ and an upper half sphere, such that outside S is inside $V_1$. Let the radius of the half sphere go to infinity and assume as before the appropriate radiation conditions. We get:

$$\left.\begin{array}{l}\vec{E}_1(\vec{r})\\ \vec{0}\end{array}\right\} = -\nabla\times\iint_{S_0}\hat{n}_0\times\vec{E}_1^-(\vec{r}')G(\vec{r},\vec{r}';k_1)\,ds' -$$

$$-k_1^{-2}\nabla\times\left\{\nabla\times\iint_{S_0}\hat{n}_0\times(\nabla'\times\vec{E}_1^-(\vec{r}'))G(\vec{r},\vec{r}';k_1)\,ds'\right\} +$$

$$+\nabla\times\iint_{S_1}\hat{n}_1\times\vec{E}_1^+(\vec{r}')\,G(\vec{r},\vec{r}';k_1)\,ds' +$$

$$+k_1^{-2}\nabla\times\left\{\nabla\times\iint_{S_1}\hat{n}_1\times(\nabla'\times\vec{E}_1^+(\vec{r}'))G(\vec{r},\vec{r}';k_1)\,ds'\right\} \quad \left\{\begin{array}{l}\vec{r}\in V_1\\ \vec{r}\notin V_1\end{array}\right. \tag{15}$$

We will primarily use the equation above when $\vec{r}$ is outside $V_1$. First we introduce suitable expansions of the surface fields $\hat{n}_0\times\vec{E}_1^-$ and $\hat{n}_1\times\vec{E}_2^-$. In [4] the following expansions were introduced:

$$\hat{n}_0\times\vec{E}_1^-(\vec{r}') = \int_0^{2\pi}d\beta_1\;\hat{n}_0\times\left\{\int_{C_-}\vec{\alpha}(\vec{k}_1)+\int_{C_+}\vec{\beta}(\vec{k}_1)\right\}e^{i\vec{k}_1\cdot\vec{r}'}\sin\alpha_1\,d\alpha_1 \tag{16}$$

$$\hat{n}_1\times\vec{E}_2^-(\vec{r}') = \sum_n \alpha_n\,\hat{n}_1\times\mathrm{Re}\,\vec{\psi}_n(k_2\vec{r}') \tag{17}$$

The expansion on the surface $S_0$ is an expansion in plane waves, both up- and downgoing while on $S_1$ we use an expansion in regular spherical vector waves. Furthermore, the determination of the coefficients in the related expansions of the relevant derivatives of these fields can be discussed in a way which is directly analogous to the lossless case. (However, we emphasize that considerable work remains to be done in order to determine the class of surfaces $S_0$ and $S_1$ for which the required relations between the expansion coefficients is rigorously valid [1], [3], [4].) The completeness of these expansions in the lossless case is found in [4] and in the lossy case the derivation is formally analogous.

The application of the integral representation Eq. (15) inside $V_2$ and above $S_o$ is formally the same as in the lossless case and for details in this matter we refer to Ref. [4]. The boundary conditions, Eq. (2), the surface field expansions, Eq. (16)-(17), and the expansion of the Green's dyadic in spherical vector waves, Eq. (9), are applied and we get:

$$\int_0^{2\pi} d\beta_1 \int_{C_+} \vec{\beta}(\vec{k}_1)\, e^{i\vec{k}_1\cdot\vec{r}} \sin\alpha_1\, d\alpha_1 = -i \sum_{nn'} \vec{\Psi}_n(k_1\vec{r})\, Q_{nn'}(Re,Re)\, \alpha_{n'} \qquad (18)$$

$$\int_0^{2\pi} d\beta_1 \int_{C_-} \vec{\alpha}(\vec{k}_1)\, e^{i\vec{k}_1\cdot\vec{r}} \sin\alpha_1\, d\alpha_1 = i \sum_{nn'} Re\vec{\Psi}_n(k_1\vec{r})\, Q_{nn'}(Out,Re)\, \alpha_{n'} \qquad (19)$$

The first equation holds for all $\vec{r}$ above $S_o$ and outside the circumscribing sphere of $S_1$, while the Eq. (19) holds for all $\vec{r}$ inside the inscribed sphere of $S_1$. The derivation of these two equations in the lossless case relies on a "limiting absorption principle", i.e. the wave number has a small imaginary part, which eventually goes to zero. However, in the situation treated here, when losses are present no such limiting procedure is necessary. The derivation of the Eqs. (18)-(19) are otherwise analogous to the lossless case. Furthermore, we have introduced the $Q_{nn}'$-matrices of the scatterer $S_1$,

$$Q_{nn'}(Out,Re) \equiv k_1 \iint_{S_1} \hat{n}_1\cdot\Big\{ (\nabla'\times\vec{\Psi}_n(k_1\vec{r}'))\times Re\vec{\Psi}_{n'}(k_2\vec{r}') + c_1\vec{\Psi}_n(k_1\vec{r}')\times(\nabla'\times Re\vec{\Psi}_{n'}(k_1\vec{r}'))\Big\}\, dS' \qquad (20)$$

and $Q_{nn'}(Re,Re)$ is analogously defined but with regular spherical vector waves in all places.

The next step will be to eliminate the $\vec{r}$ dependence in the two equations above. In the first equation we make use of the transformation between plane and spherical vector waves, see Eq. (6). As was pointed out above, the integral over $\alpha, \beta$ on both sides are a two-dimensional Fourier integral in $k_x, k_y$ and by use of the inverse transform on a plane z=constant we obtain

$$\vec{\beta}(\vec{k}_1) = \frac{1}{2\pi} \sum_{nn'} \vec{B}_n(\hat{k}_1)\, Q_{nn'}(Re, Re)\, \alpha_{n'} \qquad \vec{k}_1 \in C_+ \tag{21}$$

In Eq. (19) we first make a scalar multiplication on both sides with $\vec{A}_n(\hat{r})$ followed by an integration over the unit sphere. We get by use of Eq. (10)

$$\int_0^{2\pi} d\beta_1 \int_{C_-} \vec{\alpha}(\vec{k}_1) \cdot \vec{B}_n^{\dagger}(\hat{k}_1) \sin\alpha_1 \, d\alpha_1 = -\frac{1}{4\pi} \sum_{n'} Q_{nn'}(Out, Re)\, \alpha_{n'} \tag{22}$$

where $\vec{B}_n^{\dagger}(\hat{k})$ is identical to $\vec{B}_n(\hat{k})$, but with $(-i)^{\ell+2-\tau}$ exchanged with $i^{\ell+2-\tau}$.

The derivation of Eq. (21) relies on the inverse two-dimensional Fourier transform and it should here be noted the importance of expanding the transformation, see Eq. (6), and the surface field Eq. (16) in terms of integrals with the same contours $C_\pm$. For both this transformation and the surface fields we have the possibility of choosing different integration contours (cf. the appendix for a discussion of the more detailed choice of contours). However, when applying the inverse transform we then have to pay attention to the analytic properties of the integrands, since when the contours differ from $C_\pm$, $k_x$ and $k_y$ are not real everywhere. In this paper we will always use the $C_\pm$ contours, i.e. the contours where both $k_x$ and $k_y$ are real, and in this way we can avoid any discussion of the analytic properties.

In [4] the field $\vec{E}_0^{sc.}$ was calculated. In the case where losses are present the derivation of this field is formally identical, and will therefore only be outlined here without any details. Here we will focus on the field in $V_1$, which in e.g. many prospecting applications is of great importance.

The field in $V_1$ is given by Eq. (15). The elimination of the surface fields in this equation will be quite analogous to the computation of the field $\vec{E}_0^{sc.}$. It is also obvious that the derivation below are valid also in the lossless case. The most straightforward region to compute the field $\vec{E}_1$ in is outside the circumscribing sphere of $S_1$, i.e. when $r > \max_{\vec{r}' \in S_1} |r'|$. In this special case we can make use of the same expansion of the Green's dyadic over the whole surface $S_1$, i.e. in spherical waves. However, we note that calculations of the scattered field in $V_1$ inside the circumscribing sphere of $S_1$ can also be made, with the appropriate modifications (for "near-field" calculations within the T-matrix approach see e.g. Bringi and Seliga [9]). The surface field expansions, Eq. (16)-(17), are inserted and we get (the derivation of this equation is analogous to the Eq. (19), and again, as pointed out above, we do not rely on any "limiting absorption principle" in this lossy case):

$$\vec{E}_1(\vec{r}) = \int_0^{2\pi} d\beta_1 \int_{C_-} \vec{\alpha}(\vec{k}_1)\, e^{i\vec{k}_1\cdot\vec{r}} \sin\alpha_1\, d\alpha_1 - i \sum_{nn'} \vec{\psi}_n(k_1\vec{r})\, Q_{nn'}(Re, Re)\, \alpha_{n'} \tag{23}$$

The prescribed incoming field amplitude $\vec{a}(\vec{k}_0)$ in Eq. (14) is now used to eliminate the surface field expansion amplitudes $\vec{\alpha}(\vec{k}_1)$, $\vec{\beta}(\vec{k}_1)$ and $\alpha_n'$. We insert Eq. (2) and (16) in Eq. (14) and get:

$$a_j(\vec{k}_0) = i\sum_{j'=1}^{2}\int_0^{2\pi} d\beta_1 \left\{ \int_{C_-} \alpha_{j'}(\vec{k}_1) + \int_{C_+} \beta_{j'}(\vec{k}_1) \right\} Q_{jj'}(\vec{k}_0,\vec{k}_1) \sin\alpha_1 \, d\alpha_1 \qquad \vec{k}_0 \in C_- \tag{24}$$

Here we have introduced the generalization of the $Q_{nn'}$-matrix Eq. (20) to the infinite surface $S_0$:

$$Q_{jj'}(\vec{k}_0,\vec{k}_1) \equiv \frac{k_0}{8\pi^2} \iint_{S_0} \left\{ (\hat{n}\times\hat{a}_{j'}) \cdot (i\vec{k}_0 \times \hat{a}_j) - c_0[\hat{n}_0 \times (i\vec{k}_1 \times \hat{a}_{j'})] \cdot \hat{a}_j \right\} e^{i(\vec{k}_0-\vec{k}_1)\cdot\vec{r}'} dS' \tag{25}$$

The formal solution of Eq. (21), (22) and (24) will be found by an elimination of the $\alpha_{n'}$-coefficients in Eq. (21) and (22)

$$\vec{\beta}(\vec{k}_1) = 2\sum_{nn'} \vec{B}_n(\hat{k}_1) T_{nn'} c_{n'} \tag{26}$$

where

$$c_n \equiv \int_0^{2\pi} d\beta_1 \int_{C_-} \vec{\alpha}(\vec{k}_1) \cdot \vec{B}_n^{\dagger}(\hat{k}_1) \sin\alpha_1 \, d\alpha_1 \tag{27}$$

$$T_{nn'} \equiv -\sum_{n''} Q_{nn''}(Re,Re)\, Q^{-1}_{n''n'}(Out,Re) \tag{28}$$

The $c_n$-quantity is the spherical projection of the plane wave amplitude $\vec{\alpha}(\vec{k}_1)$ later determined by a matrix equation. The T matrix characterize the scatterer $S_1$, i.e. it contains its shape, boundary conditions etc. If we encounter a situation where we have several buried scatterers or inhomogeneous ones the formal structure of the equation is the same, but with the relevant T-matrix inserted.

We proceed by formally inverting Eq. (24) (this can be done algebraically when $S_0$ is a plane, see below) and we get with Eq. (26)

$$\alpha_j(\vec{k}_1) = -i\sum_{j'}\int_0^{2\pi} d\beta_0 \int_{C_-} a_{j'}(\vec{k}_0)\, Q^{-1}_{jj'}(\vec{k}_1,\vec{k}_0)\,\sin\alpha_0\, d\alpha_0 +$$

$$+ 2\sum_{nn'j'}\int_0^{2\pi} d\beta_1' \int_{C_+} B_{nj'}(\hat{k}_1')\, T_{nn'}\, c_{n'}\, \mathcal{R}_{jj'}(\vec{k}_1,\vec{k}_1')\,\sin\alpha_1'\, d\alpha_1' \qquad \vec{k}_1\in C_- \tag{29}$$

Here $B_{nj}\equiv\vec{B}_n\cdot\hat{a}_j$ and similarly $B^\dagger_{nj}\equiv\vec{B}^\dagger_n\cdot\hat{a}_j$ is defined. The reflection coefficient $\mathcal{R}_{jj'}(\vec{k}_1,\vec{k}_1')$ for the surface $S_0$ from below is analogous to the T-matrix for the scatterer $S_1$ and is defined as:

$$\mathcal{R}_{jj'}(\vec{k}_1,\vec{k}_1') \equiv -\sum_{j''}\int_0^{2\pi} d\beta_0 \int_{C_-} Q^{-1}_{jj''}(\vec{k}_1,\vec{k}_0)\, Q_{j''j'}(\vec{k}_0,\vec{k}_1')\,\sin\alpha_0\, d\alpha_0 \qquad \vec{k}_1'\in C_+,\ \vec{k}_1\in C_- \tag{30}$$

We construct the basic matrix equation for determining the coefficients $c_n$ by multiplying Eq. (29) with $B^\dagger_{nj}(\hat{k}_1)$ and sum over j and integrate over $\beta_1$ and $\alpha_1$ over a $C_-$ contour. We get:

$$c_n + \sum_{n'} A_{nn'}c_{n'} = d_n \tag{31}$$

where

$$A_{nn'} \equiv -2\sum_{n''jj'}\int_0^{2\pi} d\beta_1 \int_{C_-}\sin\alpha_1\, d\alpha_1 \int_0^{2\pi} d\beta_1' \int_{C_+}\sin\alpha_1'\, d\alpha_1' \times$$

$$\times B_{n''j'}(\hat{k}_1')\, T_{n''n'}\, \mathcal{R}_{jj'}(\vec{k}_1,\vec{k}_1')\, B^\dagger_{nj}(\hat{k}_1) \tag{32}$$

$$d_n \equiv -i\sum_{jj'}\int_0^{2\pi} d\beta_1 \int_{C_-}\sin\alpha_1\, d\alpha_1 \int_0^{2\pi} d\beta_0 \int_{C_-}\sin\alpha_0\, d\alpha_0\, a_{j'}(\vec{k}_0)\, Q^{-1}_{jj'}(\vec{k}_1,\vec{k}_0)\, B^\dagger_{nj}(\hat{k}_1) \tag{33}$$

Thus we obtain an equation which is formally the same as in the lossless case. Again the iteration of Eq. (31) reflect the multiple scattering phenomenon between the surface $S_o$ and the scatterer $S_1$, and we refer to [4] for more details on this subject.

The final expression for the field $\vec{E}_1$ outside the circumscribing sphere of $S_1$, Eq. (23), can eventually be written in terms of the $c_n$-vector by introducing the Eqs. (29), (22), (27) and (28)

$$\vec{E}_1(\vec{r}) = \sum_{jj'} \int_0^{2\pi} d\beta_1 \int_{C_-} \hat{a}_j e^{i\vec{k}_1\cdot\vec{r}} \Big\{ -i \int_0^{2\pi} d\beta_0 \int_{C_-} a_{j'}(\vec{k}_0)\, Q^{-1}_{jj'}(\vec{k}_0,\vec{k}_1) \sin\alpha_0\, d\alpha_0 +$$
$$+ 2\sum_{nn'} \int_0^{2\pi} d\beta_1' \int_{C_+} B_{nj'}(\hat{k}_1')\, T_{nn'}\, c_{n'}\, R_{jj'}(\vec{k}_1,\vec{k}_1') \sin\alpha_1'\, d\alpha_1' \Big\} \sin\alpha_1\, d\alpha_1 -$$
$$- 4\pi i \sum_{nn'} \vec{\Psi}_n(k_1\vec{r})\, T_{nn'}\, c_{n'} =$$
$$= \vec{E}_1^{\,dir.}(\vec{r}) + \sum_{nn'} \vec{\mathfrak{F}}_n(\vec{r})\, T_{nn'}\, c_{n'} - 4\pi i \sum_{nn'} \vec{\Psi}_n(k_1\vec{r})\, T_{nn'}\, c_{n'} \tag{34}$$

The total field $\vec{E}_1$ has been divided into three terms with the following definitions:

$$\vec{E}_1^{\,dir.}(\vec{r}) \equiv -i \sum_{jj'} \int_0^{2\pi} d\beta_1 \int_{C_-} \hat{a}_j e^{i\vec{k}_1\cdot\vec{r}} \int_0^{2\pi} d\beta_0 \int_{C_-} a_{j'}(\vec{k}_0)\, Q^{-1}_{jj'}(\vec{k}_0,\vec{k}_1) \sin\alpha_0\, d\alpha_0 \sin\alpha_1\, d\alpha_1 \tag{35}$$

$$\vec{\mathfrak{F}}_n(\vec{r}) \equiv 2\sum_{jj'} \int_0^{2\pi} d\beta_1 \int_{C_-} \hat{a}_j e^{i\vec{k}_1\cdot\vec{r}} \int_0^{2\pi} d\beta_1' \int_{C_+} B_{nj'}(\hat{k}_1')\, R_{jj'}(\vec{k}_1,\vec{k}_1') \sin\alpha_1'\, d\alpha_1' \sin\alpha_1\, d\alpha_1 \tag{36}$$

The field $\vec{E}_1^{dir.}$ is the transmitted field as if no scatterer is present, while the two remaining terms reflect the presence of the scatterer; a field which could be called the scattered field $\vec{E}_1^{sc.}$. In terms of a multiple scattering interpretation, which is discussed in more detail in [4], the second term on the right hand side of Eq. (34) can be interpreted as the sum of all those contributions, which is reflected the last time at the surface $S_o$. Similarly, the third term corresponds to the contributions reflected the last time from the inhomogeneity. Of course both these contributions added contain all the multiple scattering effects between $S_o$ and the inhomogeneity, via $c_n$, the solution of Eq. (31).

As we have seen, the consideration of the field $\vec{E}_1$ is quite analogous to the derivation of the scattered field $\vec{E}_0^{sc.}$ in $V_0$, which in the lossless case is found in [4]. In the lossy case the formal derivation of $\vec{E}_o^{sc.}$ is the same and we here just state the result.

$$\vec{E}_o^{sc.}(\vec{r}) = \vec{E}_o^{sc.,dir.}(\vec{r}) + \vec{E}_o^{sc.,anom.}(\vec{r}) = \vec{E}_o^{sc.,dir.}(\vec{r}) + \sum_{nn'} \vec{F}_n(\vec{r})\, T_{nn'}\, c_{n'} \tag{37}$$

where

$$\vec{E}_o^{sc.,dir.}(\vec{r}) \equiv \sum_{jj'} \int_0^{2\pi} d\beta_o \int_{C_+} \hat{a}_j e^{i\vec{k}_o\cdot\vec{r}} \int_0^{2\pi} d\beta_o' \int_{C_-} R_{jj'}(\vec{k}_o,\vec{k}_o')\, a_{j'}(\vec{k}_o') \sin\alpha_o' d\alpha_o' \sin\alpha_o d\alpha_o$$

$$\vec{F}_n(\vec{r}) \equiv -2i \sum_{jj'} \int_0^{2\pi} d\beta_o \int_{C_+} \hat{a}_j e^{i\vec{k}_o\cdot\vec{r}} \int_0^{2\pi} d\beta_1 \int_{C_+} B_{nj'}(\hat{k}_1) \Big[ Q_{jj'}(\vec{k}_o,\vec{k}_1) +$$

$$+ \sum_{j''} \int_0^{2\pi} d\beta_1' \int_{C_-} Q_{jj''}(\vec{k}_o,\vec{k}_1')\, R_{j''j'}(\vec{k}_1',\vec{k}_1) \sin\alpha_1' d\alpha_1' \Big] \sin\alpha_1 d\alpha_1 \sin\alpha_o d\alpha_o \tag{38}$$

$$R_{jj'}(\vec{k}_o,\vec{k}_o') \equiv -\sum_{j''} \int_0^{2\pi} d\beta_1 \int_{C_-} Q_{jj''}(\vec{k}_o,\vec{k}_1)\, Q^{-1}_{j''j'}(\vec{k}_1,\vec{k}_o') \sin\alpha_1 d\alpha_1 \qquad \vec{k}_o \in C_+ ,\ \vec{k}_o' \in C_- \tag{39}$$

$R_{jj'}(\vec{k}_o,\vec{k}_o')$ is the reflection coefficient for the surface $S_o$ from above, cf. Eq. (30).

So far we have considered the general case, and have not introduced any specific geometry. The inversion of Eq. (24) in the general case, when $S_o$ is rough, is indeed a difficult problem. However, when one has a "finite hill" one can find an algorithm so that the interaction between the hill and the inhomogeneity is taken into account [10]. The numerical computations in this paper will only consider a plane surface $S_o$ and for this case most of the equations can be simplified. We get for a plane surface $S_o$ ($z=z_o$=constant) (cf. the lossless case):

$$\vec{E}_1^{dir.}(\vec{r}) = \sum_j \frac{2}{k_0} \int_0^{2\pi} d\beta_1 \int_{C_-} \hat{a}_j e^{i\vec{k}_1\cdot\vec{r}} a_j(\vec{k}_1^{*-}) e^{iz_0((k_1^2-\lambda_1^2)^{1/2}-(k_0^2-\lambda_1^2)^{1/2})} \frac{k_1(k_1^2-\lambda_1^2)^{1/2}}{\Delta_j(\lambda_1)} \sin\alpha_1 d\alpha_1 \tag{40}$$

$$\vec{\mathcal{F}}_n(\vec{r}) = -2\sum_j \int_0^{2\pi} d\beta_1 \int_{C_-} \hat{a}_j e^{i\vec{k}_1\cdot\vec{r}} B_{nj}(\hat{k}^-) R_j(\lambda_1) e^{2iz_0(k_1^2-\lambda_1^2)^{1/2}} \sin\alpha_1 d\alpha_1 \tag{41}$$

$$A_{nn'} = 2\sum_{n''j} \int_0^{2\pi} d\beta_1 \int_{C_-} B_{nj}^{\dagger}(\hat{k}_1) B_{n''j}(\hat{k}_1^-) R_j(\lambda_1) e^{2iz_0(k_1^2-\lambda_1^2)^{1/2}} \sin\alpha_1 d\alpha_1 T_{n''n'} \tag{42}$$

$$d_n = \sum_j \int_0^{2\pi} d\beta_0 \int_{C_-} a_j(\vec{k}_0) B_{nj}^{\dagger}(\hat{k}_0^{*-}) e^{iz_0((k_1^2-\lambda_0^2)^{1/2}-(k_0^2-\lambda_0^2)^{1/2})} \frac{2(k_0^2-\lambda_0^2)^{1/2}}{\Delta_j(\lambda_0)} \sin\alpha_0 d\alpha_0 \tag{43}$$

$$\vec{E}_0^{sc.,dir.}(\vec{r}) = \sum_j \int_0^{2\pi} d\beta_0 \int_{C_+} a_j(\vec{k}_0^-) R_j(\lambda_0) e^{-2iz_0(k_0^2-\lambda_0^2)^{1/2}+i\vec{k}_0\cdot\vec{r}} \hat{a}_j \sin\alpha_0 d\alpha_0 \tag{44}$$

$$\vec{F}_n(\vec{r}) = \frac{4c_0k_0}{k_1} \sum_j \int_0^{2\pi} d\beta_0 \int_{C_+} e^{i\vec{k}_0\cdot\vec{r}+iz_0((k_1^2-\lambda_0^2)^{1/2}-(k_0^2-\lambda_0^2)^{1/2})} B_{nj}(\hat{k}_0^{*+}) \hat{a}_j \frac{(k_0^2-\lambda_0^2)^{1/2}}{\Delta_j(\lambda_0)} \sin\alpha_0 d\alpha_0 \tag{45}$$

The Fresnel reflection coefficients are [11] :

$$R_1(\lambda) \equiv \frac{N_1(\lambda)}{D_1(\lambda)} \equiv \frac{c_0 k_1 \left(1-(\lambda/k_0)^2\right)^{1/2} - k_0 \left(1-(\lambda/k_1)^2\right)^{1/2}}{c_0 k_1 \left(1-(\lambda/k_0)^2\right)^{1/2} + k_0 \left(1-(\lambda/k_1)^2\right)^{1/2}} \tag{46}$$

$$R_2(\lambda) \equiv \frac{N_2(\lambda)}{D_2(\lambda)} \equiv \frac{(k_0^2-\lambda^2)^{1/2} - c_0\,(k_1^2-\lambda^2)^{1/2}}{(k_0^2-\lambda^2)^{2} + c_0\,(k_1^2-\lambda^2)^{1/2}}$$

$R_1$ is the reflection coefficient for an incoming wave polarized along $\hat{a}_1$ and $R_2$ for a wave polarized along $\hat{a}_2$. $D_j(\lambda)$ and $N_j(\lambda)$ are the denominator and nominator in the reflection coefficient respectively, and $\lambda_i = k_i \sin\alpha_i$; $i=0,1$. All square roots are defined such that $\mathrm{Im}(k_i^2-\lambda_i^2)^{1/2} \geqslant 0$. We have also introduced the follwoing notation for the transformed arguments in Eq. (40)-(45)

$$\vec{k}_1^{*-} \equiv k_1 \left(\sin\alpha_1 \cos\beta_1, \sin\alpha_1 \sin\beta_1, -\left(\left(\frac{k_0}{k_1}\right)^2 - \sin^2\alpha_1\right)^{1/2}\right)$$

$$\hat{k}_1^{-} \equiv \left(\sin\alpha_1 \cos\beta_1, \sin\alpha_1 \sin\beta_1, -\cos\alpha_1\right)$$

$$\vec{k}_0^{-} \equiv k_0 \left(\sin\alpha_0 \cos\beta_0, \sin\alpha_0 \sin\beta_0, -\cos\alpha_0\right)$$

$$\hat{k}_0^{*\pm} \equiv \frac{k_0}{k_1} \left(\sin\alpha_0 \cos\beta_0, \sin\alpha_0 \sin\beta_0, \pm\left(\left(\frac{k_0}{k_1}\right)^2 - \sin^2\alpha_0\right)^{1/2}\right)$$

## III. Numerical applications

The final Eqs. (40)-(45) given in the preceding section will now be applied in some numerical examples. These illustrations include both field computations above the ground (the electric field) and below the interface (the magnetic field). The source that excites the inhomogeneity is chosen to be a vertical dipole source; in the case of electric field an electric dipole, while in the magnetic case a magnetic dipole. The dipole source located at a source point $P \equiv (\rho_t, 0, z_t)$ is given by [11]:

$$\left.\begin{matrix} \vec{E}_0^{inc.}(\vec{r}) \\ \vec{H}_0^{inc.}(\vec{r}) \end{matrix}\right\} = \frac{1}{k_0^2} \nabla \times \left\{ \nabla \times \left[ \hat{z} \frac{e^{ik_0|\vec{r}-\vec{r}_t|}}{k_0|\vec{r}-\vec{r}_t|} \right] \right\} \tag{47}$$

If we place the source on the surface $S_o$ $(z_t = z_o)$ then Eq. (43) simplifies into

$$d_n = \frac{2k_1 \gamma_n}{i k_0 \sqrt{\ell(\ell+1)}} \int_{i\infty}^{1} \frac{i^{m-\ell} k_0 t \, dt}{k_0 \left(1-\left(\frac{k_0}{k_1}\right)^2 (1-t^2)\right)^{1/2} + c_0 k_1 t} \, e^{ik_1 z_0 \left(1-\left(\frac{k_0}{k_1}\right)^2 (1-t^2)\right)^{1/2}} \times$$

$$\times J_m\left(k_0 \rho_t (1-t^2)^{1/2}\right) \left\{ i m \delta_{\sigma o} \delta_{\tau 1} P_\ell^m\left(\left(1-\left(\frac{k_0}{k_1}\right)^2 (1-t^2)\right)^{1/2}\right) - \right.$$

$$\left. -\delta_{\sigma e} \delta_{\tau 2} \left[ (\ell-m+1) P_{\ell+1}^m\left(\left(1-\left(\frac{k_0}{k_1}\right)^2 (1-t^2)\right)^{1/2}\right) - (\ell+1)\left(1-\left(\frac{k_0}{k_1}\right)^2 (1-t^2)\right)^{1/2} P_\ell^m\left(\left(1-\left(\frac{k_0}{k_1}\right)^2 (1-t^2)\right)^{1/2}\right) \right] \right\} \tag{48}$$

Here the definitions of spherical harmonics and normalization follow Ref. [4].

The numerical examples are separated into two groups; one in which we compute the electric field above the ground and one where we focus on the magnetic field below the interface. In all numerical examples we have chosen a moderate contrast in the parameters between $V_o$ and $V_1$. These values are chosen here since they are believed to illustrate "worst case" applications of the above formalism (i.e. using the full structure of the equations, without further specializations). Numerical computations for values of the parameters corresponding to high contrasts and losses will be performed in the future. In this case it is possible to introduce further simplications by using asymptotic methods for the strongly oscillating integrals. Work in this direction is in progress. Parameters in common in all the numerical examples are:

$$k_1^2/k_0^2 = 10+5i$$

$$\mu_{1r}/\mu_{0r} = 1$$

$$k_0 z_0 = 0.8$$

The inhomogeneity, completely specified by its T-matrix, is here taken as a perfectly conducting spheroid. The semiaxis in the direction of rotational symmetry is a and the semiaxis in the perpendicular direction is b. The T-matrix for the spheroid is generated numerically for an orientation of the rotational symmetry axis along the z-axis, and is then rotated by means of the three-dimensional rotation matrices [12] to an arbitrary orientation. The orientation of the symmetry axis is given in spherical angles $\phi, \chi$. This procedure allows us to calculate the T-matrix for inhomogeneities which are asymmetrically oriented both with respect to the interface and to the source position. The generation of the

T-matrix for a large class of asymmetrical scatterers can thus be performed efficiently, since the rotational matrices are fairly easy to generate numerically. To get the result for a different orientation of the scatterer only these matrices have to be recalculated; the T-matrix along the symmetry axis is the same as before. The steps of computation of the scattered field for a given inhomogeneity are in short:

1) Compute the $d_n$-vector, Eq. (48) for a given source position P.
2) Compute the $A_{nn'}$-matrix, Eq. (42).
3) Compute the field vector $\vec{F}_n(\vec{r})$ or $\vec{\mathfrak{F}}_n(\vec{r})$, Eq. (41) or (45) for a given array of field points $\vec{r}$.
4) Generate the $T_{nn'}$-matrix of the inhomogeneity oriented along the z-axis.
5) Rotate the $T_{nn'}$-matrix by applying the rotation matrices.
6) Solve Eq. (31) for the $c_n$-vector.
7) Combine the quantities in Eq. (37) or (34).

A variation of a single parameter or a different choice of inhomogeneity (concerning e.g. both shape and orientation) does not affect all steps above; most of them need not be repeated. Only a few items have to be recalculated and this feature makes the formalism efficient in situations where one is interested exploring the effect of these types of parameter variations.

Many of the quantities contain an integral over a $C_+$ or $C_-$ contour. These integrals have to be computed numerically, and we use a fast, improved quadrature, which in a subdivision of the integration interval uses the previously computed function values. The integrand usually contains an exponential factor, which makes the convergence very rapid. In those integrals where such an exponential factor is absent or is small, we use a different method. Since the integrals have oscillating integrands we divide the integration interval into parts, according to e.g. the nulls of the integrand so that the total integral becomes an alternating series. We then apply an Euler transformation to the series, which improves the convergence of the series very efficiently.

The computer time required in the various steps above varies considerably and only a rough estimate can be given. The steps 1), 2), 4), 5), 7) have usually an execution time of less than 2 min. c.p.u. on an IBM 370/3031 or IBM 360/65. Item 3) is the most time-consuming step, which for an array size $\vec{r}$ and a truncation order used in the numerical examples considered here, takes about 10 min. c.p.u. The radial and the azimuthal dependence in the pertinent integrals can be separated in such a way that all azimuthal dependence is a common factor outside a remaining integral, which only depends on the $\rho$ and z coordinates. Step 6) takes only a couple of seconds c.p.u. to evaluate. However, it should be noted that the various execution times here given highly depend on the truncation size of the matrices, prerequired accuracy in the evaluation of the integrals, array size of measuring points in $\vec{F}_n(\vec{r})$ or $\vec{\mathcal{F}}_n(\vec{r})$ etc., and we should also observe that the constituent parts in the scheme above can be used again in various combinations which reduces the computational costs considerably. The above c.p.u. times refer to computations to about three significant figures in the final results. In many practical applications one does not of course need such a high accuracy and the c.p.u. time requirements are reduced accordingly.

In a series of plots, Figures 3-6, we illustrate the anomalous scattered electric field on the surface $S_o$ $(z=z_o)$ in a region close to the inhomogeneity. These plots show computer interpolated surfaces of constant amplitude of the anomalous scattered electric field $E_o^{sc.,anom.}$. Due to the computer interpolation algorithm these figures contain some irregularities, which are not present in the original computations. These figures show different field components for various scatterers, which are perfectly conducting spheroids - both prolate and oblate - of diverse orientations, both with respect to the surface $S_o$ and to the source position. In Fig. 7 we show the quotient of the vertical components of the anomalous scattered electric field and the incoming electric field along the x-axis. The pattern these plots exhibit are fairly complex, but seem to fit reasonably well to the radiation pattern of a simple dipole, which replaces the scatterer. In order to achieve this the orientation of the dipole has to be adjusted. The resulting optimal orientation was found to agree reasonably well with what was to be expected from the relevant source, treated as a source in a homogeneous space (no interface $S_o$). The present more accurate computations can be used to investigate when these simple dipole-excitation models are valid.

The second part of the numerical illustrations given in this paper is the magnetic field below the surface $S_o$. Here we have calculated the magnetic field - both the direct and scattered - along a straight line with a specific direction. Along these "drillholes" the field component along the line is depicted in

Fig. 8-10. The "drillhole" starts at the coordinates $(x_o, y_o, z_o)$ and has a direction given by the spherical angles $(\eta, \psi)$. The source in these calculations is a vertical magnetic dipole located on the surface $S_o$ at $(\rho_t, 0, z_o)$. In each plot we illustrate the field component along the drillhole for various scatterers - perfectly conducting spheroid of different orientations. As expected we get the highest response from the obstacle at the position closest to the inhomogeneity. Although the scatterer is rather small and the drillhole does not come too near the obstacle, we have a rather high signal return from the inhomogeneity, which in some situations is more than 10%.

Acknowledgements

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## Appendix

## Transformation relations between plane and spherical vector waves.

The transformation between spherical waves and plane waves for a real wave number is discussed in detail in [13]. The extension to complex wave numbers is found in this appendix.

Consider the following integral for $z>0$:

$$\int_0^{2\pi} d\beta \int_{C_+} Y_n(\hat{k})\, e^{i\vec{k}\cdot\vec{r}} \sin\alpha\, d\alpha \tag{A.1}$$

Here we have adopted the notations

$$\vec{r} \equiv (x,y,z) \equiv r\hat{r}$$

$$\hat{r} \equiv (\sin\theta\ \cos\phi,\ \sin\theta\ \sin\phi,\ \cos\theta) \tag{A.2}$$

$$\vec{k} \equiv (k_x, k_y, k_z) \equiv k\hat{k} \equiv |k|e^{i\varkappa}\hat{k}$$

$$\hat{k} \equiv (\sin\alpha\ \cos\beta,\ \sin\alpha\ \sin\beta,\ \cos\alpha)$$

The complex contour $C_+$ in the $\alpha$-plane (see Figure 2) is a contour from $\alpha=0$ to $\alpha=\pi/2-\varkappa-i\infty$ subject to ($\alpha=\alpha'+i\alpha''$, where $\alpha', \alpha''$ are real numbers).

$$\tanh\alpha'' + \tan\alpha' \tan\varkappa = 0 \tag{A.3}$$

We will here assume $\varkappa \in (0,\pi/4)$. This restriction can be relaxed but it is sufficiently general for our purpose (remember $\mathrm{Im}k^2 \geqslant 0$, $\mathrm{Re}k^2 > 0$). The case $\varkappa=0$ is excluded since that is the situation analysed by Danos and Maximon [13].

The integral in Eq. (A.1) is essentially a two-dimensional Fouriertransform in $(k_x, k_y)$; first rewritten in polar coordinates $\lambda = k \sin\alpha$ and $\beta$ and then finally transformed into the spherical angles $\alpha, \beta$. For more details see Baños [14]. We have here explicitly assumed $z>0$ $(\phi<\pi/2)$ but for $z<0$ the $C_+$-contour is replaced by the $C_-$-contour and a similar analysis will hold. Note that on both these contours we have

$$\begin{aligned} &\mathrm{Im}(k \sin\alpha) = 0 \\ &\mathrm{Im}(zk \cos\alpha) \gtrsim 0 \\ &\mathrm{Re}(k \sin\alpha) \gtrsim 0 \end{aligned} \tag{A.4}$$

The $C_+$-contour defined by Eq. (A.3) transformed to the $t \equiv \cos\alpha$-plane is a contour (see Figure 11) from $t = \infty e^{i(\pi/2-\kappa)}$ to $t=1$ subject to $(t = t' + it'')$

$$(t' \tan\kappa + t'')(t' - t'' \tan\kappa) = \tan\kappa \tag{A.5}$$

The integral (A.1) is easily shown to be uniformly convergent for all $\vec{r}$ when $z \geq c > 0$, where c is any positive number (cf. the exponential decaying factor). We also note that the integrand is an analytic function except for the branch points $t = \pm 1$ of the associated Legendre functions $P_\ell^m(\cos\alpha)$ in $Y_n(\hat{k})$ and for the essential singularity at infinity for the exponential function. It is permitted to deform the contour $C_+$ to any contour $\Gamma$ provided no singularities are crossed. For convenience we will deform the $C_+$-contour to $\Gamma$ as depicted in Figure 12 or in the $\cos\alpha$-plane, see Fig. 13. The constant $\alpha_0 = \pi/2 - i\alpha_0''$ is any complex number where $\alpha_0''$

satisfies

$$\sinh \alpha_0'' \geq 1/\cos\theta \tag{A.6}$$

The reason for this deformation of $C_+$ is to simplify the analysis given below. We note that the first part of $\Gamma$ (from $\alpha=0$ to $\alpha=\alpha_0$) is identical to the corresponding part of the contour used in [13].

The evaluation of the integral in Eq. (A.1) is done by first making a rotation of the coordinate system $(\alpha,\beta)$ to a coordinate system $(\eta,\psi)$. Here $\eta$ is the new polar angle, specified as the angle between $\vec{k}$ and $\vec{r}$, and $\psi$ is the new azimuthal angle (see Fig. 14). The spherical harmonics $Y_n(\hat{k})$ is transformed according to [12]:

$$Y_n(\alpha,\beta) = \sum_{m'} \mathcal{D}^{(\ell)}_{m'm}(\phi,\theta,0)\, Y_{\ell m'}(\eta,\psi) \tag{A.7}$$

This relation holds for real angles $(\alpha,\beta)$, but can be analytically continued to complex angles $(\alpha,\beta)$ (note that the $m'$ summation is finite). We have here changed from real combinations of the spherical harmonics to complex ones which have an azimuthal dependence $e^{im\beta}$ (this is merely a matter of convenience and in accordance with references [12] and [13]). The final result holds for real combinations too. Furthermore, we can without loss of generality take $\phi=0$. The case $\phi\neq0$ is an azimuthal rotation with a trivial shift in the $\beta$ interval. The relation between the two coordinate systems $(\alpha,\beta)$ and $(\eta,\psi)$ is [12] $(\phi=0)$:

$$\begin{aligned} \cos\eta &= \sin\theta\sin\alpha\cos\beta + \cos\theta\cos\alpha \\ \sin\eta\cos\psi &= \cos\theta\sin\alpha\cos\beta - \sin\theta\cos\alpha \\ \sin\eta\sin\psi &= \sin\alpha\sin\beta \end{aligned} \tag{A.8}$$

The Eq. (A.1) can be rewritten as

$$\int_0^{2\pi} d\beta \int_\Gamma Y_n(\hat{k})\, e^{i\vec{k}\cdot\vec{r}} \sin\alpha\, d\alpha =$$

$$= \sum_{m'} \mathcal{D}^{(\ell)}_{m'm}(0,\theta,0) \int_{\gamma'} d\psi \int_{\Gamma'} Y_{\ell m'}(\eta,\psi)\, e^{ikr\cos\eta} \left|\frac{\partial(\alpha,\beta)}{\partial(\eta,\psi)}\right| \sin\alpha\, d\eta \quad \text{(A.9)}$$

The Jacobian $\left|\frac{\partial(\alpha,\beta)}{\partial(\eta,\psi)}\right|$ is easily found:

$$\left|\frac{\partial(\alpha,\beta)}{\partial(\eta,\psi)}\right| \sin\alpha = \left[\frac{\partial\alpha}{\partial\eta}\frac{\partial\beta}{\partial\psi} - \frac{\partial\alpha}{\partial\psi}\frac{\partial\beta}{\partial\eta}\right]\sin\alpha = \sin\eta \, .$$

We observe that the Jacobian is non-zero for $|\cos\eta| \neq 1$, and thus the transformation non-singular, see [13] and below.

$$\int_0^{2\pi} d\beta \int_\Gamma Y_n(\hat{k})\, e^{i\vec{k}\cdot\vec{r}} \sin\alpha\, d\alpha =$$

$$= \sum_{m'} \mathcal{D}^{(\ell)}_{m'm}(0,\theta,0) \int_{\gamma'} d\psi \int_{\Gamma'} Y_{\ell m'}(\eta,\psi)\, e^{ikr\cos\eta} \sin\eta\, d\eta \quad \text{(A.10)}$$

The contours $\gamma'$ and $\Gamma'$ are the transformed contours of the $(\alpha,\beta)$ variables given by Eq. (A.8). (Note that $\Gamma'$ is a function of $\psi$.)

Our goal is now to deform the contours in the right hand side of Eq. (A.10) to the original contours ($\psi\epsilon[0,2\pi]$ and $\eta\epsilon\Gamma$ or equivalently $\eta\epsilon C_+$). For this step to be valid we have to make sure that no singularities of the integrand are crossed. The endpoints of the integral also have to be the same.

In the discussion of the contour deformations it is convenient to divide $\Gamma$ into two parts: one from the origin to the point $\alpha_o$ and from $\alpha_o$ to infinity respectively. The deformation of the first part of the $\Gamma$ contour, where $\alpha$ is real and $\alpha=\pi/2+i\alpha''$, $-\alpha''_o \leq \alpha'' \leq 0$

is found in [13]. There a detailed analysis is found of how to rearrange the integrations in the real region, and how to let the starting point of the $\eta$ variable remain at $\eta=0$ ($\alpha\to 0$ corresponds to $\eta\to\theta$). We see that the map of the real part of $\Gamma$ is real, and the rearrangement can be interpreted as a transformation on the unit sphere. A discussion of how to circumvent the singular points $\cos\eta=\pm 1$ is also found in [13]. The rearrangement discussed above can also be performed for the original contour $C_+$, but then much harder to illustrate.

We now discuss the final part of the $\Gamma'$ contour, i.e. $\alpha=\alpha'-i\alpha_o''$, $\arctan\left(\frac{\tanh\alpha_o''}{\tan\varkappa}\right)\leq\alpha'\leq\pi/2$ and $\alpha\epsilon C_+$, $\alpha''\leq-\alpha_o''$. First we analyse the asymptotic behaviour of $\Gamma'$ as $\alpha\to\pi/2-\varkappa-i\infty$. According to Eq. (A.8) we have:

$$\cos\eta=\sin\theta\,\sin\alpha\,\cos\beta+\cos\theta\,\cos\alpha$$

Asymptotically we have

$$\arg(\cos\eta)=-\varkappa+\pi/2-\arctan(\tan\theta\cos\beta)\qquad\alpha''\to-\infty\qquad\text{(A.11)}$$

and we see that $\arg(\cos\eta)\in[\pi/2-\varkappa-\theta,\ \pi/2-\varkappa+\theta]$, see Fig. 15. Furthermore we have

$$\arg(k\cos\eta)=\pi/2-\arctan(\tan\theta\cos\beta)\qquad\alpha''\to-\infty\qquad\text{(A.12)}$$

and we conclude that $\arg(k\cos\eta)\in[\pi/2-\theta,\ \pi/2+\theta]$ as $\alpha''\to-\infty$. Thus for $\theta<\pi/2$ we have $\arg(k\cos\eta)\in(0,\pi)$ as $\alpha''\to-\infty$, which ensures the correct convergence properties at infinity. In the $\eta$-plane this corresponds to (see Figure 16)

$$\eta \to -\varkappa + \pi/2 - \arctan(\tan\theta\cos\beta) - i\infty \qquad \alpha \to \pi/2 - \varkappa - i\infty$$

The remaining step is to make sure that we do not cross the singular points $\cos\eta=\pm 1$ as we deform the remaining part of $\Gamma'$. We have, see Eq. (A.8)

$$\cos\eta = \sin\alpha \sin\theta \cos\beta + \cos\alpha \cos\theta$$

It is fairly easy to see that for $\alpha=\alpha'+i\alpha''$ with $\alpha'$ arbitray and $\alpha'' \leqslant -\alpha''_0$, where $\alpha''_0$ is given by Eq. (A.6) we have

$$|\cos\eta| > 1$$

Thus we see that the last part of $\Gamma'$ ($\alpha=\alpha'+i\alpha''$, $\alpha''\leqslant-\alpha''_0$) does not enclose the singular points $\cos\eta=\pm 1$. Neither does the last part of $\Gamma'$ enclose both $\cos\eta=1$ and $\cos\eta=-1$, since the curve never enters the third quadrant of the $\cos\eta$ plane. (The last part of $\Gamma'$ starts in the first or second quadrant, depending on $\beta$, and ends in the sector $\arg(\cos\eta)\,\epsilon\,[\pi/2-\varkappa-\theta,\ \pi/2-\varkappa+\theta]$, and the first part of $\Gamma'$, where $\alpha$ is real or $\alpha=\pi/2+i\alpha''$; $-\alpha''_0 \leqslant \alpha'' \leqslant 0$, has only values in the first and second quadrant).

The integration variable $\psi$ can then easily be changed to the real interval $\psi\,\epsilon\,[0,2\pi]$. The endpoints of the deformed curve remain the same; we have $\psi=n\pi$ when $\beta=n\pi$. From Eq. (A.8) we can solve for $\psi$, and we get ($\psi=\psi'+i\psi''$ and see Fig. 17):

$$\begin{cases} \tan 2\psi' = \dfrac{2A\sin\beta}{A^2+B^2-\sin^2\beta} \\ \tanh 2\psi'' = \dfrac{2B\sin\beta}{A^2+B^2+\sin^2\beta} \end{cases} \qquad \text{(A.13)}$$

where

$$\begin{cases} A \equiv \cos\theta\cos\beta - \sin\theta\tan\alpha'\left(1-\tanh^2\alpha''\right)\left(\tanh^2\alpha''+\tan^2\alpha'\right)^{-1} \\ B = -\sin\theta\tanh\alpha''\left(1+\tan^2\alpha'\right)\left(\tanh^2\alpha''+\tan^2\alpha'\right)^{-1} \end{cases}$$

Thus we have analysed the contours $\gamma'$ and $\Gamma'$ and justified the deformation of these contours to the original $\psi\epsilon[0,2\pi]$ and $\eta\epsilon\Gamma$ or $\eta\epsilon C_+$ and showed that this deformation is valid. We have, see Eq. (A.10)

$$\int_0^{2\pi} d\beta \int_{C_+} Y_n(\hat{k})\, e^{i\vec{k}\cdot\vec{r}} \sin\alpha\, d\alpha = \sum_{m'} \mathcal{D}^{(\ell)}_{m'm}(\phi,\theta,0) \int_0^{2\pi} d\psi \int_{C_+} Y_{\ell m'}(\eta,\psi)\, e^{ikr\cos\eta} \sin\eta\, d\eta$$

$$= \mathcal{D}^{(\ell)}_{0m}(\phi,\theta,0)\, 2\pi\gamma_{\ell 0} \int_{C_+} P_\ell(\cos\eta)\, e^{ikr\cos\eta} \sin\eta\, d\eta =$$

$$= 2\pi i^\ell h^{(1)}_\ell(kr)\, Y_n(\hat{r}) \equiv 2\pi i^\ell \psi_n(k\vec{r}) \qquad \text{(A.14)}$$

In this last equation we have returned to the more general situation where $\phi \neq 0$, see the discussion above, and furthermore we have used the integral representation of $h^{(1)}_\ell(z)$ [15] and the definitions of $\mathcal{D}^{(\ell)}_{m'm}(\phi, \theta, 0)$ [12], i.e.

$$h^{(1)}_\ell(z) = i^{-\ell} \int_{\infty\exp(iv)}^{1} P_\ell(t)\, e^{izt}\, dt \qquad v\epsilon(-\arg z, \pi - \arg z)$$

$$\mathcal{D}^{(\ell)}_{0m}(\phi,\theta,0) = Y_n(\hat{r})/\gamma_{\ell 0} \qquad \gamma_{\ell 0} = \sqrt{\frac{2\ell+1}{4\pi}}$$

A completely analogous analysis can be made for $z<0$ but this time with the contour $C_-$ and we can here conclude:

$$\psi_n(k\vec{r}) = \frac{1}{2\pi i^\ell} \int_0^{2\pi} d\beta \int_{C_\pm} Y_n(\hat{k})\, e^{i\vec{k}\cdot\vec{r}} \sin\alpha\, d\alpha \qquad z \gtrless 0 \qquad \text{(A.15)}$$

For an $z \neq 0$ the integral in Eq. (a.15) is a uniformly convergent integral and differentiation under the integral sign can be justified. The spherical vector waves $\vec{\psi}_n(k\vec{r})$ can thus be formed

$$\vec{\Psi}_{\tau n}(k\vec{r}) \equiv \frac{1}{\sqrt{\ell(\ell+1)}} \left(\frac{1}{k}\nabla \times\right)^{\tau} \left(k\vec{r}\,\Psi_n(k\vec{r})\right) \qquad \tau = 1,2 \tag{A.16}$$

In proving the transformation between spherical vector waves and plane vector waves we will essentially follow the presentation found in [8]. First we consider:

$$\vec{\Psi}_{1n}(k\vec{r}) = h_\ell^{(1)}(kr)\,\vec{A}_{1n}(\hat{r}) \tag{A.17}$$

For definition of the vector spherical harmonics see [4]. We will also need the following identity [6]. (Note that we for convenience are using the complex combination of the spherical harmonics, i.e. no $\sigma$ index.)

$$\begin{aligned}\vec{A}_{1n}(\hat{r}) = \vec{A}_{1m\ell}(\hat{r}) &= \frac{1}{\sqrt{\ell(\ell+1)}}\nabla Y_{\ell m}(\hat{r}) \times \hat{r} = \\ &= a_- Y_{\ell m+1}(\hat{r})\,\hat{\varepsilon}_- + a_+ Y_{\ell m-1}(\hat{r})\,\hat{\varepsilon}_+ + a_z Y_{\ell m}(\hat{r})\,\hat{z}\end{aligned} \tag{A.18}$$

where $a_-, a_+, a_z$ are contants depending only on $\ell$ and m and where

$$\begin{cases}\hat{\varepsilon}_+ = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y}) \\ \hat{\varepsilon}_- = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y})\end{cases} \tag{A.19}$$

We get with Eq. (A.15)

$$\vec{\Psi}_{1n}(k\vec{r}) = h_\ell^{(1)}(kr)\left[a_- Y_{\ell m+1}(\hat{r})\hat{\varepsilon}_- + a_+ Y_{\ell m-1}(\hat{r})\hat{\varepsilon}_+ + a_z Y_{\ell m}(\hat{r})\hat{z}\right] =$$

$$= \frac{1}{2\pi i^\ell}\int_0^{2\pi} d\beta \int_{C_\pm} \vec{A}_{1n}(\hat{k})\, e^{i\vec{k}\cdot\vec{r}} \sin\alpha\, d\alpha \qquad z \gtrless 0 \tag{A.20}$$

The spherical vector wave for $\tau=2$ is found by differentiation under the integral sign, and we get

$$\vec{\Psi}_{2n}(k\vec{r}) = \frac{1}{k}\nabla\times\vec{\Psi}_{1n}(k\vec{r}) =$$

$$= \frac{i}{2\pi i^\ell}\int_0^{2\pi} d\beta \int_{C_\pm} \vec{A}_{2n}(\hat{k})\, e^{i\vec{k}\cdot\vec{r}} \sin\alpha\, d\alpha \qquad z \gtrless 0 \tag{A.21}$$

Thus finally we have

$$\vec{\Psi}_{n}(k\vec{r}) = \frac{i}{2\pi}\int_0^{2\pi} d\beta \int_{C_\pm} \vec{B}_{n}(\hat{k})\, e^{i\vec{k}\cdot\vec{r}} \sin\alpha\, d\alpha \qquad z \gtrless 0 \tag{A.22}$$

where $\vec{B}_n(\hat{k})$ is defined in Eq. (7).

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Figure captions

Fig. 1 Geometry and notations.

Fig. 2 The integration contours $C_+$ and $C_-$.

Fig. 3 The amplitude of the x-component of the anomalous scattered field $|\vec{E}_o^{sc.,anom.} \cdot \hat{x}|$ on the surface $S_o$ ($k_o z_o = 0.8$) for a buried perfectly conducting spheroid. The semi axis in the direction of rotational symmetry is $k_o a = 0.3$ and the other semi axis is $k_o b = 0.15$. The orientation of the symmetry axis is $\theta = \chi = \pi/6$. The source is located at $k_o \rho_t = 6$ and the scale factor is $10^{-6}$.

Fig. 4 The same as Fig. 3 but the z-component $|\vec{E}_o^{sc.,anom.} \cdot \hat{z}|$ shown and $\theta = \pi/3$, $\chi = \pi/2$, $k_o \rho_t = 3$ and the scale factor is $10^{-5}$.

Fig. 5 The same as Fig. 3 but $k_o a = 0.15$, $k_o b = 0.3$ and $\theta = \pi/6$, $\chi = 3\pi/4$, $k_o \rho_t = 3$ and the scale factor is $10^{-5}$.

Fig. 6 The same as Fig. 3 but $k_o a = 0.15$, $k_o b = 0.3$, $k_o \rho_t = 0$ and the scalar factor is $10^{-4}$.

Fig. 7 The variation in $|\vec{E}_o^{sc.,anom.} \cdot \hat{z}| / |\vec{E}_o^{inc.} \cdot \hat{z}|$ for data as in Fig. 5 along the x-axis. The scale is in percent.

Fig. 8 The amplitude and phase variation of the field component of the scattered magnetic field $\hat{n} \cdot \vec{H}_1^{sc.}$ along a drillhole (t is a parameter along the drillhole). The drillhole starts at $k_o(x_o, y_o, z_o) = (1, 0, 0.8)$ and has the direction $\eta = 5\pi/6$, $\psi = \pi$. The source is located at $k_o \rho_t = 3$. Data for the obstacle $k_o a = 0.1$, $k_o b = 0.2$

a) $\theta = \pi/2$, $\chi = 0$

b) $\theta = \pi/6$, $\chi = \pi/4$

Fig. 9 The amplitude of the field component of the scattered mangetic field $|\hat{n}\cdot\vec{H}_1^{sc.}|$ along the drillhole $k_o(x_o,y_o,z_o)=(-1,1,0.8)$, $\eta=5\pi/6$, $\psi=3\pi/2$, $k_o\rho_t=3$ $k_oa=0.1$, $k_ob=0.2$ and a) $\theta=\pi/6$, $\chi=\pi/4$, b) $\theta=\pi/3$, $\chi=\pi/2$.

Fig. 10 The amplitude and phase variation of the field component of the total magnetic field $\hat{n}\cdot\vec{H}_1$ along the drillhole for data as in Fig. 8 b). The dashed line is field component for a homogeneous ground.

Fig. 11 The $C_+$ contour in the complex $\cos\alpha$-plane ($\varkappa=\pi/6$).

Fig. 12 The $\Gamma$ contour ($\varkappa=\pi/6$).

Fig. 13 The $\Gamma$ contour in the complex $\cos\alpha$-plane

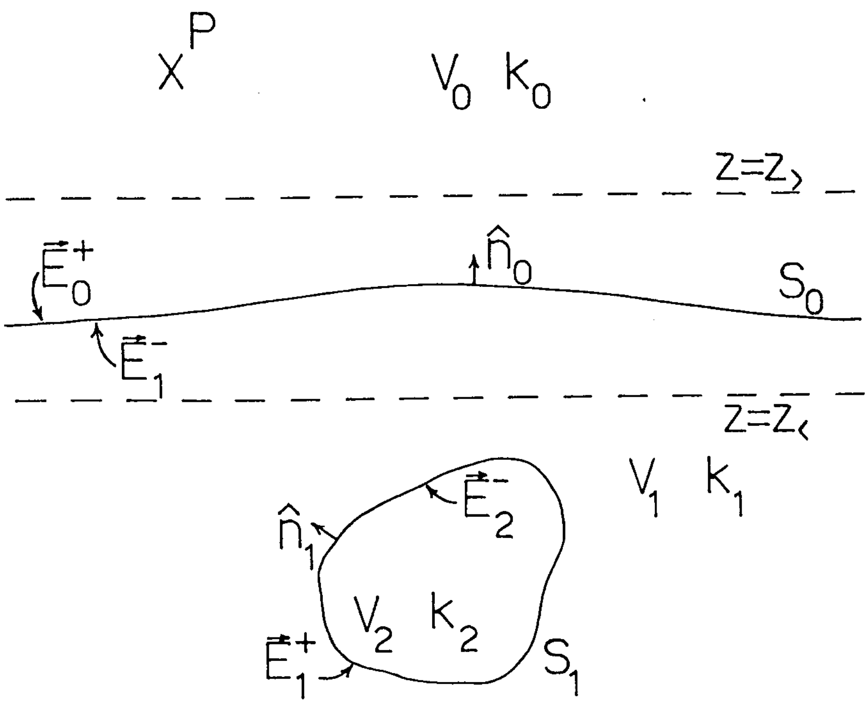
Fig. 14 Notations of angles in the rotated coordinate system.

Fig. 15 The variation in $\cos\eta$ as a function of $\alpha$ for fixed $\theta$ ($\alpha$ varies along the complex part of $\Gamma$ ; the dotted line is the variation of $\alpha$ along $\alpha=\pi/2-i\nu$, $\nu=[0,\alpha_0'']$; $\theta=5\pi/12$, $\varkappa=\pi/6$ a) $\cos\beta=-1$, b) $\cos\beta=-1/2$, c) $\cos\beta=0$, d) $\cos\beta=1/2$, e) $\cos\beta=1$.

Fig. 16 The variation of $\eta$ as a function of $\alpha$ for fixed $\theta$. For data see Fig. 16.

Fig. 17 The variation of $\psi$ as a function of $\beta$ for fixed $\alpha$ ($\theta=5\pi/12$, $\varkappa=\pi/6$, $\sinh\alpha''=-1/\cos\theta$, $\tan\alpha'=-\tanh\alpha''/\tan\varkappa$ ).

Figure 1



P
$V_0$ $k_0$
$z=z_>$
$\vec{E}_0^+$
$\hat{n}_0$
$S_0$
$\vec{E}_1^-$
$z=z_<$
$\vec{E}_2^-$
$V_1$ $k_1$
$\hat{n}_1$
$V_2$ $k_2$
$\vec{E}_1^+$
$S_1$

Figure 2

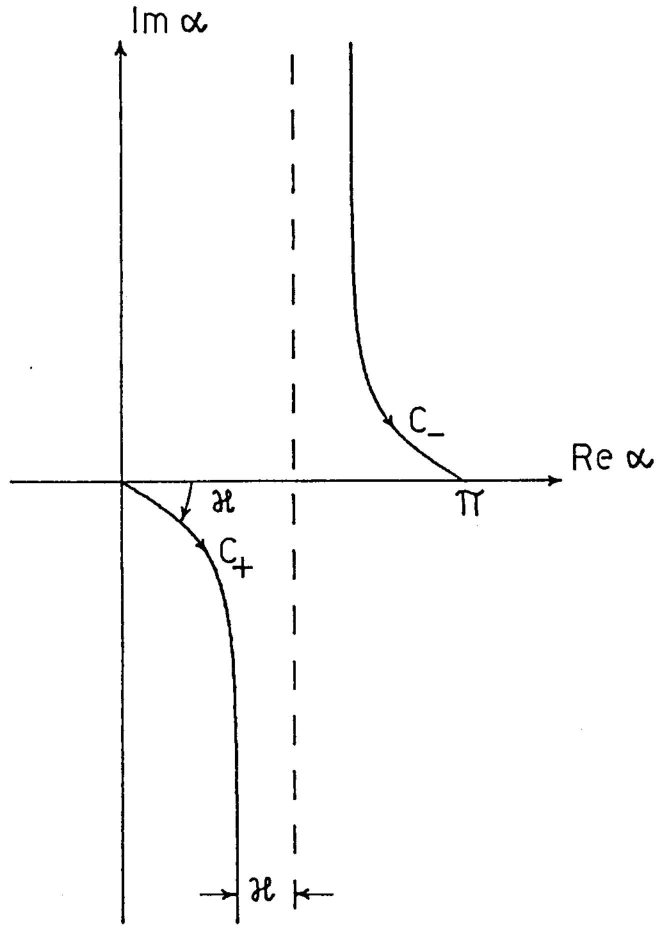


Figure 3

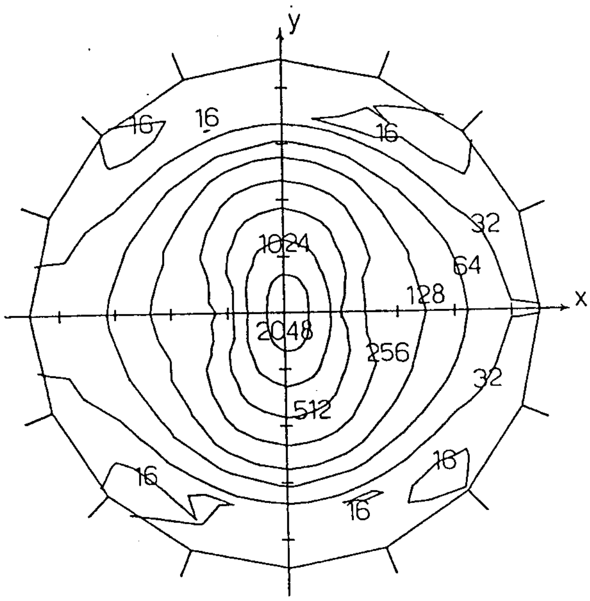



Figure 4

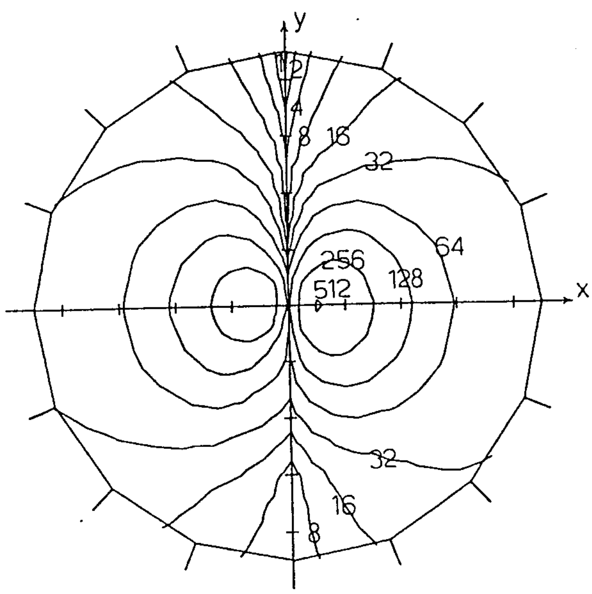



Figure 5

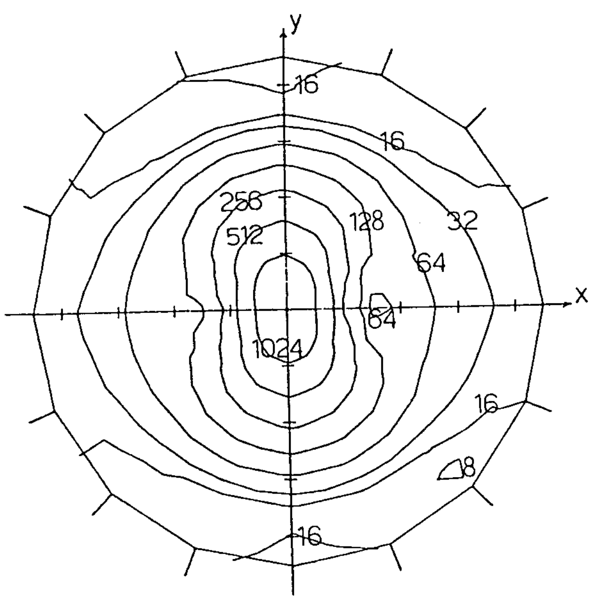



Figure 6

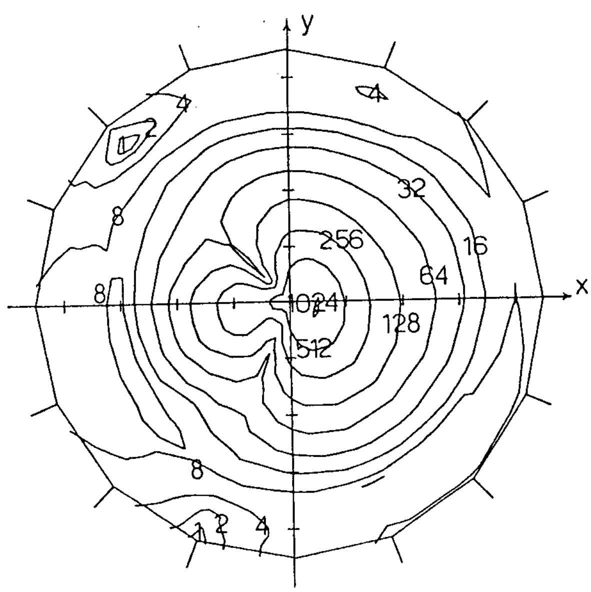


Figure 7

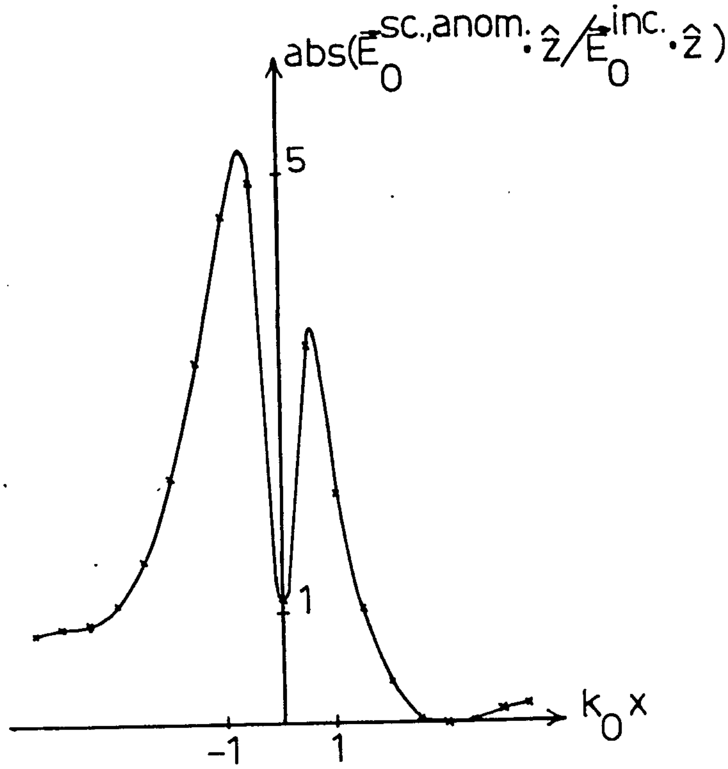
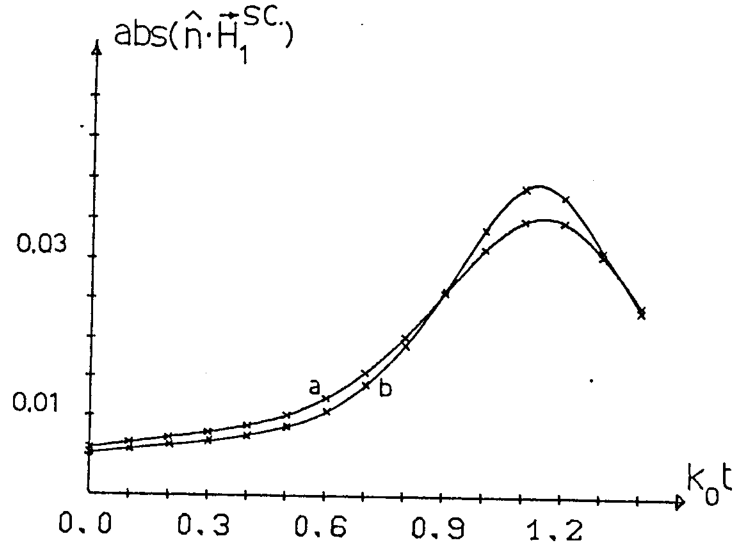
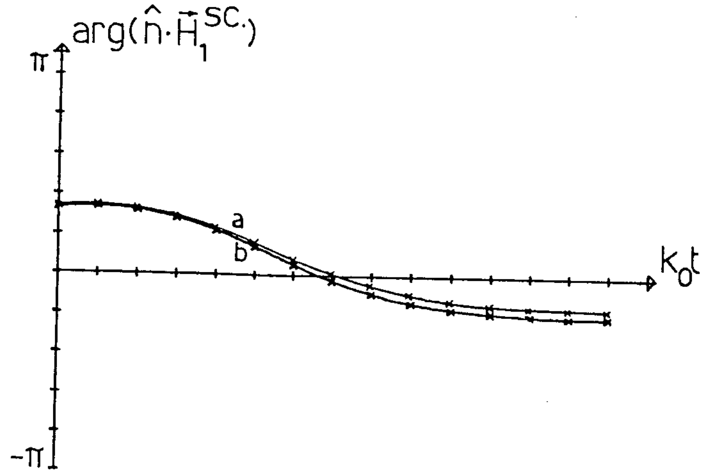


Figure 8

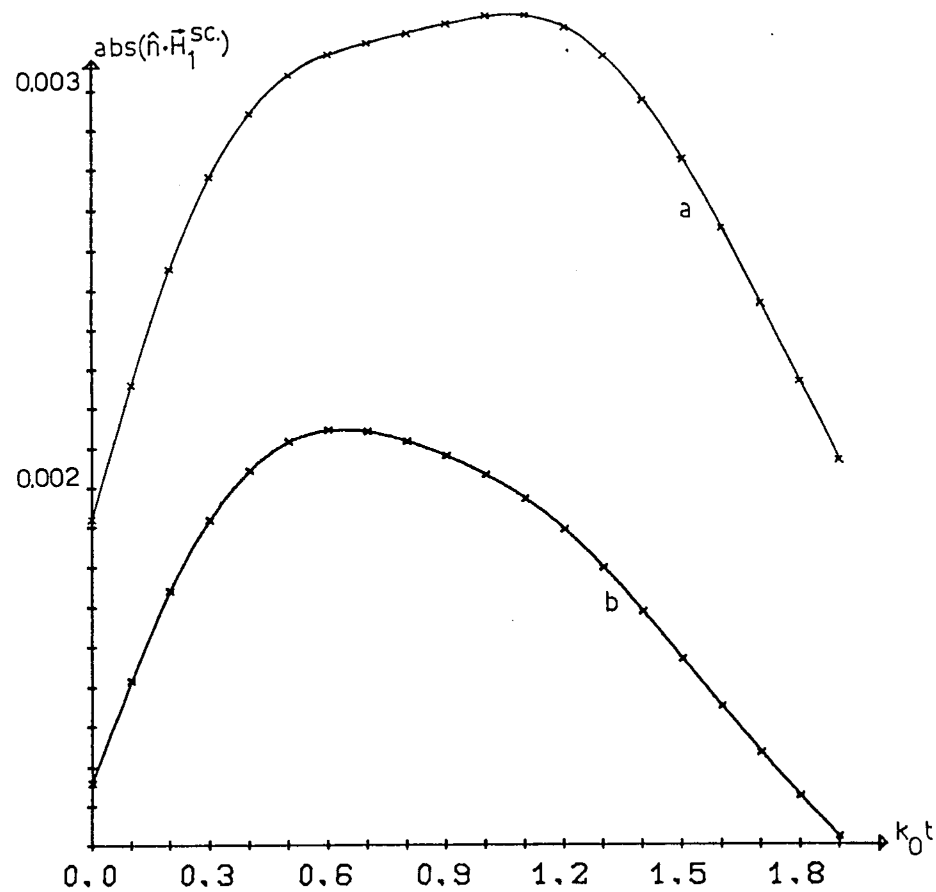


abs($\hat{n}\cdot\vec{H}_1^{sc.}$)
0.03
0.01
a
b
$k_0t$
0.0
0.3
0.6
0.9
1.2

arg($\hat{n}\cdot\vec{H}_1^{sc.}$)
π
a
b
$k_0t$
-π

Figure 9



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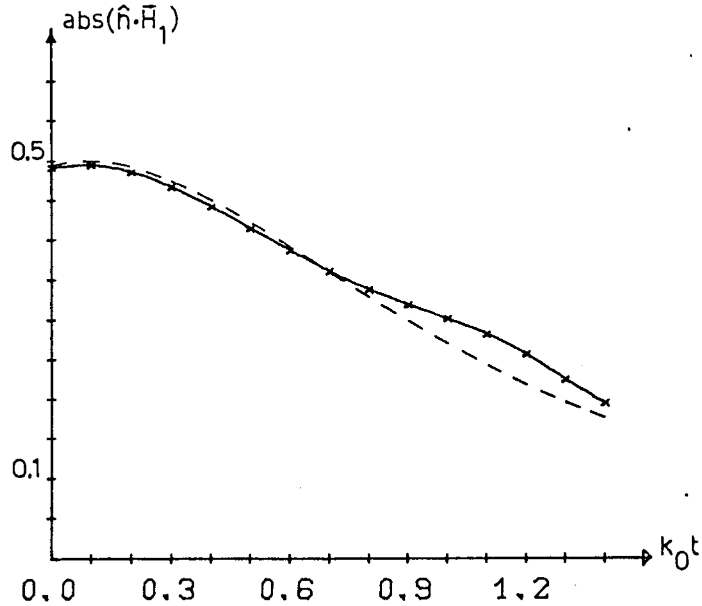


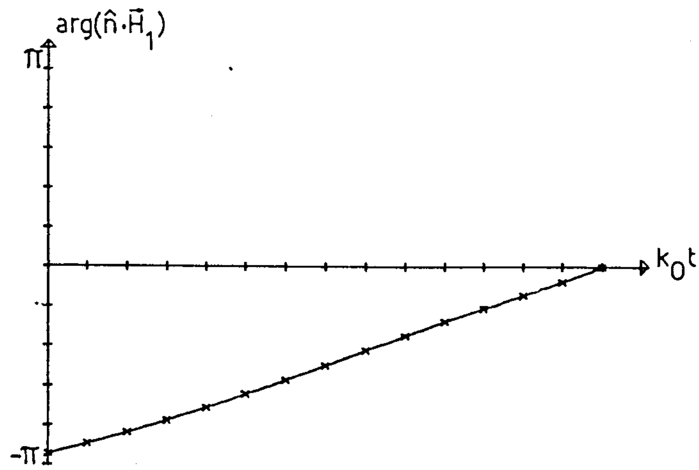



Figure 11

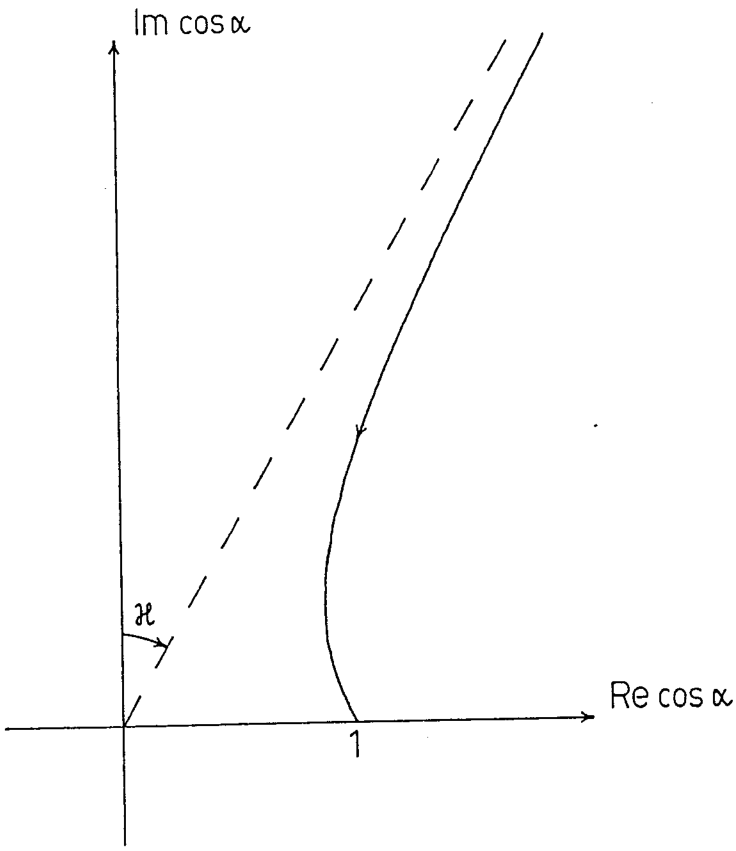


Figure 12

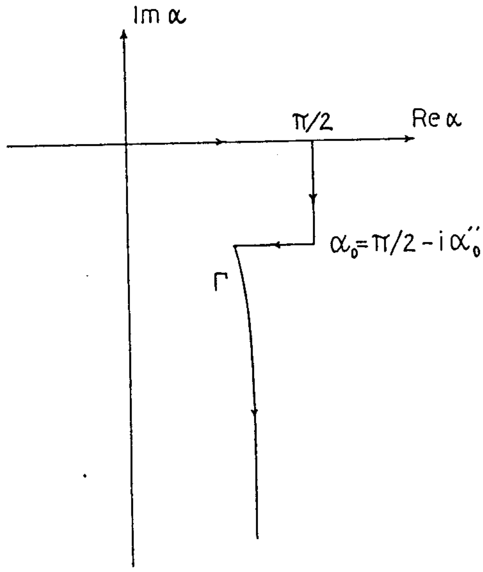



Figure 13

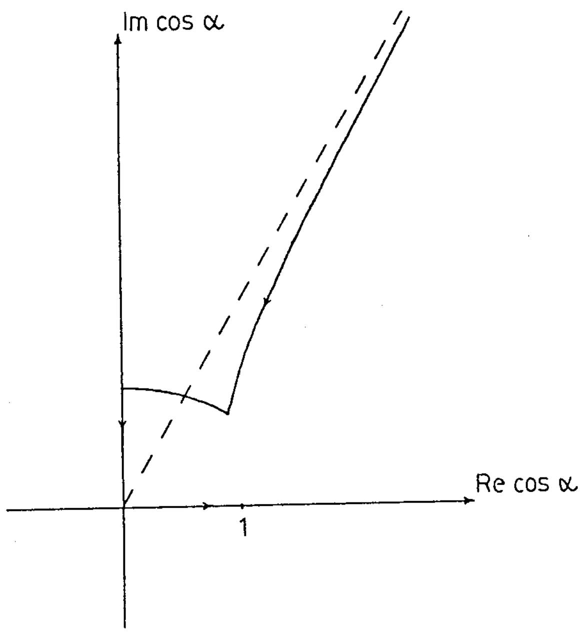


Figure 14

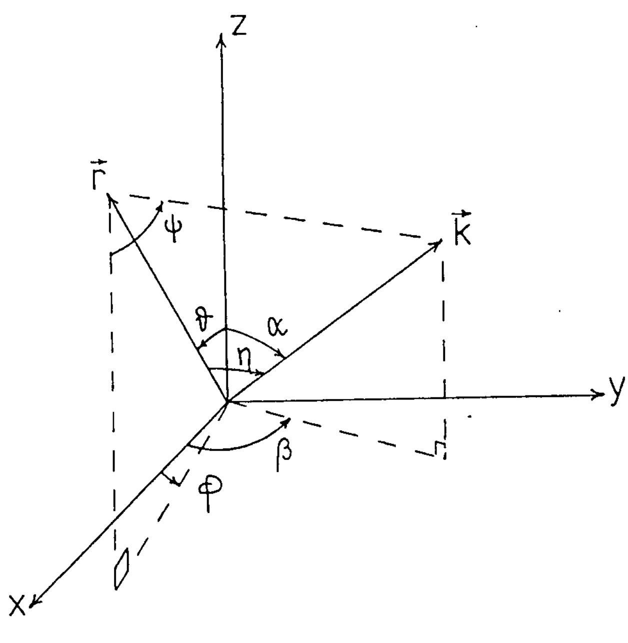


Figure 15

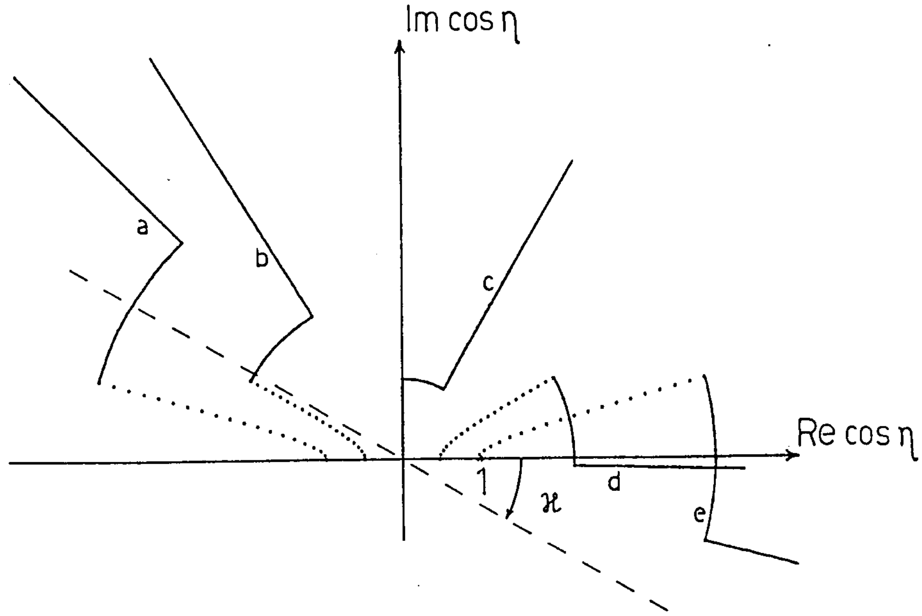



Figure 16

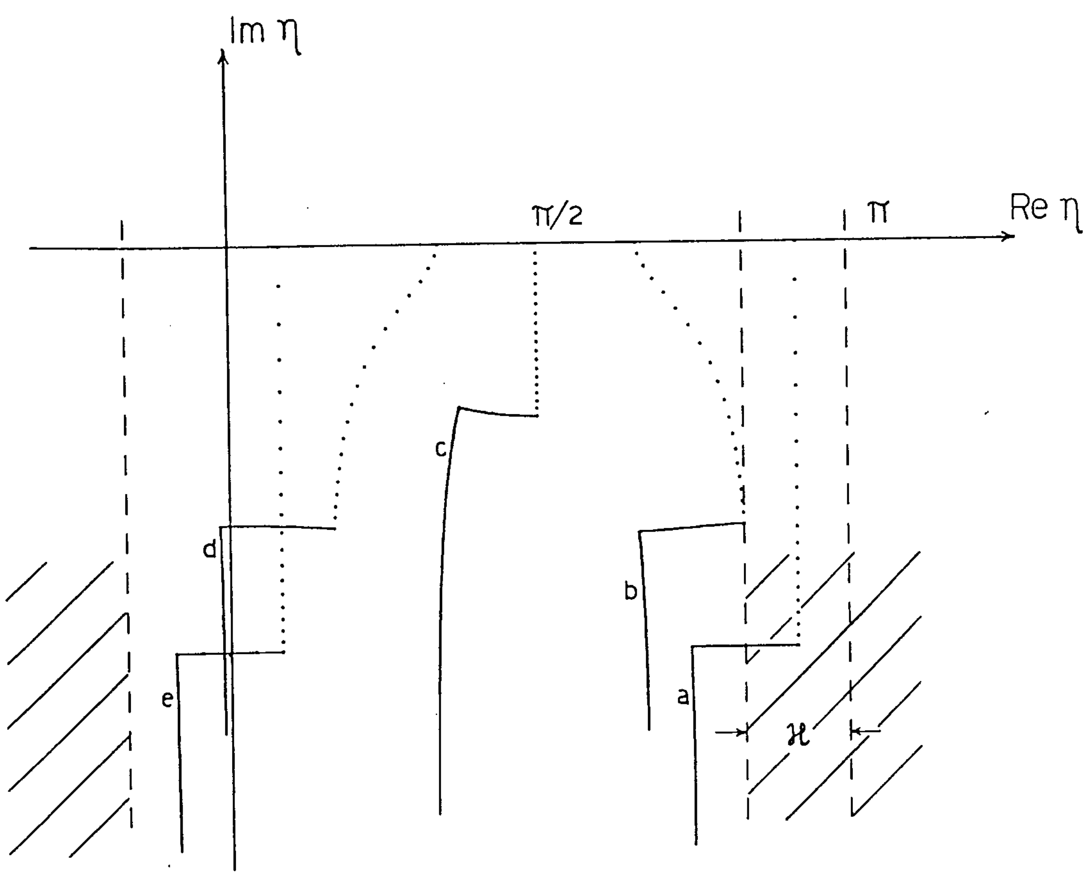
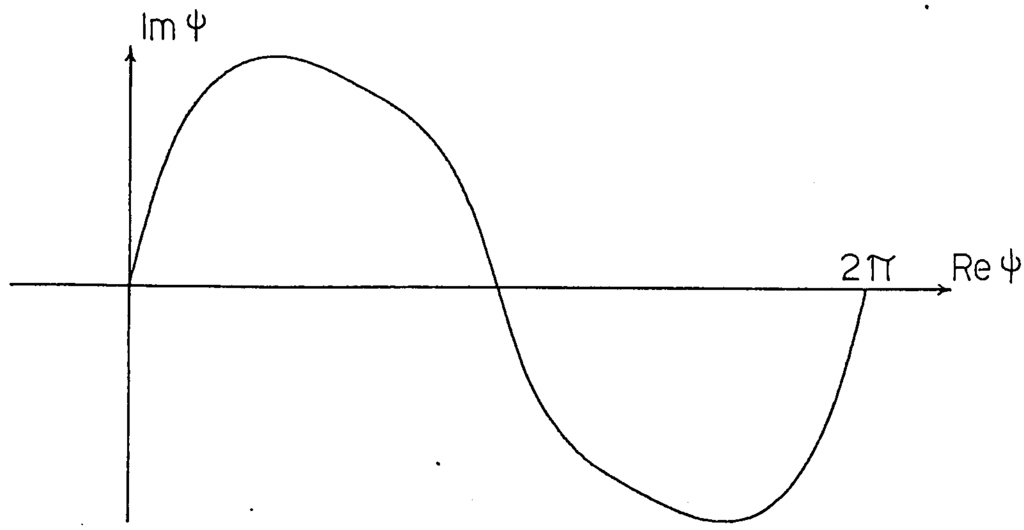


Figure 17



# Paper IV

79-33

October 1979

# Elastic wave scattering by a three-dimensional inhomogeneity in an elastic half space[†]

by

Anders Boström and Gerhard Kristensson

[†]This work is supported by the National Swedish Board for Technical Development (STU).

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Abstract

In this paper we will consider scattering of elastic waves in a half space. The half space is an isotropic, linear and homogeneous medium except for a finite inhomogeneity. The T-matrix method (also called the "extended boundary condition method" or "null field approach") is extended to derive expressions for the elastic field inside the half space and the surface field on the interface. The assumptions on the source that excites the half space are fairly weak. In the numerical applications found in this paper we assume a Rayleigh surface wave to be the incoming field, and we only compute the surface displacements. We make illustrations on some simple types of scatterers (spheres and spheroids; the latter ones can be arbitrarily oriented).

## I Introduction

In this paper we consider scattering of stationary elastic waves from a buried inhomogeneity. We will adopt a formalism recently developed for scattering of stationary scalar (acoustic) and electromagnetic waves in a geometry similar to the one treated here [1-2]. This formalism is an extension of the T-matrix method (also called the "extended boundary condition method" or "null field approach") first given by Waterman [3-5] (for the elastic case see also Ref. 6) for scattering from finite scatterers.

To our knowledge, very few results exist for elastic waves, apart from purely numerical ones, which apply to the geometry we consider in this paper. In a number of papers Datta et al. [7-8] analyse the two-dimensional scattering of elastic waves by a crack (buried or edge) and a cylindrical inhomogeneity in a half space. Some authors have also considered the two-dimensional scattering of a pulsed Rayleigh surface wave by a cylindrical obstacle, see Ref. 9 and further references given there.

Scattering of elastic waves in a layered half space (no finite inhomogeneities present) has been studied intensively and a number of textbooks [10-11] analyse this scattering problem thoroughly. Different types of incoming fields, which excite the layered half space, and their effects upon scattering, are usually considered in detail. Furthermore, the timedependent scattering solution of a layered earth structure is of primary interest in many applications and also this topic is thoroughly studied in these textbooks.

The formalism considered in this paper starts by applying a surface integral representation of the field. The free space

Green's dyadic is expanded in both plane and spherical vector waves. The pertinent surface fields are expanded in these basic waves; the plane wave system applies to the infinite interface, while the spherical waves are more adapted to the finite inhomogeneity. The surface integral representation will give us a matrix equation, and the formal inversion of the matrix in a power series can be interpreted as the multiple scattering contributions from the interface and the obstacle.

In this paper we assume that the sources are located below the interface, but no further assumptions are made. A similar formulation can be derived when we have the sources located above the interface or inside the inhomogeneity. However, we will not pursue these situations any further in this context, but refer to Refs. 1-2 for some additional details concerning these types of source locations. Since losses usually are fairly small for elastic waves in the ground, we consider in this paper only the lossless case. However, an analogous formulation can be derived when losses are present, and as found in the electromagnetic case [12], losses do not introduce any further complications. In modelling elastic wave propagation in the ground it is usually a good approximation to take the upper half space as vacuum. The general case, when we have elastic waves propagating in the upper as well as the lower one, can with appropriate modifications be obtained from this specialized situation and we emphasize that the formal structure of the theory remains unchanged. Furthermore, we will derive the basic equations when the inhomogeneity is a cavity, but any obstacle can be considered, as long as its T-matrix is known (for more details see Refs. 5, 6, 13 and 14).

## II Basis functions

In this preparatory section we introduce the vector basis functions corresponding to the equation of motion for the displacement $\vec{u}$ in an elastic solid

$$(1/k_p^2)\,\nabla\nabla\cdot\vec{u} - (1/k_s^2)\,\nabla\times\nabla\times\vec{u} + \vec{u} = \vec{0} \tag{1}$$

We have here assumed stationary conditions with the time factor $\exp(-i\omega t)$ suppressed. The transverse and longitudinal wave numbers are $k_s^2=\rho\omega^2/\mu$ and $k_p^2=\rho\omega^2/(\lambda+2\mu)$, respectively, where $\rho$ is the density and $\lambda$ and $\mu$ are the Lamé parameters of the elastic solid.

We will need both the plane and spherical vector basis functions of Eq. (1), and also the transformations between these two sets. The plane waves are the natural set for expanding the Green's dyadic and the surface field in the integral over the infinite surface in the integral representation and the spherical waves are in the same way natural when dealing with the bounded inhomogeneity.

We define the plane vector waves to be

$$\begin{aligned}
\vec{\phi}_1(\hat{\gamma};\vec{r}) &\equiv \hat{\alpha}/4\pi \;\exp(ik_s\hat{\gamma}\cdot\vec{r})\\
\vec{\phi}_2(\hat{\gamma};\vec{r}) &\equiv \hat{\beta}/4\pi \;\exp(ik_s\hat{\gamma}\cdot\vec{r})\\
\vec{\phi}_3(\hat{\gamma};\vec{r}) &\equiv \hat{\gamma}/4\pi \;(k_p/k_s)^{3/2}\exp(ik_p\hat{\gamma}\cdot\vec{r})
\end{aligned} \tag{2}$$

where $\hat{\gamma}=(\sin\alpha\cos\beta,\ \sin\alpha\ \sin\beta,\ \cos\alpha)$ and where $\hat{\alpha}$ and $\hat{\beta}$ are the other two spherical unit vectors. $\beta$ belongs to the real interval $[0,2\pi]$ and $\alpha$ belongs to some contour in the complex plane (see further below). The first of the basis functions is transverse, vertically polarized (SV), the second is transverse, horizontally polarized (SH), and the third is longitudinal (P). We will further need the functions $\vec{\phi}_j^{\dagger}(\hat{\gamma};\vec{r})$ which have the i in the exponent changed to -i.

The outgoing spherical vector waves are defined as

$$\vec{\Psi}_{1\sigma m\ell}(\vec{r}) \equiv [\ell(\ell+1)]^{-1/2}\,\nabla\times[\vec{r}\,h_\ell^{(1)}(k_s r)\,Y_{\sigma m\ell}(\hat{r})] = \vec{A}_{1\sigma m\ell}(\hat{r})\,h_\ell^{(1)}(k_s r)$$

$$\vec{\Psi}_{2\sigma m\ell}(\vec{r}) \equiv [\ell(\ell+1)]^{-1/2}(1/k_s)\nabla\times\nabla\times[\vec{r}\,h_\ell^{(1)}(k_s r)\,Y_{\sigma m\ell}(\hat{r})] = \vec{A}_{2\sigma m\ell}(\hat{r})\,\frac{d}{d(k_s r)}[k_s r\,h_\ell^{(1)}(k_s r)]/k_s r + [\ell(\ell+1)]^{1/2}\vec{A}_{3\sigma m\ell}(\hat{r})\,h_\ell^{(1)}(k_s r)/k_s r \qquad (3)$$

$$\vec{\Psi}_{3\sigma m\ell}(\vec{r}) \equiv (k_p/k_s)^{3/2}(1/k_p)\nabla[h_\ell^{(1)}(k_p r)\,Y_{\sigma m\ell}(\hat{r})] = (k_p/k_s)^{1/2}\left\{\frac{d\,h_\ell^{(1)}(k_p r)}{d(k_p r)}\vec{A}_{3\sigma m\ell}(\hat{r}) + [\ell(\ell+1)]^{1/2}\vec{A}_{2\sigma m\ell}(\hat{r})\,\frac{h_\ell^{(1)}(k_p r)}{k_p r}\right\}$$

where $h_\ell^{(1)}$ is the spherical Hankel function of the first kind. The normalized real spherical harmonics are:

$$Y_{\sigma m\ell}(\hat{r}) \equiv (-1)^m\sqrt{\varepsilon_m\,\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\;P_\ell^m(\cos\theta)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix}$$

$\tau=1,2,3$, $\sigma=e,o$, $m=0,1,\ldots,\ell$, $\ell=0,1,\ldots$, $\varepsilon_m=2-\delta_{m0}$, and the vector spherical harmonics are:

$$\vec{A}_{1\sigma m\ell}(\hat{r}) \equiv [\ell(\ell+1)]^{-1/2}\,\nabla\times[\vec{r}\,Y_{\sigma m\ell}(\hat{r})] = -\hat{r}\times\vec{A}_{2\sigma m\ell}(\hat{r})$$

$$\vec{A}_{2\sigma m\ell}(\hat{r}) \equiv [\ell(\ell+1)]^{-1/2}\,r\nabla Y_{\sigma m\ell}(\hat{r}) = \hat{r}\times\vec{A}_{1\sigma m\ell}(\hat{r}) \tag{4}$$

$$\vec{A}_{3\sigma m\ell}(\hat{r}) \equiv \hat{r}\,Y_{\sigma m\ell}(\hat{r})$$

When convenient we will abbreviate the indices as follows:

$$\vec{\psi}_n \equiv \vec{\psi}_{\tau\sigma m\ell} \qquad\qquad \vec{A}_n \equiv \vec{A}_{\tau\sigma m\ell}$$

We will also need the regular functions $\mathrm{Re}\vec{\psi}_n$ obtained by taking spherical Bessel rather than Hankel functions.

Transformations between plane and spherical vector waves are given by Danos and Maximon [15] for the scalar case (i.e. the longitudinal part) and Devaney and Wolf [16] for the transverse part. Here we use the following compact way of writing these transformations:

$$\vec{\varphi}_j(\hat{\gamma};\vec{r}) = -i\sum_n B^{\dagger}_{nj}(\hat{\gamma})\,\mathrm{Re}\vec{\psi}_n(\vec{r}) \tag{5}$$

$$\mathrm{Re}\vec{\psi}_n(\vec{r}) = i\sum_j\int_0^{\pi} d\hat{\gamma}\; B_{nj}(\hat{\gamma})\,\vec{\varphi}_j(\hat{\gamma};\vec{r}) \tag{6}$$

$$\vec{\psi}_n(\vec{r}) = 2i\sum_j\int_{C_\pm} d\hat{\gamma}\,B_{nj}(\hat{\gamma})\,\vec{\varphi}_j(\hat{\gamma};\vec{r}) \qquad z \gtrless 0 \tag{7}$$

The sum over n indicates a sum over $\tau$, $\sigma$, m and $\ell$ and j is summed from 1 to 3. The integrals are in fact double integrals; we have introduced the shorthand notation $\int d\hat{\gamma} \equiv \int_0^{2\pi} d\beta \int \sin\alpha d\alpha$. The integration contours $C_+$ and $C_-$ are given in Fig. 1, $C_+$ is relevant if $z>0$ and $C_-$ if $z<0$. The transformation functions are essentially the components of the vector spherical harmonics [2):]

$$B_{n1}(\hat{\gamma}) \equiv i^{-\ell-\delta_{\tau,1}} \hat{\alpha}\cdot\vec{A}_n(\hat{\gamma})$$
$$B_{n2}(\hat{\gamma}) \equiv i^{-\ell-\delta_{\tau,1}} \hat{\beta}\cdot\vec{A}_n(\hat{\gamma}) \tag{8}$$
$$B_{n3}(\hat{\gamma}) \equiv i^{-\ell-\delta_{\tau,1}} \hat{\gamma}\cdot\vec{A}_n(\hat{\gamma})$$

and $B^{\dagger}_{nj}$ has the first factor changed to $i^{\ell+\delta_{\tau,1}}$. We note that $B_{nj}$ is zero for $\tau=3$ and $j=1,2$ or $\tau=1,2$ and $j=3$ so that the transverse and longitudinal waves do not couple - as of course they cannot.

The free space Green's dyadic corresponding to Eq. (1) satisfies ($\overleftrightarrow{I}$ is the unit dyadic)

$$(1/k_p^2)\nabla\nabla\cdot\overleftrightarrow{G}(\vec{r},\vec{r}\,') - (1/k_s^2)\nabla\times\nabla\times\overleftrightarrow{G}(\vec{r},\vec{r}\,') + \overleftrightarrow{G}(\vec{r},\vec{r}\,') = -1/k_s^3\overleftrightarrow{I}\,\delta(\vec{r}-\vec{r}\,') \tag{9}$$

and radiation conditions at infinity. By considering the transverse and longitudinal parts separately it is easy to obtain the expansion of the Green's dyadic in plane vector waves (see also Ref. 2)

$$\overleftrightarrow{G}(\vec{r},\vec{r}\,') = 2i\sum_j \int_{C_\pm} d\hat{\gamma}\, \vec{\Phi}_j(\hat{\gamma};\vec{r})\,\vec{\Phi}^{\dagger}_j(\hat{\gamma};\vec{r}\,') \qquad z \gtrless z' \tag{10}$$

The corresponding expansion in spherical vector waves can be found in e.g. Ref. 13:

$$\overleftrightarrow{G}(\vec{r},\vec{r}') = i\sum_n \mathrm{Re}\vec{\psi}_n(\vec{r}_<)\,\vec{\psi}_n(\vec{r}_>) \tag{11}$$

Apart from the Green's dyadic we will also use the Green's stress triadic - or rather the Green's surface traction dyadic. It has the same relation to the Green's dyadic as the surface traction has to the displacement (cf. Eq. (16)):

$$\hat{n}\cdot\overline{\overline{\Sigma}} = \lambda\hat{n}\,\nabla\cdot\overleftrightarrow{G} + 2\mu\frac{\partial}{\partial n}\overleftrightarrow{G} + \mu\hat{n}\times(\nabla\times\overleftrightarrow{G}) \tag{12}$$

where $\hat{n}$ is the normal to the surface under consideration. From this equation and Eqs. (10) and (11) we can then obtain the expansions of the Green's surface traction dyadic in plane and spherical vector waves (cf. the symmetry properties of $\overleftrightarrow{G}$):

$$\hat{n}\cdot\overline{\overline{\Sigma}}(\vec{r},\vec{r}') = 2i\sum_j\int_{C_\pm} d\hat{\gamma}\;\vec{t}(\vec{\phi}_j^{\dagger}(\hat{\gamma};\vec{r}'))\,\vec{\phi}_j(\hat{\gamma};\vec{r}) \qquad z \gtrless z' \tag{13}$$

$$\begin{aligned}\hat{n}\cdot\overline{\overline{\Sigma}}(\vec{r},\vec{r}') &= i\sum_n \vec{t}(\vec{\psi}_n(\vec{r}'))\,\mathrm{Re}\vec{\psi}_n(\vec{r}) && r<r' \\ &= i\sum_n \vec{t}(\mathrm{Re}\vec{\psi}_n(\vec{r}'))\,\vec{\psi}_n(\vec{r}) && r'<r\end{aligned} \tag{14}$$

where (cf. Eqs. (12) and (16))

$$\vec{t}(\vec{\psi}_n) = \lambda\hat{n}\,\nabla\cdot\vec{\psi}_n + 2\mu\frac{\partial}{\partial n}\vec{\psi}_n + \mu\hat{n}\times(\nabla\times\vec{\psi}_n)$$

and similarly for $\vec{t}(\vec{\phi}_j^{\dagger})$ and $\vec{t}(\mathrm{Re}\vec{\psi}_n)$.

## III General formalism

This section contains a more detailed description of the geometry and develops the fundamental equations of the formalism. The formalism is in many respects a generalization of the similar acoustic and electromagnetic scattering problems [1-2]. However, a number of problems not appearing in these cases emerge here, and these will be discussed in the appropriate contexts. With a few exceptions we use the notations and symbols (or generalized versions of these) found in Refs. 1-2.

We study a geometry of the scattering problem as depicted in Figure 2. We consider two half spaces $V_o$ and $V_1$ whose interface is $S_o$ and for simplicity we take $V_o$ as vacuum. The volume $V_1$ is assumed to consist of an isotropic and linear medium of Lamé parameters $\lambda$, $\mu$, density $\rho$ and wave numbers $k_p$ and $k_s$. Furthermore it is assumed homogeneous, except for a finite inhomogeneity $V_2$ (bounded by $S_1$), which we for simplicity take as a cavity. We assume the surface $S_o$ to be bounded by two parallel planes $z=z_>$ and $z=z_<$, and the direction perpendicular to these planes is defined as the z-direction. No further assumptions on the surfaces $S_o$ and $S_1$ are made, besides the rather implicit assumption that they are sufficiently regular to allow an application of the divergence theorem on every finite part of $V_1$. The formalism does not depend explicitly on the location of the origin, but we assume we can pick an origin inside $V_2$ and furthermore such that $0<z_<$.

This simplified situation (but still sufficiently general for the structure of the basic equations to be the same as in the general case), which we analyse here have only wave propagation in one of the volumes, namely $V_1$. Thus we study the displacement field $\vec{u}=\vec{u}^i+\vec{u}^{sc}$. in $V_1$, where the incoming field $\vec{u}^i$ has sources in $V_1$, and the boundary conditions appropriate to the present problem:

$$\begin{cases} \vec{t}_-(\vec{r}') = \vec{0} & \text{on } S_0 \\ \vec{t}_+(\vec{r}') = \vec{0} & \text{on } S_1 \end{cases} \qquad (15)$$

where the surface traction $\vec{t}$ is defined as:

$$\vec{t}(\vec{r}) = \lambda \hat{n}\, \nabla\cdot\vec{u}(\vec{r}) + 2\mu \frac{\partial}{\partial n}\vec{u}(\vec{r}) + \mu \hat{n}\times(\nabla\times\vec{u}(\vec{r})) \qquad (16)$$

The basic equations of the formalism are derived from the following integral representation [6]:

$$\vec{u}^i(\vec{r}) - \frac{k_s}{\mu}\iint_{S_0} \vec{u}_-(\vec{r}')\cdot(\hat{n}\cdot\overline{\overline{\Sigma}}(\vec{r},\vec{r}'))\,dS' + \frac{k_s}{\mu}\iint_{S_1} \vec{u}_+(\vec{r}')\cdot(\hat{n}\cdot\overline{\overline{\Sigma}}(\vec{r},\vec{r}'))\,dS' = \begin{cases} \vec{u}(\vec{r}) & \vec{r}\in V_1 \\ \vec{0} & \vec{r}\notin V_1 \end{cases} \qquad (17)$$

Here the boundary conditions have already been applied. The definition of the Green's surface traction is given by Eq. (12). In the derivation of Eq. (17) we have assumed the displacement field $\vec{u}$ to satisfy appropriate radiation conditions, which eventually will give zero contribution from a surface integral over a lower half sphere as the radius increases to infinity.

The next step in the formalism will be the introduction of suitable expansions of the surface fields $\vec{u}_\pm$. Since our special choice of boundary conditions, cf. Eq. (15), does not introduce any derivatives of the surface fields, we have a greater freedom in choosing suitable expansion systems. The expansions here adopted are:

$$\vec{u}_+(\vec{r}') = \sum_n \alpha_n \operatorname{Re}\vec{\psi}_n(\vec{r}') \tag{18}$$

$$\vec{u}_-(\vec{r}') = \sum_j \int_{C_+} d\hat{\gamma}\, \alpha_j(\hat{\gamma})\, \vec{\phi}_j(\hat{\gamma};\vec{r}') \tag{19}$$

This is an expansion in regular spherical vector waves on the surface $S_1$ and a plane vector wave expansion for the surface field on $S_o$. For the plane wave expansion we have chosen to expand in up-going plane waves, i.e. an expansion on the contour $C_+$.

The prescribed incoming field can also be expanded in suitable expansion systems. We apply these expansions in two regions; one near the origin and one above the plane $z=z_>$. In these regions we have the following convergent expansions of the incoming field (which has its sources located in $V_1$):

$$\vec{u}^i(\vec{r}) = \sum_n a_n \operatorname{Re}\vec{\psi}_n(\vec{r}) \qquad r < \min_{\vec{r}' \in S_1} |\vec{r}'| \tag{20}$$

$$\vec{u}^i(\vec{r}) = \sum_j \int_{C_+} d\hat{\gamma}\, a_j(\hat{\gamma})\, \vec{\phi}_j(\hat{\gamma};\vec{r}) \qquad z > z_> \tag{21}$$

The expansion coefficients $a_n$ and $a_j(\hat{\gamma})$ can be considered as prescribed and our primary interest in this paper will be to find an expression for the surface field $\vec{u}_-$ (or the amplitude

$\alpha_j(\hat{\gamma})$ in Eq. (19)).

Consider now an $\vec{r}$ inside the inscribed sphere of $S_1$ and such that $r<z_<$. In the integral representation Eq. (17) we insert the appropriate expansions of the Green's surface traction dyadic, see Eqs. (13)-(14) and furthermore we use the surface field and incoming field expansions, see Eqs. (18)-(20), and get:

$$\sum_n a_n \mathrm{Re}\vec{\Psi}_n(\vec{r}) = i\sum_{jj'} \int_{C_-} d\hat{\gamma}\, \vec{\Phi}_j(\hat{\gamma};\vec{r}) \int_{C_+} d\hat{\gamma}'\, \alpha_{j'}(\hat{\gamma}')\, Q_{jj'}(\hat{\gamma},\hat{\gamma}') - \\ - i\sum_{nn'} Q_{nn'}\alpha_{n'}\, \mathrm{Re}\vec{\Psi}_n(\vec{r}) \tag{22}$$

Here we have introduced the $Q_{nn'}$-matrix for the surface $S_1$ and the corresponding quantity $Q_{jj'}(\hat{\gamma},\hat{\gamma}')$ for the infinite surface $S_o$. They are defined as:

$$Q_{nn'} \equiv \frac{k_s}{\mu} \iint_{S_1} \mathrm{Re}\vec{\Psi}_{n'}(\vec{r}') \cdot \vec{t}\,(\vec{\Psi}_n(\vec{r}'))\, dS' \tag{23}$$

$$Q_{jj'}(\hat{\gamma},\hat{\gamma}') \equiv 2\frac{k_s}{\mu} \iint_{S_0} \vec{\Phi}_{j'}(\hat{\gamma}';\vec{r}') \cdot \vec{t}\,(\vec{\Phi}_j^{\dagger}(\hat{\gamma};\vec{r}'))\, ds' \tag{24}$$

For an $\vec{r}$ above $S_o$ satisfying $z>z_>$ and $|\vec{r}|>\max_{\vec{r}'\in S_1}|\vec{r}'|$ we can again apply the integral representation Eq. (17) and we get as above:

$$\sum_j \int_{C_+} d\hat{\gamma}\, a_j(\hat{\gamma})\vec{\Phi}_j(\hat{\gamma};\vec{r}) = i\sum_{jj'} \int_{C_+} d\hat{\gamma}\, \vec{\Phi}_j(\hat{\gamma};\vec{r}) \int_{C_+} d\hat{\gamma}'\, \alpha_{j'}(\hat{\gamma}')\, Q_{jj'}(\hat{\gamma},\hat{\gamma}') - \\ - i\sum_{nn'} \mathrm{Re}\,Q_{nn'}\alpha_{n'}\, \vec{\Psi}_n(\vec{r}) \tag{25}$$

Here the $ReQ_{nn'}$-matrix is defined as in Eq. (23) but with regular functions in both places. We note that $Q_{jj'}(\hat{\gamma},\hat{\gamma}')$ appearing in Eqs. (22) and (25) have different ranges for $\hat{\gamma}$ in the two equations.

The next step is to eliminate the $\vec{r}$ dependence in the Eqs. (22) and (25), so as to get two relations between the unknown coefficients $\alpha_j(\hat{\gamma})$ and $\alpha_n$. To obtain these relations we make transformations between the spherical and plane wave system. Thus in Eq. (25) we introduce the transformation between the outgoing spherical and plane vector waves, see Eq. (7), and in Eq. (22) the transformation between the plane and the regular spherical vector waves, see Eq. (5). By the linear independence of the basic expansion systems we get (notice that the integral contour $\int_{C_\pm} d\hat{\gamma}$ is essentially a two-dimensional Fourier integral in the horizontal components of $\hat{\gamma}$ and we merely apply the two-dimensional invers transform on Eq. (25)):

$$a_n = \sum_{jj'} \int_{C_-} d\hat{\gamma}\, B^{\dagger}_{nj}(\hat{\gamma}) \int_{C_+} d\hat{\gamma}'\, \alpha_{j'}(\hat{\gamma}')\, Q_{jj'}(\hat{\gamma},\hat{\gamma}') - i\sum_{n'} Q_{nn'}\alpha_{n'} \tag{26}$$

$$a_j(\hat{\gamma}) = i\sum_{j'} \int_{C_+} d\hat{\gamma}'\, \alpha_{j'}(\hat{\gamma}')\, Q_{jj'}(\hat{\gamma},\hat{\gamma}') + 2\sum_{nn'} B_{nj}(\hat{\gamma})\, ReQ_{nn'}\alpha_{n'} \qquad \hat{\gamma}\in C_+ \tag{27}$$

These two equations give the relation between the two unknowns $\alpha_j(\hat{\gamma})$ and $\alpha_n$, so formally the surface field $\vec{u}_-$ can be computed by an inversion of these equations. In the general case of a rough interface $S_o$ this is a laborious problem, and we will in this context only give the formal solution of the equations. However, in the plane surface case we can give more explicit expressions.

Finally we focus on the scattered field $\vec{u}^{sc.}$ in $V_1$, which is given by the upper part of Eq. (17). We here give the explicit expression of the field in the region below the plane $z=z_<$ and outside the circumscribing sphere of $S_1$. In this region it is possible to introduce the same expansions of the Green's surface traction over the entire surfaces $S_o$ and $S_1$, respectively. We get in analogy with the derivation above:

$$\vec{u}^{sc.}(\vec{r}) = -i\sum_{jj'}\int_{C_-} d\hat{\gamma}\,\vec{\Phi}_j(\hat{\gamma};\vec{r})\int_{C_+} d\hat{\gamma}'\,\alpha_{j'}(\hat{\gamma}')\,Q_{jj'}(\hat{\gamma},\hat{\gamma}') + \\ + i\sum_{nn'}\vec{\psi}_n(\vec{r})\,\mathrm{Re}\,Q_{nn'}\,\alpha_{n'} \tag{28}$$

## IV Computation of the scattered and surface fields

The Eqs. (26) and (27) will in this section be inverted formally and the final expressions for the scattered and surface fields in $V_1$ given. In the later part of the section we will specialize the equations to a flat surface, which is the configuration used in the numerical computations.

Formally we can introduce the inverse of $Q_{jj'}(\hat{\gamma},\hat{\gamma}')$, $\hat{\gamma},\hat{\gamma}'\epsilon C_+$, and solve Eq. (27) for $\alpha_j(\hat{\gamma})$. The Eqs. (26) and (27) can be rewritten:

$$a_n = i \sum_{jj'} \int_{C_-} d\hat{\gamma}\, B^{\dagger}_{nj}(\hat{\gamma}) \int_{C_+} d\hat{\gamma}'\, R_{jj'}(\hat{\gamma},\hat{\gamma}') \Big\{ a_{j'}(\hat{\gamma}') - - 2 \sum_{nn'} B_{nj'}(\hat{\gamma}') \operatorname{Re} Q_{nn'} \alpha_{n'} \Big\} - i \sum_{n'} Q_{nn'} \alpha_{n'} \tag{29}$$

$$\alpha_j(\hat{\gamma}) = -i \sum_{j'} \int_{C_+} d\hat{\gamma}'\, Q^{-1}_{jj'}(\hat{\gamma},\hat{\gamma}') \Big\{ a_{j'}(\hat{\gamma}') - 2 \sum_{nn'} B_{nj'}(\hat{\gamma}') \operatorname{Re} Q_{nn'} \alpha_{n'} \Big\} \quad \hat{\gamma}\epsilon C_+ \tag{30}$$

Here we have introduced the formal reflection coefficient from below for the surface $S_o$:

$$R_{jj'}(\hat{\gamma},\hat{\gamma}') \equiv - \sum_{j''} \int_{C_+} d\hat{\gamma}''\, Q_{jj''}(\hat{\gamma},\hat{\gamma}'')\, Q^{-1}_{j''j'}(\hat{\gamma}'',\hat{\gamma}') \qquad \hat{\gamma}\epsilon C_-,\ \hat{\gamma}'\epsilon C_+ \tag{31}$$

Eq. (29) is now a matrix equation in the unknown surface field amplitudes $\alpha_n$. We can write this equation in a more attractive way, by defining the following spherical projections:

$$d_n \equiv a_n - i \sum_{jj'} \int_{C_-} d\hat{\gamma}\, B^{\dagger}_{nj}(\hat{\gamma}) \int_{C_+} d\hat{\gamma}'\, R_{jj'}(\hat{\gamma},\hat{\gamma}')\, a_{j'}(\hat{\gamma}') \tag{32}$$

$$R_{nn'} \equiv 2 \sum_{jj'} \int_{C_-} d\hat{\gamma}\, B^{\dagger}_{nj}(\hat{\gamma}) \int_{C_+} d\hat{\gamma}'\, R_{jj'}(\hat{\gamma},\hat{\gamma}')\, B_{n'j'}(\hat{\gamma}') \tag{33}$$

$$c_n \equiv -i \sum_{n'} Q_{nn'} \alpha_{n'} \tag{34}$$

We get:

$$c_n - \sum_{n'n''} R_{nn''} T_{n''n'} c_{n'} = d_n \tag{35}$$

The T-matrix $T_{nn'}$ for the scatterer $S_1$ is defined in the conventional way,

$$T_{nn'} \equiv -\sum_{n''} \mathrm{Re}\, Q_{nn''} Q^{-1}_{n''n'} \tag{36}$$

The known quantity $d_n$ defined in Eq. (32) contains two parts, one, $a_n$, which is the expansion coefficients of the incoming field in spherical vector waves, and a second part, which can be interpreted as the incoming field reflected once at the surface $S_o$ from below, and then transformed to the spherical basis by the transformation function $B^{\dagger}_{nj}(\hat{\gamma})$. The matrix quantity $R_{nn'}$ can also be interpreted as a projection of the reflection coefficient $R_{jj'}(\hat{\gamma},\hat{\gamma}')$ on the spherical basis.

We can finally write down the solution of the surface field amplitude $\alpha_j(\hat{\gamma})$ as a function of $c_n$ - the solution of Eq. (35).

$$\alpha_j(\hat{\gamma}) = -i \sum_{j'} \int_{C_+} d\hat{\gamma}'\, Q^{-1}_{jj'}(\hat{\gamma},\hat{\gamma}') \left\{ a_{j'}(\hat{\gamma}') + 2i \sum_{nn'} B_{nj'}(\hat{\gamma}') T_{nn'} c_{n'} \right\} \quad \hat{\gamma} \in C_+ \tag{37}$$

The final expression for the surface field $\vec{u}_-$, see Eq. (19), can then be found, and we also give the expression of the scattered field $\vec{u}^{sc.}$ in the region discussed in connection with Eq. (28).

$$\vec{u}_-(\vec{r}) = \vec{u}_-^{dir.}(\vec{r}) + \vec{u}_-^{anom.}(\vec{r}) = \vec{u}_-^{dir.}(\vec{r}) + \sum_{nn'} \vec{F}_n(\vec{r}) T_{nn'} c_{n'} \tag{38}$$

$$\vec{u}^{sc.}(\vec{r}) = \vec{u}^{sc.,dir.}(\vec{r}) + \vec{u}^{sc.,anom.}(\vec{r}) =$$

$$= \vec{u}^{sc.,dir.}(\vec{r}) + \sum_{nn'} \vec{\mathcal{F}}_n(\vec{r}) T_{nn'} c_{n'} + \sum_{nn'} \vec{\Psi}_n(\vec{r}) T_{nn'} c_{n'} \tag{39}$$

The new fields defined in the equations above are:

$$\vec{u}_-^{dir.}(\vec{r}) \equiv -i \sum_{jj'} \int_{C_+} d\hat{\gamma}\, \vec{\Phi}_j(\hat{\gamma};\vec{r}) \int_{C_+} d\hat{\gamma}'\, Q_{jj'}^{-1}(\hat{\gamma},\hat{\gamma}')\, a_{j'}(\hat{\gamma}') \tag{40}$$

$$\vec{F}_n(\vec{r}) \equiv 2 \sum_{jj'} \int_{C_+} d\hat{\gamma}\, \vec{\Phi}_j(\hat{\gamma};\vec{r}) \int_{C_+} d\hat{\gamma}'\, Q_{jj'}^{-1}(\hat{\gamma},\hat{\gamma}')\, B_{nj'}(\hat{\gamma}') \tag{41}$$

$$\vec{u}^{sc.,dir.}(\vec{r}) \equiv \sum_{jj'} \int_{C_-} d\hat{\gamma}\, \vec{\Phi}_j(\hat{\gamma};\vec{r}) \int_{C_+} d\hat{\gamma}'\, R_{jj'}(\hat{\gamma},\hat{\gamma}')\, a_{j'}(\hat{\gamma}') \tag{42}$$

$$\vec{\mathcal{F}}_n(\vec{r}) \equiv 2i \sum_{jj'} \int_{C_-} d\hat{\gamma}\, \vec{\Phi}_j(\hat{\gamma};\vec{r}) \int_{C_+} d\hat{\gamma}'\, R_{jj'}(\hat{\gamma},\hat{\gamma}')\, B_{nj'}(\hat{\gamma}') \tag{43}$$

We can here identify the fields $\vec{u}_-^{dir.}$ and $\vec{u}^{sc.dir.}$ as the total field and total scattered field, respectively, as if no inhomogeneity were present. The anomalous part of the fields reflects the presence of the inhomogeneity.

In the analysis found above we have explicitly assumed the inhomogeneity to be a cavity. This assumption can readily be relaxed and a more complex inhomogeneity can be considered, e.g. a layered obstacle or an assembly of scatterers, see Refs. 5, 6, 13 and 14. Formally the theory remains identical, but with the appropriate T-matrix inserted.

The expressions above are of limited practical value if the inverse of $Q_{jj'}(\hat{\gamma},\hat{\gamma}')$ cannot be found explicitly, and we therefore give corresponding formulas when we have a plane interface $S_o$, defined by $z=z_o$. In this case Eq. (27) is reduced to an algebraic expression, since $Q_{jj'}(\hat{\gamma},\hat{\gamma}')$ essentially becomes a two-dimensional delta function, so that Eq. (27) can easily be solved for $\alpha_j(\hat{\gamma})$. For this special case we get the following more explicit version of Eqs. (32), (33), (40) and (41).

$$d_n = a_n - i \sum_{jj'} \int_{C_-} d\hat{\gamma}\, B^{\dagger}_{nj}(\hat{\gamma})\, R_{jj'}(\hat{\gamma})\, a_{j'}(\hat{\gamma}^*) \tag{44}$$

$$R_{nn'} = 2 \sum_{jj'} \int_{C_-} d\hat{\gamma}\, B^{\dagger}_{nj}(\hat{\gamma})\, R_{jj'}(\hat{\gamma})\, B_{n'j'}(\hat{\gamma}^*) \tag{45}$$

$$\vec{u}_-^{\,dir.}(\vec{r}) = -\sum_{jj'} i \int_{C_+} d\hat{\gamma}\, \vec{\Phi}_j(\hat{\gamma};\vec{r})\, Q^{-1}_{jj'}(\hat{\gamma})\, a_{j'}(\hat{\gamma}^*) \tag{46}$$

$$\vec{F}_n(\vec{r}) = 2 \sum_{jj'} \int_{C_+} d\hat{\gamma}\, \vec{\Phi}_j(\hat{\gamma};\vec{r})\, Q^{-1}_{jj'}(\hat{\gamma})\, B_{nj'}(\hat{\gamma}^*) \tag{47}$$

We have here introduced the following notations

$$R_{11}(\hat{\gamma}) \equiv [4q_s^2 h_{ps} h_{ss} - (k_s^2 - 2q_s^2)^2] \exp(2iz_0 h_{ss}) / D_s(\alpha)$$

$$R_{13}(\hat{\gamma}) \equiv -(k_s/k_p)^{1/2} 4q_s h_{ss} (k_s^2 - 2q_s^2) \exp(iz_0(h_{ss}+h_{ps})) / \Delta_s(\alpha)$$

$$R_{22}(\hat{\gamma}) \equiv \exp(2iz_0 h_{ss}) \tag{48}$$

$$R_{31}(\hat{\gamma}) \equiv (k_p/k_s)^{1/2} 4q_p h_{pp} (k_s^2 - 2q_p^2) \exp(iz_0(h_{pp}+h_{sp})) / \Delta_p(\alpha)$$

$$R_{33}(\hat{\gamma}) \equiv [4q_p^2 h_{pp} h_{sp} - (k_s^2 - 2q_p^2)^2] \exp(2iz_0 h_{pp}) / \Delta_p(\alpha)$$

$$R_{12}(\hat{\gamma}) = R_{21}(\hat{\gamma}) = R_{23}(\hat{\gamma}) = R_{32}(\hat{\gamma}) \equiv 0$$

which are essentially the reflection coefficients for a flat elastic - vacuum interface, see e.g. Ref. 9. Furthermore,

$$Q_{11}^{-1}(\hat{\gamma}) \equiv i N_s(\alpha) / \Delta_s(\alpha)$$

$$Q_{13}^{-1}(\hat{\gamma}) \equiv -i(k_p/k_s)^{3/2}(q_s/k_p^2 h_{ps})[k_s^2 - 2q_s^2 - 2h_{ss}h_{ps}] e^{iz_0(h_{ps}-h_{ss})} N_s(\alpha)/\Delta_s(\alpha)$$

$$Q_{22}^{-1}(\hat{\gamma}) \equiv 2i \tag{49}$$

$$Q_{31}^{-1}(\hat{\gamma}) \equiv i(k_p/k_s)^{3/2}(q_p/k_p k_s h_{sp})[k_s^2 - 2q_p^2 - 2h_{sp}h_{pp}] e^{iz_0(h_{sp}-h_{pp})} N_p(\alpha)/\Delta_p(\alpha)$$

$$Q_{33}^{-1}(\hat{\gamma}) \equiv i N_p(\alpha) / \Delta_p(\alpha)$$

$$Q_{21}^{-1}(\hat{\gamma}) = Q_{12}^{-1}(\hat{\gamma}) = Q_{23}^{-1}(\hat{\gamma}) = Q_{32}^{-1}(\hat{\gamma}) \equiv 0$$

The following quantities appear in $R_{jj'}(\hat{\gamma})$ and $Q^{-1}_{jj'}(\hat{\gamma})$

$$D_i(\alpha) \equiv 4q_i^2 h_{pi} h_{si} + (k_s^2 - 2q_i^2)^2 \qquad i = p,s$$

$$N_i(\alpha) \equiv 2k_s^4 h_{si} h_{pi} / (q_i^2 + h_{si} h_{pi}) \qquad i = p,s \tag{50}$$

$$q_i \equiv k_i \sin\alpha \qquad i = p,s$$

$$h_{ii'} \equiv (k_i^2 - q_{i'}^2)^{1/2} \qquad i, i' = p,s \quad \text{and } \operatorname{Im}(\quad)^{1/2} \geq 0$$

Finally $\hat{\gamma}^*$ in Eqs. (44)-(47) indicates that $\alpha$ should be exchanged with

| j \ j' | 1 | 2 | 3 |
|---|---|---|---|
| 1 | $\pi - \alpha$ | $-$ | $\pi - \arcsin[(k_s/k_p)\sin\alpha]$ |
| 2 | $-$ | $\pi - \alpha$ | $-$ |
| 3 | $\pi - \arcsin[(k_p/k_s)\sin\alpha]$ | $-$ | $\pi - \alpha$ |

(51)

where j and j' are as in Eqs. (44)-(47). This exchange is due to the mode conversion and change in direction of propagation on reflection of a plane wave by the flat elastic - vacuum interface.

## V Numerical applications

To show the applicability of the theory in the preceding sections we now give some numerical examples. We then let the infinite surface $S_o$ be the flat surface $z=z_o$ and we take the upper half space $z>z_o$ as vacuum. On the other hand we do not specify the bounded inhomogeneity - in fact we will treat both a cavity and an elastic obstacle.

The incoming field will be taken as a Rayleigh surface wave with unit amplitude in the z-direction travelling in the positive x-direction. Apart from being an example of great interest this choice also has the advantage that the sum of the incoming and directly scattered field (cf. Eq. (46)) is trivially known. This would not be true if we e.g. took the incoming field to be generated by a point source. We note that for an incoming Rayleigh wave, whose sources can be considered as a surface distribution on $S_o$, the integral representation, Eq. (17), has to be interpreted differently. The surface field $\vec{u}_-$ in the integral over $S_o$ is only the scattered surface field and not the total surface field. Furthermore, for the incoming Rayleigh wave the directly scattered field, Eq. (46), vanishes (since this incoming wave satisfies the boundary condition on $S_o$).

There is another way to fit an incoming Rayleigh wave into the present formalism and that is to consider the Rayleigh wave to be excited by a point source on the surface (in the limit from below) pushed to infinity with an appropriate growth in strength. This approach is further developed in the appendix where we also show the equivalence of this approach and the one where we look upon the Rayleigh wave as generated by a surface source distribution.

The main steps in the numerical procedure are:

1) Compute $d_n$, Eq. (44).
2) Compute the transition matrix for the obstacle $T_{nn'}$, Eq. (36).
3) Compute the spherical projection of the reflection coefficient $R_{nn'}$, Eq. (45).
4) Compute $\vec{F}_n(\vec{r})$, Eq. (47), for an array of measuring points on $S_o$.
5) Solve Eq. (35) for $c_n$ and compute the scattered surface field, cf. Eq. (38).

We note that steps 1), 3) and 4) are independent of the obstacle, so if we keep everything fixed except the properties of the obstacle we only have to repeat steps 2) and 5) for each new obstacle, thus greatly enhancing the efficiency of the method.

The computation of $d_n$ (see Eq. (A.10) for an explicit formula) is straightforward and need not be further commented on, and for comments on the computation of $T_{nn'}$ we refer to Refs. 5, 6, 13 and 14.

The $\beta$-integral in $R_{nn'}$ and $\vec{F}_n(\vec{r})$ can be performed analytically and this essentially gives diagonality in m for $R_{nn'}$ and a cylindrical Bessel function for $\vec{F}_n(\vec{r})$; the remaining $\alpha$-integral along $C_+$ must be computed numerically. We note that the integration contour must first be deformed to avoid the Rayleigh pole (the zero of $D_s(\alpha)$ and $D_p(\alpha)$), which is situated on $C_+$. In fact we perform the integrations in the q-plane (cf. Eq. (50)) from the origin along a half-circle in the fourth quadrant and then out along the real axis. First, $\vec{F}_n(\vec{r})$ is written in cylindrical coordinates as the angle dependence then only appears as trigonometric functions multiplying the integrals. Thus $\vec{F}_n(\vec{r})$ needs only to be computed along a ray as a function of the radius $\rho$ on the surface. We have chosen to compute $\vec{F}_n(\vec{r})$ in 35 points: for $0 \leq k\rho \leq 1$ in steps of 0.1, for $1 < k\rho \leq 3$ in steps of 0.2 and for $3 < k\rho \leq 10$ in steps of 0.5. In between we use interpolation techniques - the irregularities shown in the level plots and for $k_\rho$ close to -10 in the other plots are due to limitations in these techniques. The last item in the numerical scheme given above only involves matrix manipulations. The matrix inversion required in solving Eq. (35) for $c_n$ is usually performed by a simple iteration procedure [1].

In the numerical examples we have kept the depth $z_o$ of the origin and the Poisson ratio $\nu$ of the half space fixed: $kz_o=1$ and $\nu=0.3$. We remark that $kz_o=1$ corresponds to about a third of the Rayleigh wavelength, so that the inhomogeneity is quite close to the surface $S_o$. When we in two examples below consider other depths of the obstacle this is accomplished not by changing $kz_o$ (which requires the recalculation of $d_n$, $R_{nn'}$ and $\vec{F}_n(\vec{r})$ as remarked above), but instead by a translation of the transition matrix[13]. Below we also show examples where the symmetry axis of the obstacle is not in the z-direction. As in the acoustic [1] and electromagnetic [2] cases the transition matrices are then obtained by first calculating it for the symmetry axis in the z-direction and then rotate it. The rotation matrices [17] are diagonal in the $\tau$-index (and in the $\ell$-index) so they are in fact the same for the elastic case as for the acoustic and electromagnetic cases.

In Fig. 3 we show the surface displacements in the x- and z-directions along the x-axis (remember that the incoming Rayleigh wave travels in the positive x-direction) for a spherical cavity of radius ka=0.3 and depth $kz_o$=1. In Fig. 3a the absolute values of the total fields $|u_x|$ and $|u_z|$ are shown, and as in all other plots we show we note that the deviations from the undisturbed values $|u_x|=0.656$ and $|u_z|=1$ (obtained from Eqs. (A.9) and (A.6) for $\nu=0.3$) is concentrated around kx=0 as expected, but also in the backward direction where we get ripples with a wavelength which is half of the Rayleigh wavelength. These ripples are what we should expect from two waves with the same wavelength travelling in opposite directions, and this is born out by Figs. 3b and 3c where the absolute values and real and imaginary parts of the scattered fields are shown. The real and imaginary parts correspond to two fixed times, so these plots clearly show that apart from near the origin the scattered field on the surface is mainly an outgoing circular Rayleigh wave which we expect to fall off as the square root of the distance from the origin.

The same types of plots are shown in Fig. 4, this time for a prolate spheroidal cavity with axes ka=0.6 and kb=0.2 with the symmetry axis in the x-direction and depth $kz_o$=1. We note that Figs. 3 and 4 are quite similar although the sphere tends to scatter somewhat more than the spheroid.

We next consider some elastic obstacles. We then take the following material parameters: $\rho'/\rho=1.5$, $\mu'/\mu=4$ and $\nu'=0.28$ (where primed quantities refer to the obstacle). In Figs. 5 and 6 we give data along the x-axis for two spheres of the same radius ka=0.5 but with different depths: $kz_o=0.8$ and $kz_o=1.2$, respectively. We note that the influence of the depth is quite small and this is also true for variations in the form and orientation of the obstacle which can be seen by comparing Figs. 5 and 6 with Fig. 7, showing corresponding plots for an oblate spheroidal elastic obstacle with axes ka=0.3 and kb=0.6 whose symmetry axis makes the angle $\pi/3$ with the z-axis and the azimuth angle $\pi/4$ with the x-axis.

Finally we give some level plots showing the fields in the circle $k\rho<10$, thus giving a good overall picture. In Fig. 8 we show $|u_x|$ for the shallowly buried sphere of Fig. 5 and in Fig. 9 level plots of all three components of the scattered field of the oblate spheroid in Fig. 7; for the y-component this is also the total field. We note the asymmetries in the scattered field in Fig. 9 due to the asymmetrical obstacle. We remark that far from the origin the level plots are not so accurate, mainly because the points where we calculate the fields are a little sparse (the angle increment is $\pi/8$ and the radial was given above).

## VI Discussion

Elastic wave propagation in an elastic half space has been the subject of this paper. We have explicitly taken the half space above the ground as vacuum and furthermore we have assumed in the derivation of the equations that the inhomogeneity was a cavity. This latter property can easily be relaxed, as pointed out above, and the proper T-matrix inserted, which is relevant to the inhomogeneity considered. Furthermore, it is straightforward to relax the assumption of vacuum above the interface. Analogously to the generalization for the buried obstacle, we exchange the reflection coefficients and Q functions to the appropriate ones, and the rest of the formalism remains the same as long as we are interested in the field below the ground. For a field computation above the ground we have to employ the surface integral representation once more, but now to the upper region (for further details see Refs. 1-2).

The numerical examples in this paper have only used one type of incoming field - the Rayleigh wave - and furthermore we have only focused on the field on the interface. Another case of interest is a point source, which can be located on or below the ground. The numerical calculation of the field below the ground can also be of interest in some situations, and this field can be computed in analogy to the surface field $\vec{u}_-$.

A layered earth, with the inhomogeneity below all the layers, is a straightforward extension of the formalism above (formally an exchange of the pertinent reflection coefficients of the ground). If we, however, consider a geometry where the inhomogeneity is located between any layers, we get a more intricate scattering problem and further studies on this type of extension are required.

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## Appendix: The point source and the Rayleigh wave

This appendix shows how $d_n$ (cf. Eq. (44)) for an incoming Rayleigh wave can be obtained from a simple point source, whose location we push to infinity with a suitable increase in strength. This limiting procedure will also be compared with another more direct way of finding $d_n$ for an incoming Rayleigh wave. We then explicitly start with a Rayleigh wave as an incoming field, despite the fact that the sources of the Rayleigh wave are a surface distribution and should therefore not be treated in the same way as e.g. an incoming plane wave or point source. Nevertheless, we end up with identical expressions for $d_n$. For a dipole source at $\vec{r}_t=(x_t,0,z_o)$, polarized in the $\hat{z}$-direction and of strength C we have the following expression for the $d_n$-vector, see Eq. (44) (we have here chosen the integration variable as $q=q_s=q_p$):

$$
\begin{aligned}
d_n = {} & iCk_s^3\,\hat{z}\cdot\vec{\Psi}_n(\vec{r}_t) + C\Gamma_n\int_0^\infty dq\, J_m(qx_t)/\Delta(q)\ \times \\
& \times\Big\{\Big[\Big[im\,\delta_{\tau,1}\delta_{\sigma,0}P_\ell^m\big(\sqrt{1-q^2/k_s^2}\big) + \delta_{\tau,2}\delta_{\sigma,e}\Big((\ell-m+1)P_{\ell+1}^m\big(\sqrt{1-q^2/k_s^2}\big) - \\
& -(\ell+1)\sqrt{1-q^2/k_s^2}\ P_\ell^m\big(\sqrt{1-q^2/k_s^2}\big)\Big)\Big]\Big(q/k_s\sqrt{k_s^2-q^2}\Big)\ \times \\
& \times\exp\big(iz_0\sqrt{k_s^2-q^2}\big)\Big[(k_s^2-2q^2)^2+4\sqrt{k_s^2-q^2}\sqrt{k_p^2-q^2}\,(q^2-k_s^2)\Big] + \\
& +[\ell(\ell+1)]^{1/2}\delta_{\tau,3}\delta_{\sigma,e}(q/k_sk_p)\exp\big(iz_0\sqrt{k_p^2-q^2}\big)(k_p/k_s)^{1/2}\ \times \\
& \times P_\ell^m\big(\sqrt{1-q^2/k_p^2}\big)\Big[4q^2\sqrt{k_s^2-q^2}\sqrt{k_p^2-q^2}-(k_s^2+2q^2)(k_s^2-2q^2)\Big]\Big\}
\end{aligned}
\qquad (A.1)
$$

Here

$$\Gamma_n \equiv (-i)^{\ell+m} \sqrt{\varepsilon_m \frac{(2\ell+1)(\ell-m)!}{4\pi\ell(\ell+1)(\ell+m)!}}\; k_s^3$$

$$\Delta(q) \equiv 4q^2 \sqrt{k_s^2-q^2}\,\sqrt{k_p^2-q^2} + (k_s^2-2q^2)^2$$

and the integration contour should go just below the real axis to avoid the singularities of the integrand.

We notice that the pertinent integrals can be written as:

$$I = \int_0^\infty dq\, J_m(qx_t)\, f(q)\, P_\ell^m\left(\sqrt{1-q^2/k_s^2}\right) / \Delta(q)$$

and a similar integral, where $k_s^2$ is replaced by $k_p^2$ in $P_\ell^m(\sqrt{1-q^2/k_s^2})$. The function f(q) is odd in q, and all square roots occurring in the integrand have a non-negative imaginary part. Manipulating the Bessel functions we get [18]:

$$I = \frac{1}{2}\int_{-\infty}^{\infty} H_m^{(1)}(qx_t)\, f(q)\, P_\ell^m\left(\sqrt{1-q^2/k_s^2}\right) / \Delta(q)\, dq \tag{A.2}$$

(Notice that $P_\ell^m(\sqrt{1-q^2/k_s^2})$ contributes with a factor $(-1)^m$ when changing q to -q in the argument.)

The contour along the real axis will now be deformed and we have to consider the contributions from the branch points $q=\pm k_s$ and $q=\pm k_p$ and the poles $q=\pm k_R$ (defined below) of the integrand in some detail. The cuts in the complex q-plane are defined by $\mathrm{Im}(k_{s,p}^2-q^2)^{1/2}=0$. In the lossless case, which is the situation analysed here, these cuts degenerate to a finite part along the real axis and a semi-infinite part along the imaginary axis, see Fig. A.1. The Rayleigh pole $q=\pm k_R$ is defined by

$$\Delta(q)=0 \tag{A.3}$$

This equation has two real roots ($q=\pm k_R$), which both are located on the Riemann sheet considered here, i.e. $\mathrm{Im}(k_{s,p}^2-q^2)^{1/2}\geq 0$.

We will consider the situation where $x_t\to-\infty$. The dominant term of the Hankel function for large argument is [18]:

$$H_m^{(1)}(z)\sim(2/\pi z)^{1/2}\,e^{i(z-1/2(m\pi+\pi/2))} \qquad \arg z\in(-\pi,2\pi)$$

For large negative values of $x_t$ we close the integration contour in the lower half plane. The semi-circle does not give any contribution to the value of the integral, and we can deform the contour to two branch line integrals plus a Rayleigh pole contribution. The branch line integrals give a contribution of the order $o(|x_t|^{-1/2})$, which is comfortably seen if we rewrite the branch line integrals as:

$$\int_0^\infty f_1(q)\,e^{-q|x_t|}dq+\int_0^{k_p} f_2(q)\,e^{iq|x_t|}dq+\int_0^{k_s} f_3(q)\,e^{iq|x_t|}dq$$

For more details we refer to the monograph on asymptotic expansions by Olver [18].

Thus we have

$$I=-i\pi H_m^{(1)}(k_R|x_t|)\,f(-k_R)(-1)^m P_\ell^m\left(i\sqrt{k_R^2/k_S^2-1}\right)/\Delta'(-k_R)$$
$$+\,o\left(|x_t|^{-1/2}\right) \tag{A.4}$$

$d_n$, see Eq. (A.1), can thus be written down and we have

$$d_n = iCk_s^3\,\hat{z}\cdot\vec{\psi}_n(\vec{r}_t) - iC\,e^{i|x_t|k_R - i\pi/4}\left(\frac{8\pi k_R}{|x_t|}\right)^{1/2}\frac{k_s^4\sqrt{k_R^2-k_P^2}}{i\,D'(-k_R)}\times$$

$$\times\left\{(\delta_{\tau,1}+\delta_{\tau,2})\,B^{\dagger}_{n1}(\pi-\alpha_s,0)\,A + \delta_{\tau,3}\,B^{\dagger}_{n3}(\pi-\alpha_p,0)\,B\right\} + C\,o(|x_t|^{-1/2}) \tag{A.5}$$

Here we have introduced the symbols:

$$A = -2k_R/k_s\;e^{-z_0\sqrt{k_R^2-k_s^2}}$$

$$B = i\,(k_s^2-2k_R^2)(k_R^2-k_p^2)^{-1/2}\,(k_pk_s)^{-1/2}\;e^{-z_0\sqrt{k_R^2-k_p^2}}$$

$$k_s\sin\alpha_s = k_p\sin\alpha_p = k_R \qquad \alpha_p,\alpha_s\in C_+ \tag{A.6}$$

The strength C is now specified as

$$C = i\,D'(-k_R)\,e^{i\pi/4 - i|x_t|k_R}\sqrt{\frac{|x_t|}{8\pi k_R}}\;k_s^{-4}\Big/\sqrt{k_R^2-k_p^2} \tag{A.7}$$

and we finally get in the limit as $|x_t|\to\infty$.

$$d_n = -i\left\{(\delta_{\tau,1}+\delta_{\tau,2})\,A\,B^{\dagger}_{n1}(\pi-\alpha_s,0) + \delta_{\tau,3}\,B\,B^{\dagger}_{n3}(\pi-\alpha_p,0)\right\} \tag{A.8}$$

(The first term in Eq. (A.5) does not contribute since $\hat{z}\cdot\vec{\psi}_n(\vec{r}_t) = O(|x_t|^{-1})$).

By a very similar procedure we can also show that the directly scattered field, Eq. (46), becomes just the Rayleigh wave, Eq. (A.9) below.

To see the analogy with an incoming Rayleigh wave it is convenient to derive $d_n$ in a different way, where we introduce the Rayleigh wave explicitly into the formalism. Consider an incoming Rayleigh wave propagating in the positive x-direction and of unit magnitude in the z-direction on the surface $z=z_o$. Straightforward calculations give:

$$\vec{u}^{i}(\vec{r}) = A\hat{\alpha}\exp(ik_R x + \sqrt{k_R^2 - k_s^2}\, z) + + B\hat{\gamma}\,(k_p/k_s)^{3/2}\exp(ik_R x + \sqrt{k_R^2 - k_p^2}\, z) \quad \text{(A.9)}$$

Here A and B are given by Eq. (A.6), and the unit vectors $\hat{\alpha}$ and $\hat{\gamma}$ are specified by the spherical angles $\pi-\alpha_s$ and $\pi-\alpha_p$ $(\beta=0)$, respectively, see Eq. (A.6). It is easy to see that the z-component of the field in Eq. (A.9) has unit magnitude and it is furthermore straightforward to show that it satisfies the boundary condition $\vec{t}=\vec{0}$ on the surface $z=z_o$. If we formally insert this field in $d_n$, see Eq. (44), the contribution from the integral vanishes (as the field satisfies the boundary condition there is no wave first reflected by $S_o$), and the remaining first term, the spherical expansion in the vicinity of the origin, is found from Eq. (5). We have

$$d_n = -i\{(\delta_{\tau,1}+\delta_{\tau,2})\, A B^{\dagger}_{n1}(\pi-\alpha_s,0) + \delta_{\tau,3}\, B B^{\dagger}_{n3}(\pi-\alpha_p,0)\} \quad \text{(A.10)}$$

This expression for $d_n$ is identical to the one found from the more elaborate procedure above, in which the limiting process for the position of a point source was considered.

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Figure captions

Fig. 1 The integration contours $C_+$ and $C_-$.

Fig. 2 Geometry for a buried obstacle.

Fig. 3 Displacements for a spherical cavity with ka=0.3 and $kz_o$=1.

Fig. 4 Displacements for a spheroidal cavity with ka=0.6, kb=0.2, $kz_o$=1 and the symmetry axis along the x-axis.

Fig. 5 Displacements for an elastic sphere with ka=0.5 and $kz_o$=0.8.

Fig. 6 Displacements for an elastic sphere with ka=0.5 and $kz_o$=1.2.

Fig. 7 Displacements for an oblate elastic spheroid with ka=0.3, kb=0.6 and the symmetry axis making the angle $\pi/3$ with the z-axis and the azimuth angle $\pi/4$ with the x-axis.

Fig. 8 Level plots of $|u_x|$ for the same obstacle as in Fig. 5. The scale factor is $10^{-2}$.

Fig. 9 Level plots of a) $|u_x^{sc.}|$, b) $|u_y^{sc.}|$, c) $|u_z^{sc.}|$ for the same obstacle as in Fig. 7. The scale factor is $10^{-2}$.

Fig. A.1 Cuts, poles and integration contour for Eq. (A.2).

Figure 1

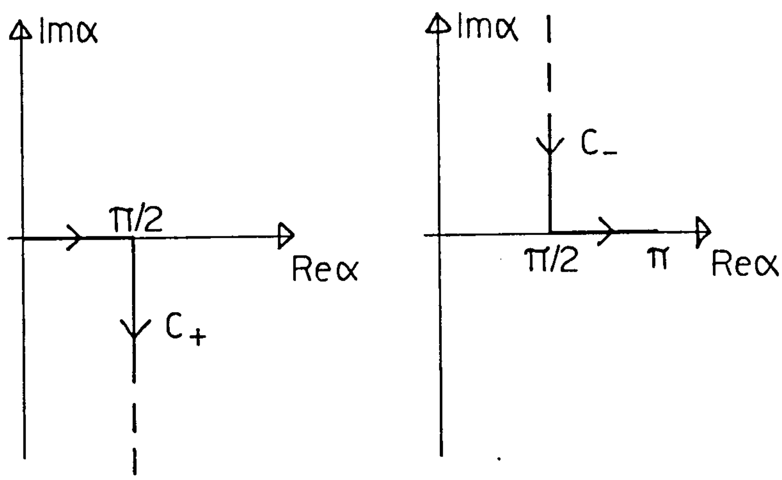



Figure 2

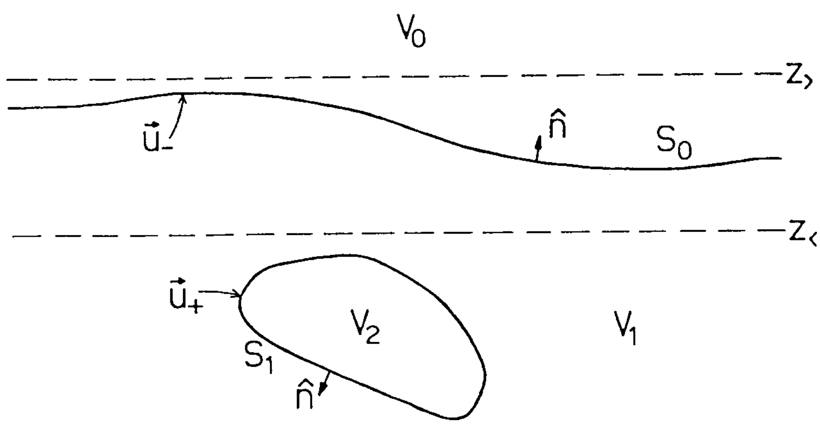


Figure 3

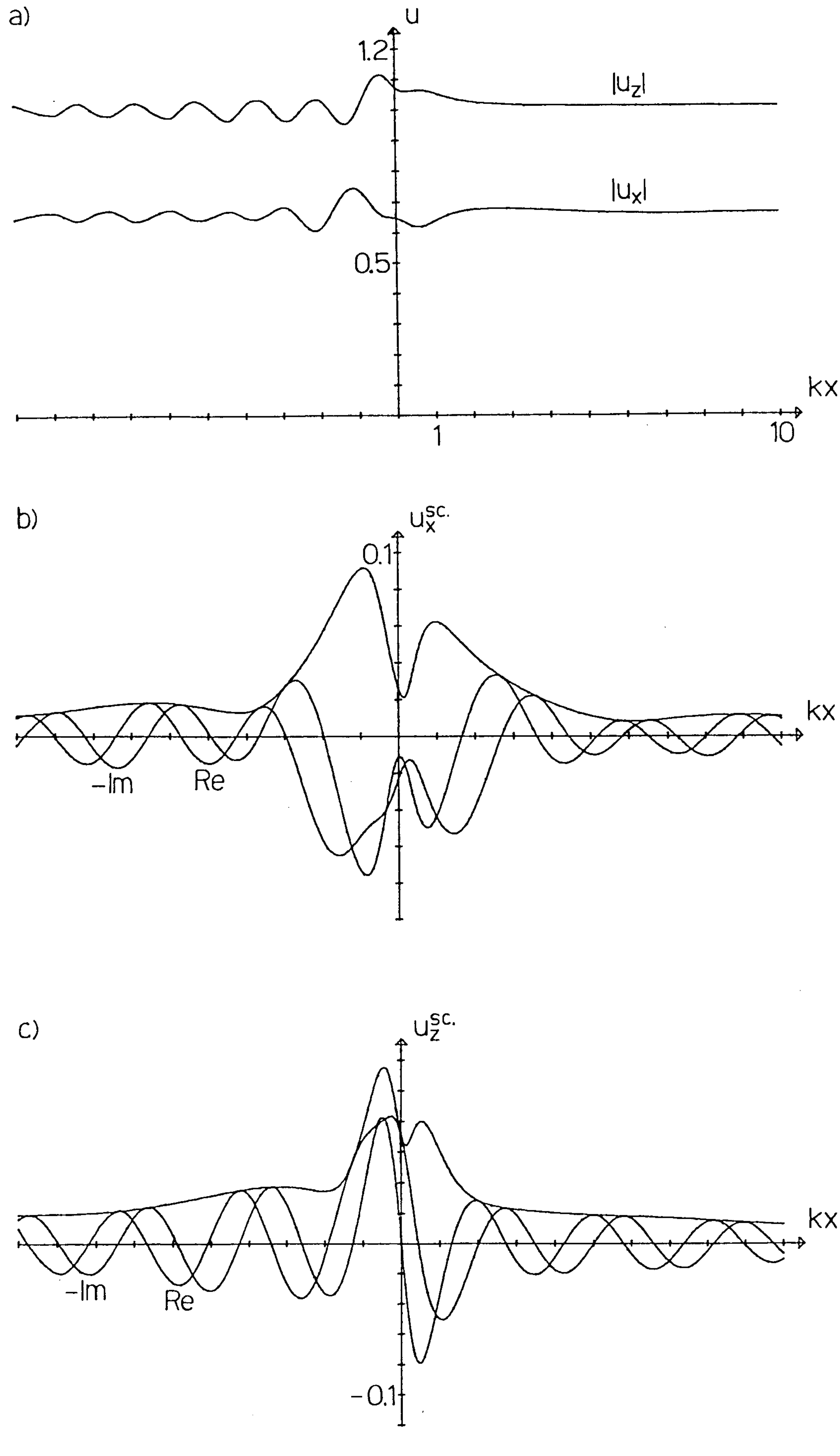


Figure 4

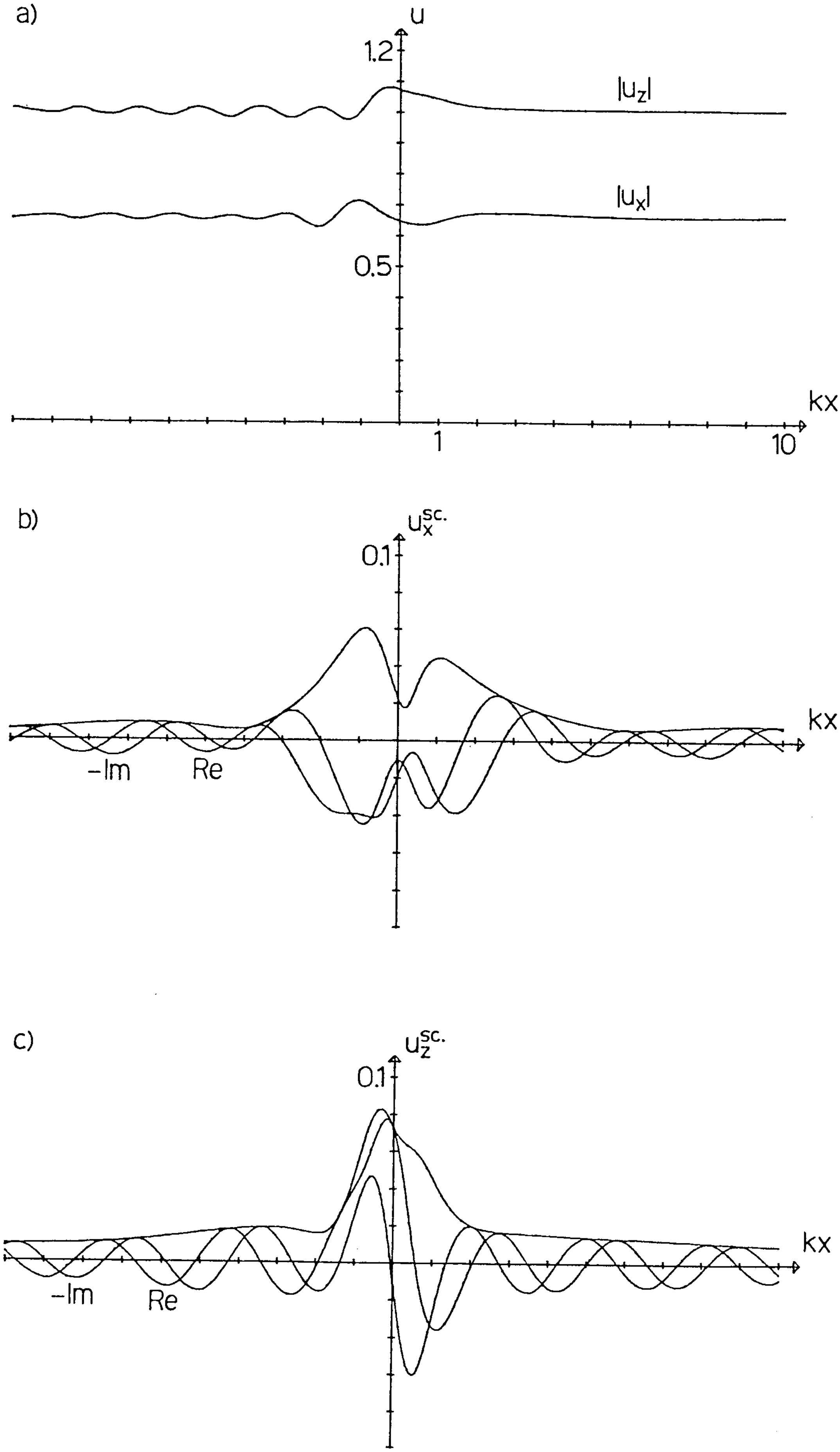


Figure 5

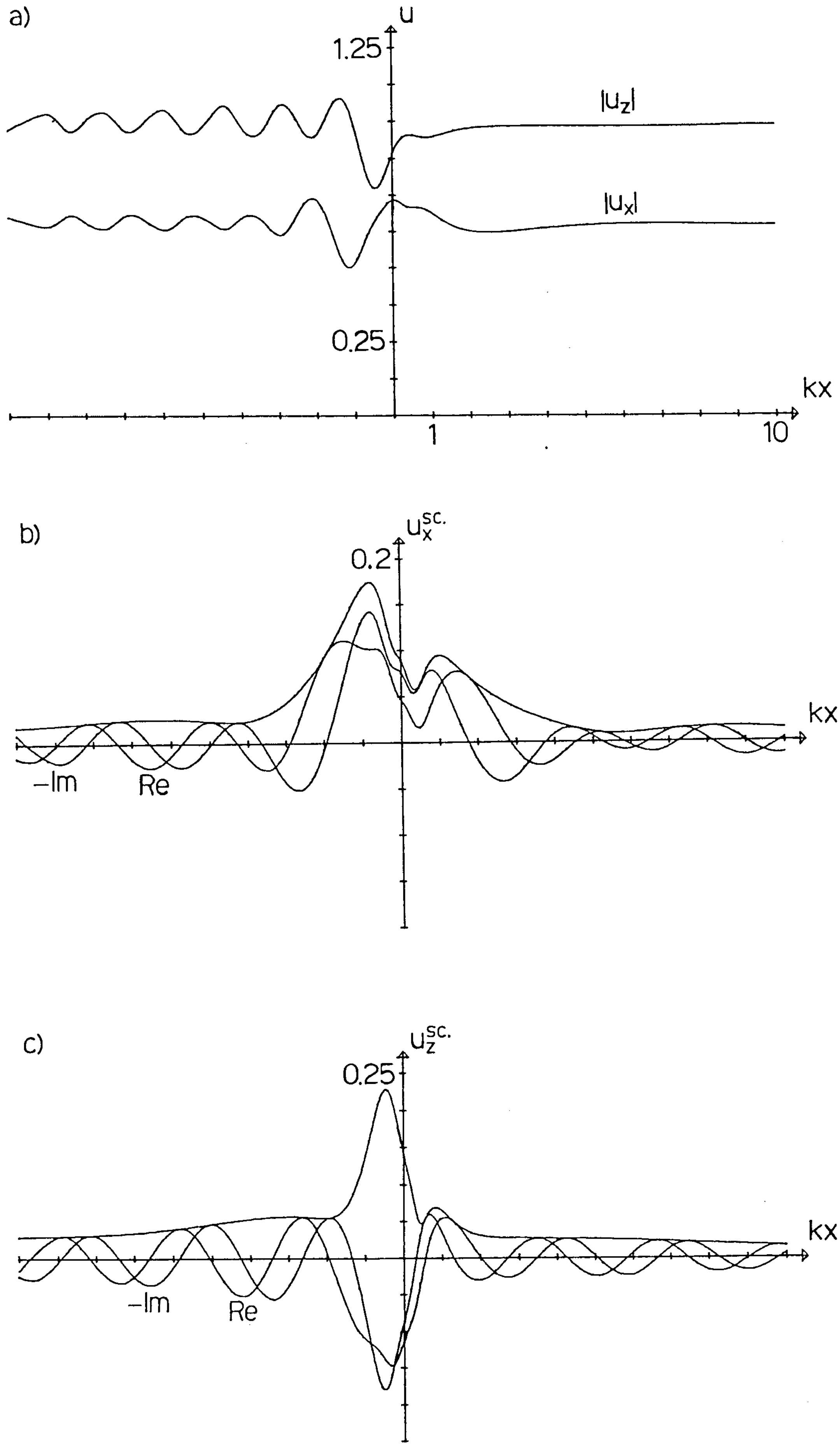


Figure 6

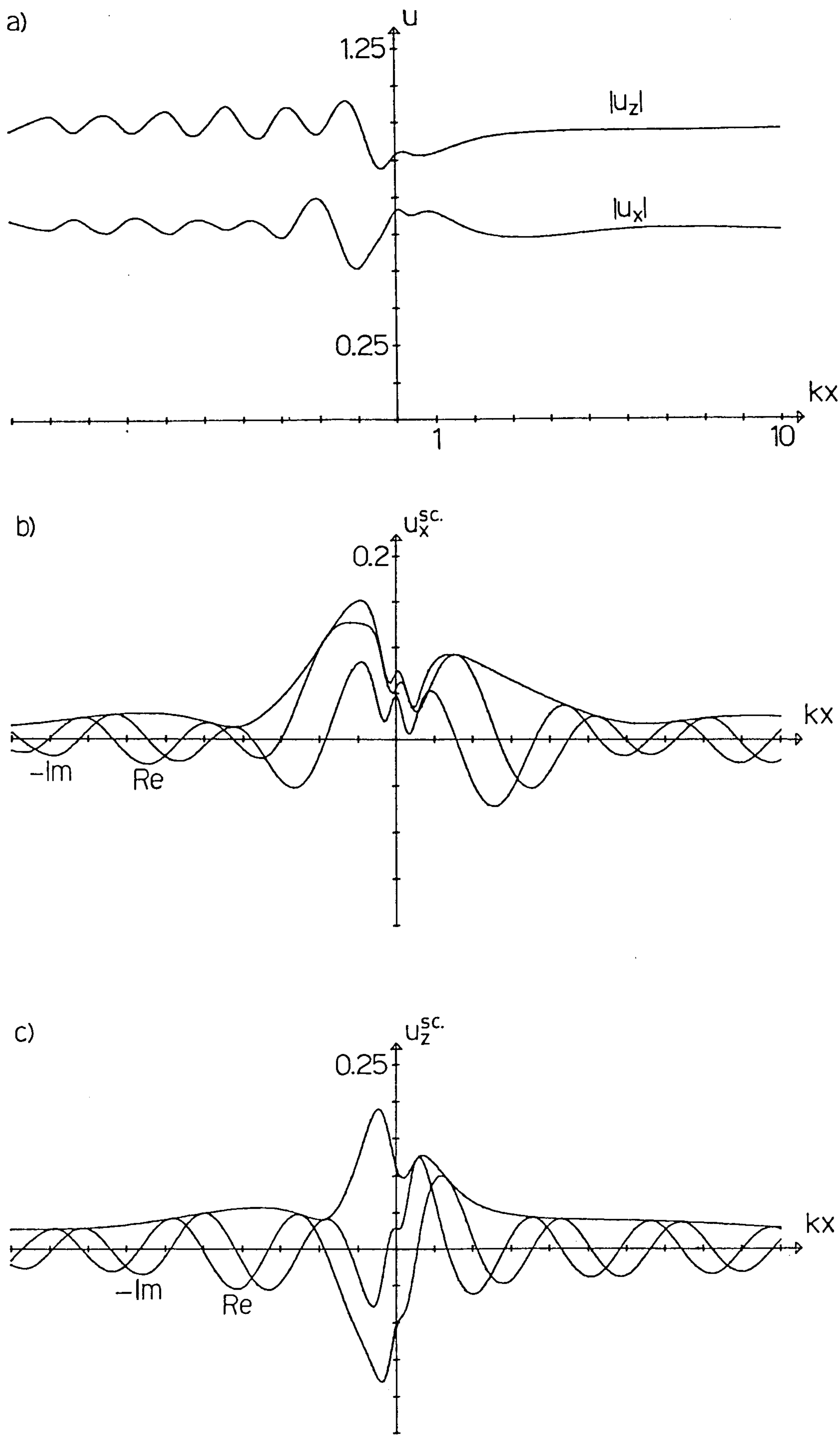


Figure 7

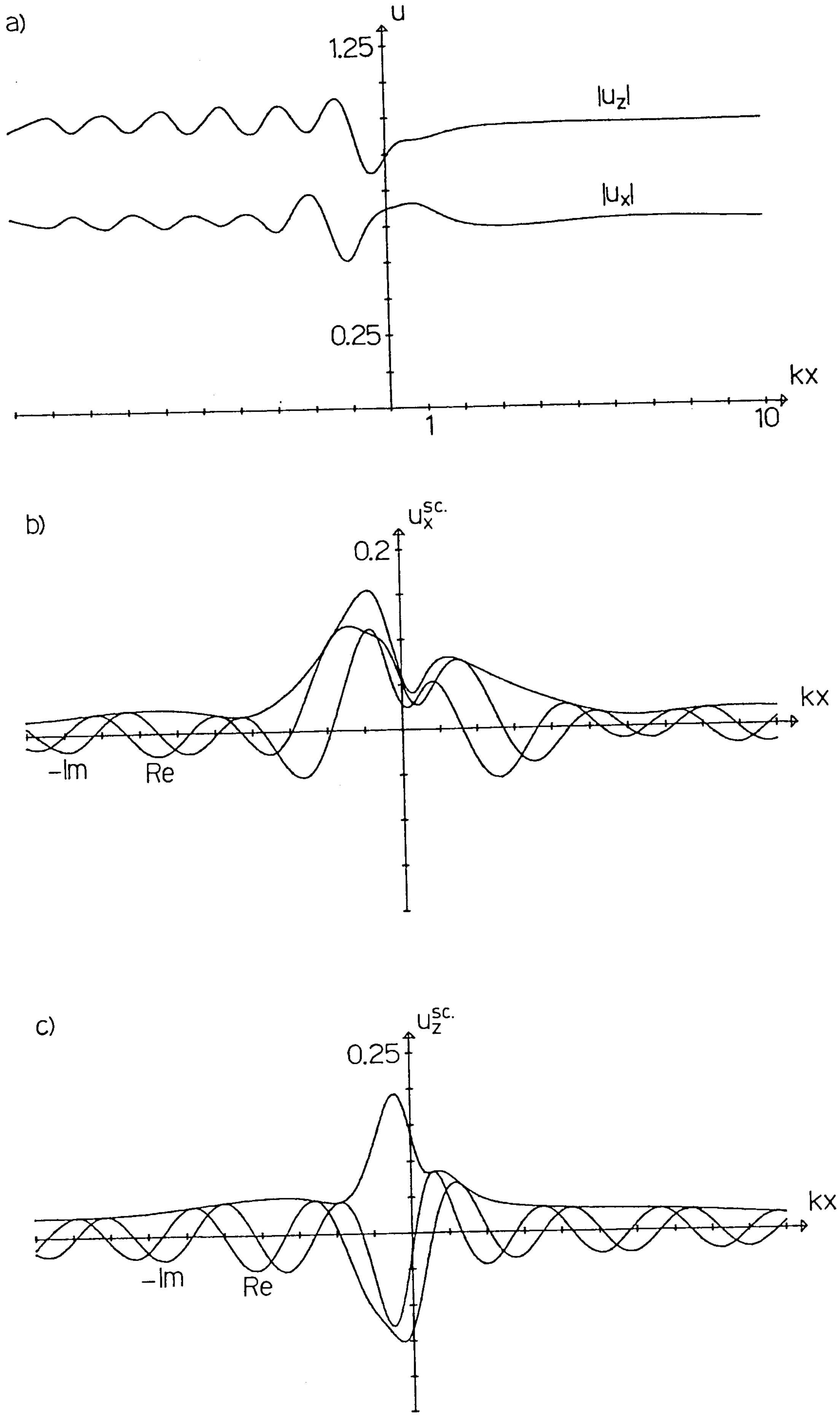


Figure 8

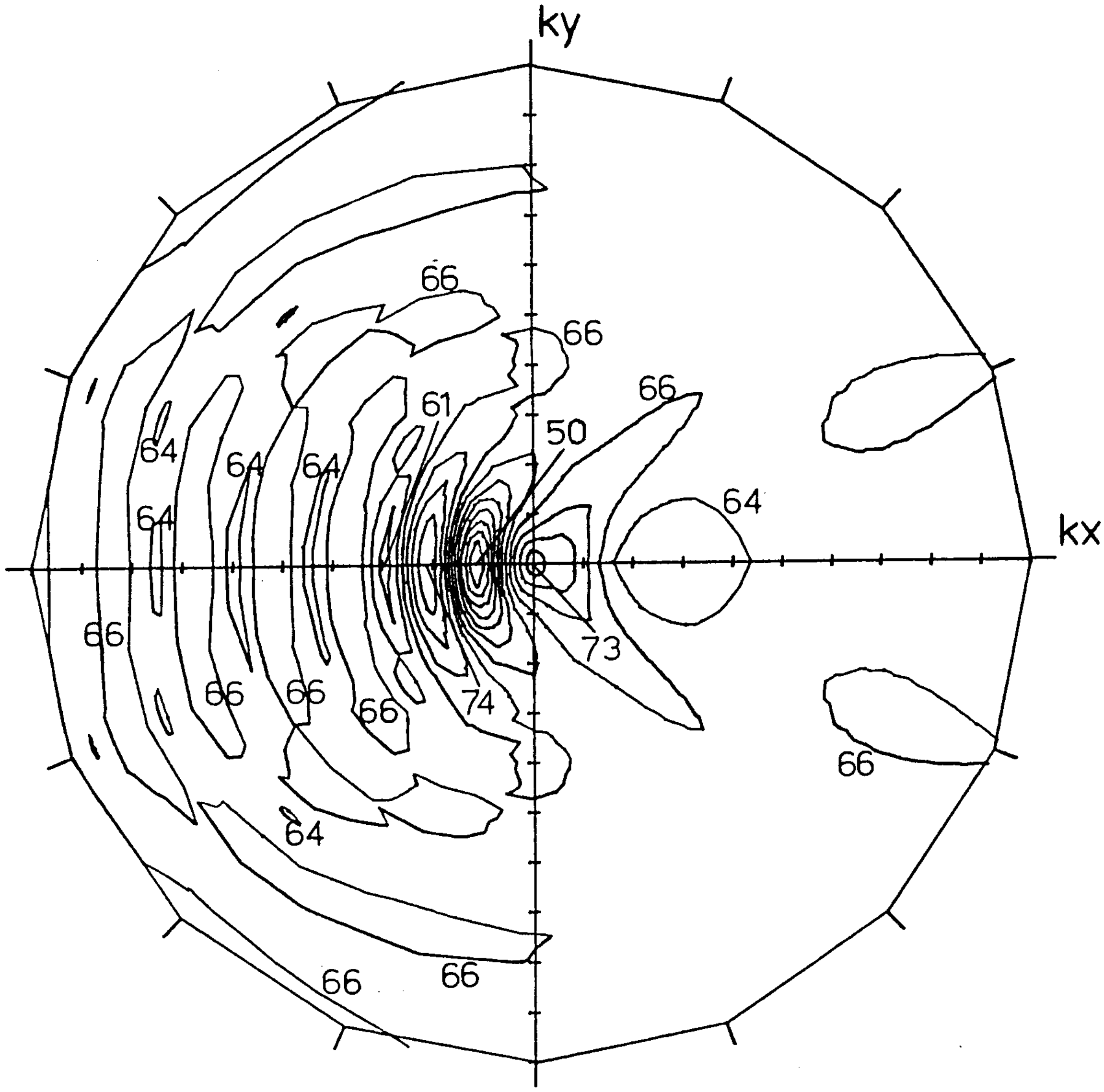


Figure 9

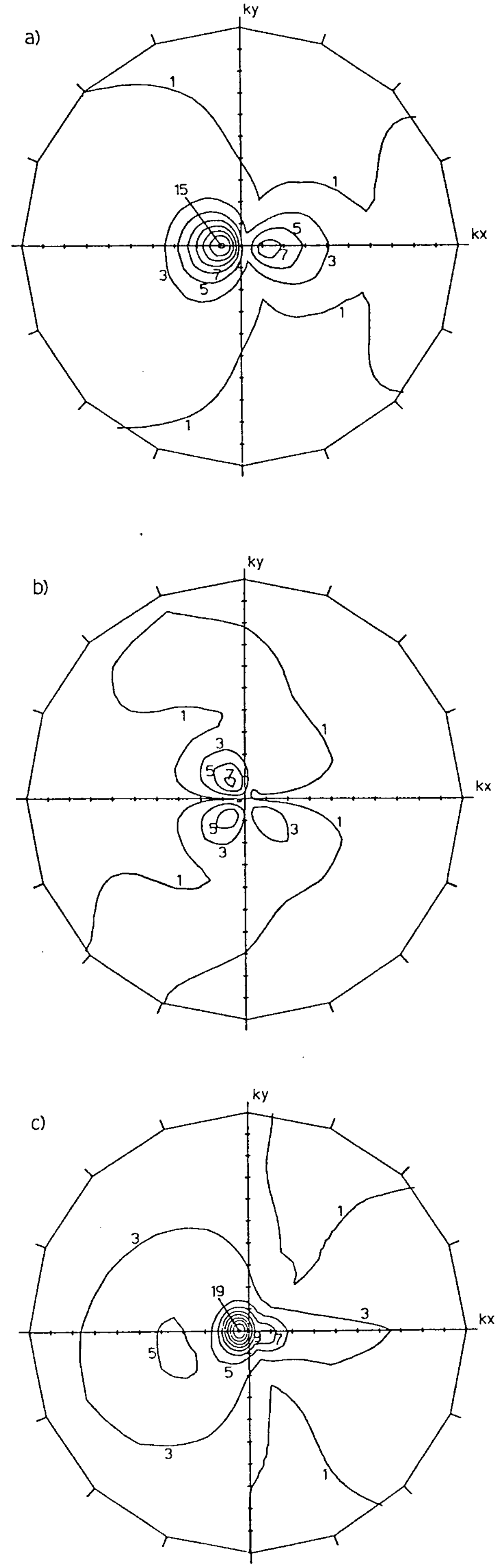
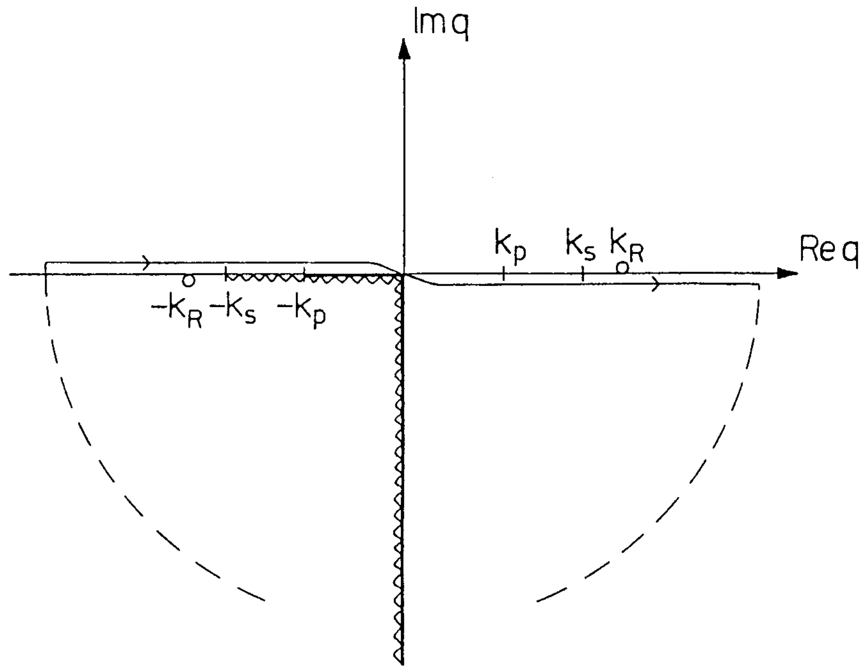


Figure A.1



# Paper V

79-10

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# A uniqueness theorem for the Helmholtz equation: Penetrable media with an infinite interface

by

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Abstract

In this paper we will prove the uniqueness of a solution to the Helmholtz equation for two halfspaces of different media in n dimensions. The theorem allows a finite number of bounded inhomogeneities in each halfspace. The surface separating the halfspaces is assumed to be a cone of arbitrary cross section far away from the origin and is furthermore assumed to be smooth. We assume all space to be lossless and in each half-space we assume a radiation condition to be fulfilled. The boundary conditions at the interface are a general coupling in the field and its normal derivative with constant coefficients.

## I Introduction

The first uniqueness theorem for the Helmholtz equation for the exterior problem was shown by A Sommerfeld [1]. In the exterior problem the field outside a bounded surface S satisfies

$$(\nabla^2 + k^2)\psi = 0 \tag{1.1}$$

k is here a real or complex constant and at the surface S certain boundary conditions are assumed to be satisfied. To obtain a welldefined problem he introduced a radiation condition for large distances from the obstacle - a boundary value at infinity. The solutions of Eq. (1.1) separate in two classes, satisfying either

$$\frac{\partial\psi}{\partial r} - ik\psi = o(r^{-1}) \qquad r \to \infty$$

or

$$\frac{\partial\psi}{\partial r} + ik\psi = o(r^{-1}) \qquad r \to \infty$$

The first class holds for the outgoing spherical waves (if we take the conventional time dependence to be $e^{-i\omega t}$) while the second is satisfied by the ingoing spherical waves. From potential theory this property was unfamiliar, and in his paper Sommerfeld clarifies the differences between the static and the wave solution. He adopts the outgoing spherical wave, and the radiation condition thus reads

$$\frac{\partial\psi}{\partial r} - ik\psi = o(r^{-1}) \qquad \text{uniformly in all angles as } r\to\infty \tag{1.2}$$

An additional condition for large distances was also introduced:

$$\psi = O(r^{-1}) \qquad r \to \infty \tag{1.3}$$

This is the "condition of finiteness" which was later proven to be superfluous by W. Magnus [2]. In a number of papers [3] - [7] the results have been sharpened and also generalized to an arbitrary number of dimensions. Some of the papers use a slightly weakened form of Sommerfeld's radiation condition, first found in Rellich [3]:

$$\iint_{\Sigma(r)} \left| \frac{\partial \psi}{\partial r} - ik\psi \right|^2 dS = o(1) \qquad r \to \infty \tag{1.4}$$

Here $\Sigma(r)$ is a large sphere of radius r.

If the bounding surface S is infinite, we then have a more limited number of results. The pioneer paper is [3], which establishes uniqueness for the solution for Helmholtz' equation in $R^n$ when the infinite surface S intersects the plane $x_n$=constant for each $x_n$ and furthermore

$$\hat{\nu} \cdot \hat{x}_n < 0 \qquad \text{on } S \tag{1.5}$$

where $\hat{\nu}$ is the outward-directed normal to S. The radiation condition to be satisfied at infinity is a modified type of Eq. (1.4) - $\Sigma$ is now a plane $x_n$=constant and the radial derivative is replaced by $\frac{\partial \psi}{\partial x_n}$. The importance of Eq. (1.5) is also discussed and an example proving the non-uniqueness of a solution for Helmholtz' equation for a geometry violating Eq. (1.5) is given.

Further results are given in [8].

Additional results for boundary value problems, where the surface S is infinite is given in [9] - [12]. D.S. Jones [9] gives a uniqueness theorem for surfaces which for large distances are cones of arbitrary cross sections, and these results are extended by F.M. Odeh [12] who shows uniqueness if $\frac{\partial r}{\partial \nu} \leq 0$ on the surface for large r. By analytic arguments W.L. Miranker [11] shows uniqueness results for domains in which a cone with an angle greater than $\pi/2$ can be inscribed, but certain restrictions on the normal derivative which have to be introduced make the result less general. A two dimensional formulation is found in [10], where the infinite boundary is a straight line for large r. A number of Russian authors [13] - [16] have also studied various aspects of the problem, mostly extensions to differential equations of more general elliptic type and in the limiting case where the losses vanish. Some results for boundary value problems with infinite boundary for a general type of elliptic differential equation have recently been published by V. Vogelsang [17] - [18]. These theorems are essentially extensions of the results of Rellich.

The geometry in all these theorems proving uniqueness for Helmholtz' equation with an infinite boundary is such that the surface gets wider for increasing r. This is achieved by assuming conical shapes or by assuming that Eq. (1.5) is satisfied. This guarantees that the energy radiates to infinity as required by the radiation condition, e.g. Eq. (1.4).

The results for Helmholtz' equation in infinite domains are, as may be seen from the brief review above, both diverse and comprehensive. Uniqueness is established for many situations of interest in applications for both finite or infinite bounding surfaces as well as for real or complex wave numbers. Now focusing on geometries with penetrable media we find that the results here are very scarce. Werner [19] has analysed the uniqueness of the solution for Helmholtz' equation in the case where we have penetrable obstacles of finite extensions. A very specialized situation where the surface is infinite is found in [12]. Odeh here analyses two halfspaces separated by a flat interface. The boundary conditions on the interface are very restricted, but the result holds for real wave numbers.

The aim of this paper is to derive a uniqueness theorem for a more general geometry in the lossless case, and for a general type of boundary conditions. In section II we will give the principal definitions and symbols used in this paper. The lemmas and the uniqueness theorem are proven in section III, while conclusions and a discussion of applications are found in section IV.

## II Principle definitions and notations

In this section we will define the notations found in this paper and state the problem more precisely.

Consider two infinite halfspaces $V_1$ and $V_2$ in $R^n$ (the radial distance is defined in the usual way as $r^2 = \sum_{j=1}^{n} x_j^2$) separated by an infinite surface S as depicted in the figure. We will assume the interface S to be sufficiently smooth, so that an application of Green's theorem at every finite part of $V_1$ and $V_2$ is valid. Sufficient conditions for this to hold are found e.g. in [20] - [22]. The volumes $V_1$ and $V_2$ are assumed to be homogeneous and isotropic with wave numbers $k_1$ and $k_2$ respectively, except for a finite number of inhomogeneities $V_1'$ and $V_2'$ (if several inhomogeneities are present, let $V_1'$ and $V_2'$ be a notation for the sum of obstacles in each volume respectively, even though the boundary conditions and wave numbers may differ). For simplicity we take $V_1'$ and $V_2'$ homogeneous and isotropic (wave numbers $k_1'$ and $k_2'$) and bounded by $S_1$ and $S_2$ respectively. Let O be an arbitrarily chosen origin (this will be specified later), and let $V_1(R)$ and $V_2(R)$ denote the interior of a hypersphere, centred at the origin of radius R, in $V_1$ and $V_2$, respectively. The portion of the hypersphere in $V_1$ is denoted $\Sigma_1(R)$ and $\Sigma_2(R)$ is defined similarly. The intersection of the hypersphere and S is called C(R) and the part of S enclosed by the hypersphere is denoted S(R). The normal $\hat{\nu}$ of S is directed into $V_1$ while the normals $\hat{\nu}$ for $S_1$ and $S_2$ are conventionally taken as directed outwards.

We will assume all space to be sourcefree, since in proving the uniqueness theorem we study the difference between two solutions having the same sources. Thus, when we consider the difference, the source term disappears, and we will in this paper study fields satisfying the following conditions.

1.

$$(\nabla^2 + k_j^2)\psi_j(\vec{r}) = 0 \qquad \vec{r} \in V_j\,, \quad \psi_j \in C^2(V_j) \qquad j=1,2$$
$$(\nabla^2 + k_j'^2)\psi_j'(\vec{r}) = 0 \qquad \vec{r} \in V_j'\,, \quad \psi_j' \in C^2(V_j')$$

2. $k_j^2 > 0\,,\ k_j'^2 > 0 \qquad j=1,2$

3. Boundary conditions

$$\begin{cases} A_1 \dfrac{\partial \psi_1}{\partial \nu} = A_2 \dfrac{\partial \psi_2}{\partial \nu} \\ B_1 \psi_1 = B_2 \psi_2 \end{cases} \qquad \text{on } S$$

$$\begin{cases} A_j \dfrac{\partial \psi_j}{\partial \nu} = A_j' \dfrac{\partial \psi_j'}{\partial \nu} \\ B_j \psi_j = B_j' \psi_j' \end{cases} \qquad \text{on } S_j \quad j=1,2$$

$A_j, A_j', B_j, B_j'$ are complex constants subject to

$$\begin{cases} C_j = \overline{A}_j B_j = A_j \overline{B}_j \\ C_j' = \overline{A}_j' B_j' = A_j' \overline{B}_j' \end{cases}$$ $j=1,2$, where $C_j$ are nonnegative real numbers

4. The surface S is conical outside a given radius, i.e.

$\exists R_o$ such that for $r \geq R_o$ we have

a) $\hat{\nu} \cdot \hat{r} = 0$ on S (This specifies our origin O)

b) a finite number of inhomogeneities and $V_j(R_o) \supset V_j'$, j=1,2

5. Radiation conditions

$$\iint_{\Sigma_j(R)} \left| \frac{\partial \psi_j}{\partial R} - i k_j \psi_j \right|^2 dS = o(1) \qquad R \to \infty, j=1,2$$

Here and below $\bar{Z}$ will denote the complex conjugate of the complex number Z.

The second condition states that all media are lossless. The lossy case will not be analysed here, but it can be expected to be easier due to damping. The third condition gives the conditions at the boundaries, and in this paper we will assume both the field and its normal derivative to be discontinuous. The coupling constants A and B can be complex or real such that the constants $C \equiv A\bar{B}$ are nonnegative real numbers. The theory allows any A or B to be zero, and in this special case the theory is the problem treated by Jones [9]. The fourth condition requires the shape of S to be conical for large r, and also limits the number of inhomogeneities in $V_1$ and $V_2$ to be finite. The choice of origin is now also fixed due to the condition $\hat{r} \cdot \hat{\nu} = 0$ for large r. The radiation condition which will be adopted here is the radiation condition discussed by e.g. Rellich [3] and Jones [9]. See also Wilcox [23] for additional comments on the choice of radiation condition. Notice that the fifth item is non-linear. However, the Minkowski's inequality proves that the sum or difference of two fields satisfying condition five still satisfies the radiation condition.

It is convenient to work with field quantities where a specific radial dependence has been extracted, so as to make the remaining radial dependence of the fields easier to study. We will therefore adopt the following notation.

$$\begin{aligned} &\phi_j(\vec{r}) \equiv \sqrt{C_j}\, r^{\frac{n-1}{2}}\, \psi_j(\vec{r}) \\ &\phi_j'(\vec{r}) \equiv \frac{\partial}{\partial r}\phi_j(\vec{r}) \qquad j=1,2 \\ &\nabla_1 \equiv r\left(\nabla - \hat{r}\frac{\partial}{\partial r}\right) \end{aligned} \tag{2.1}$$

The $\nabla_1$ operator is defined on the unit sphere $\{\vec{r}\in R^n : |\vec{r}|=1\}$ and is independent of the radial coordinate r. A direct computation of the gradient in spherical coordinates

$$\begin{cases} x_n = r\cos\vartheta_{n-1} \\ x_{n-1} = r\sin\vartheta_{n-1}\cos\vartheta_{n-2} \\ \vdots \\ x_1 = r\sin\vartheta_{n-1}\sin\vartheta_{n-2}\cdot\ldots\cdot\sin\vartheta_1 \end{cases} \qquad \begin{aligned} &0 \le \vartheta_{n-1} \le \pi \\ &0 \le \vartheta_{n-2}, \ldots, \vartheta_1 \le 2\pi \end{aligned}$$

shows this but it is also a general result from differential geometry, (for more details see e.g. [24]). If the Laplace-Beltrami operator on the unit sphere in $R^n$ is denoted $\nabla_1^2$ and is denoted $\nabla^2$ in $R^n$ itself, we have the following relation (see [25] p. 6)

$$\nabla_1^2 = r^2\nabla^2 - r^{3-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) \tag{2.2}$$

Helmholtz' equation can be rewritten in terms of the field $\phi_j$ with $\nabla_1^2$ and radial derivatives as

$$\phi_j'' + \frac{1}{r^2}\nabla_1^2\phi_j + (k_j^2 - p_n(r))\,\phi_j = 0 \tag{2.3}$$

where

$$p_n(r) \equiv \frac{(n-1)(n-3)}{4r^2} \tag{2.4}$$

We notice that the quantity $p_n(r)$ which depends on the dimension n is nonnegative, except for n=2. We also define the solid angle $\Omega$ for a part of a hypersphere $\Sigma$ in $R^n$ as the projection on the unit sphere

$$\Omega \equiv \Sigma / r^{n-1} \tag{2.5}$$

($\Sigma$ here is used both as a notation for the surface and for its measure). This solid angle $\Omega$ for $\Sigma_j(R)$ is a constant for $r \geq R_o$, with $R_o$ chosen as in condition 4.

Green's theorem for two fields u and v defined on the hypersphere will be used extensively, see e.g. [24]

$$\frac{1}{r^2}\iint_{\Sigma}\left[v\nabla_1^2 u + \nabla_1 u\cdot\nabla_1 v\right]dS = \frac{1}{r}\int_C v\,\hat{\nu}_c\cdot\nabla_1 u\,d\ell \tag{2.6}$$

Here $\Sigma$ is a part of the hypersphere of radius r and C is its periphery defined in n-2 dimensions, and $\hat{\nu}_c$ is the outward normal to the periphery C (see figure).

In the following section we will prove the uniqueness of the fields $\psi_j$ and $\psi_j'$ j=1,2, satisfying the conditions 1-5 defined above, i.e. we will show that the only possible solution to conditions 1-5 is the trivial solution $\psi_j \equiv \psi_j' \equiv 0$ j=1,2. The main building blocks in this theorem will be four lemmas which will be proven first. The first three will make no use of the radiation condition 5, i.e. they give some general features of fields satisfying conditions 1-4 for large radial distances. These lemmas are extensions to and modifications for the present situation of lemmas given by Kato [26]. However, in this paper we will not rely on any symmetry properties on $\nabla^2$ as in [26], but will make use of Green's theorem on the hypersphere. The last lemma, which includes the radiation condition 5, will serve as a contradiction to the former, leaving just the trivial zero-solution as the only remaining solution.

Some quantities will appear often, so for convenience we define, for $r \geq R_0$, the following functions depending on the radial distance r.

$$E(r) \equiv \sum_{j=1}^{2} \iint_{\Omega_j} \left[ |\phi_j'|^2 + k_j^2 |\phi_j|^2 \right] d\Omega \tag{2.7}$$

$$G(r) \equiv E(r) - \frac{1}{r^2} \sum_{j=1}^{2} \iint_{\Omega_j} |\nabla_1 \phi_j|^2 d\Omega \tag{2.8}$$

Here $\Omega_j$ is the solid angle for $\Sigma_j(r)$, which, as pointed out earlier, is a constant for $r \geq R_0$.

## III Uniqueness theorem for permeable media

We will in this section prove the uniqueness theorem for a configuration as depicted in the figure and with the assumptions and definitions stated in section II.

Lemma 1: Consider two fields $\phi_j$ satisfying condition 1-4 in the preceding section. We then have for $r \geq R_0$:

$$G'(r) = \frac{2}{r^3} \sum_{j=1}^{2} \iint_{\Omega_j} |\nabla_1 \phi_j|^2 d\Omega + p_n(r) \frac{d}{dr} \sum_{j=1}^{2} \iint_{\Omega_j} |\phi_j|^2 d\Omega \tag{3.1}$$

Proof: Take the derivative of Eq. (2.7) for $r \geq R_0$. Then we get, since $\Omega_j$ is constant for $r \geq R_0$

$$E'(r) = 2\,\mathrm{Re} \sum_{j=1}^{2} \iint_{\Omega_j} \left[ \phi_j'' \bar{\phi}_j' + k_j^2 \phi_j \bar{\phi}_j' \right] d\Omega$$

We insert Eq. (2.3) and get

$$E'(r) = 2\,\mathrm{Re} \sum_{j=1}^{2} \iint_{\Omega_j} \left[ p_n \phi_j - \frac{1}{r^2} \nabla_1^2 \phi_j \right] \bar{\phi}_j' \, d\Omega$$

We apply Green's theorem Eq. (2.6) and have

$$E'(r) = 2\,\mathrm{Re} \sum_{j=1}^{2} \iint_{\Omega_j} \left[ p_n \phi_j \bar{\phi}_j' + \frac{1}{r^2} \nabla_1 \phi_j \cdot \nabla_1 \bar{\phi}_j' \right] d\Omega$$

The contribution from C will disappear

$$\frac{1}{r} \sum_{j=1}^{2} \int_C \bar{\phi}_j' \, \hat{\nu}_c^j \cdot \nabla_1 \phi_j \, dl = -\int_C \left[ \bar{\phi}_1' \frac{\partial \phi_1}{\partial \nu} - \bar{\phi}_2' \frac{\partial \phi_2}{\partial \nu} \right] dl \tag{3.2}$$

since for $r \geq R_o$ we have $\hat{\nu}\cdot\hat{r}=0$ and

$$\frac{1}{r}\hat{\nu}_c^j\cdot\nabla_1\phi_j = \mp\hat{\nu}\cdot\nabla\phi_j = \mp\frac{\partial\phi_j}{\partial\nu}$$

By use of the boundary conditions on S (see condition 3 in section II) and the definition in Eq. (2.1) we can show the continuity of $\bar{\phi}_j'\frac{\partial\phi_j}{\partial\nu}$ across S

$$\bar{\phi}_2'\frac{\partial\phi_2}{\partial\nu} = C_2\frac{\partial}{\partial r}\left(r^{\frac{n-1}{2}}\bar{\psi}_2\right)\frac{\partial}{\partial\nu}\left(r^{\frac{n-1}{2}}\psi_2\right) = A_2\bar{B}_2 r^{n-1}\left(\frac{n-1}{2r}\bar{\psi}_2+\frac{\partial\bar{\psi}_2}{\partial r}\right)\frac{\partial\psi_2}{\partial\nu} =$$

$$= A_1\bar{B}_1 r^{n-1}\left(\frac{n-1}{2r}\bar{\psi}_1+\frac{\partial\bar{\psi}_1}{\partial r}\right)\frac{\partial\psi_1}{\partial\nu} = \bar{\phi}_1'\frac{\partial\phi_1}{\partial\nu} \qquad (3.3)$$

since $\hat{\nu}\cdot\hat{r}=0$ for $r\geq R_o$ and $\psi_j$ can be differentiated along $\hat{r}$ on S. The contribution from Eq. (3.2) is thus zero and we have

$$E'(r) = \sum_{j=1}^{2}\left\{p_n\frac{d}{dr}\iint_{\Omega_j}|\phi_j|^2 d\Omega + \frac{1}{r^2}\frac{d}{dr}\iint_{\Omega_j}|\nabla_1\phi_j|^2 d\Omega\right\}$$

since $\nabla_1$ is independent of r.

From Eq. (2.8) we get with the expression of E'(r)

$$G'(r) = \frac{2}{r^3}\sum_{j=1}^{2}\iint_{\Omega_j}|\nabla_1\phi_j|^2 d\Omega + p_n(r)\frac{d}{dr}\sum_{j=1}^{2}\iint_{\Omega_j}|\phi_j|^2 d\Omega$$

and the lemma is proven.

The quantity F(m,r), which will be used in the following lemma, is defined as

$$F(m,r) \equiv \sum_{j=1}^{2}\iint_{\Omega_j}\left[|\phi_j'^m|^2 + \left(k_j^2-\frac{a}{r}+\frac{m(m+1)}{r^2}\right)|\phi_j^m|^2 - \frac{1}{r^2}|\nabla_1\phi_j^m|^2\right]d\Omega \qquad (3.4)$$

$\phi_j^m \equiv r^m \phi_j$ $\qquad$ j=1,2, m is an arbitrary positive integer

$\phi_j^{'m} \equiv \frac{\partial}{\partial r} \phi_j^m$ $\qquad$ j=1,2

$a \equiv R_0 \min_{j=1,2} k_j^2 \equiv R_0 \kappa^2$

Lemma 2: Let conditions 1-4 in section II be fulfilled. Then there are positive integers $m_o$, $m_1$ and a number $r_1 \geq R_o$ such that

a) $\frac{d}{dr}(r^2 F(m,r)) \geq 0$ $\qquad$ for all $m \geq m_1$ and all $r \geq R_o$

b) $F(m_0,r) > 0$ $\qquad$ for all $r \geq r_1$ unless $\phi_j \equiv 0$ $\quad$ j=1,2

Proof: For $r \geq R_o$ we have

$$\frac{d}{dr}(r^2 F(m,r)) = 2r^2 \operatorname{Re} \sum_{j=1}^{2} \iint_{\Omega_j} \Big[ \phi_j^{''m} \overline{\phi_j^{'m}} + \Big(k_j^2 - \frac{a}{r} + \frac{m(m+1)}{r^2}\Big) \phi_j^m \overline{\phi_j^{'m}} - $$
$$- \frac{1}{r^2} \nabla_1 \overline{\phi_j^{'m}} \cdot \nabla_1 \phi_j^m + \frac{1}{r}\Big(k_j^2 - \frac{a}{2r}\Big) |\phi_j^m|^2 + \frac{1}{r} |\phi_j^{'m}|^2 \Big] d\Omega$$

It is straightforward to prove, using Eq. (2.3), that $\phi_j^m$ satisfies the following differential equation

$$\phi_j^{''m} + \frac{1}{r^2} \nabla_1^2 \phi_j^m - \frac{2m}{r} \phi_j^{'m} + \frac{m(m+1)}{r^2} \phi_j^m + (k_j^2 - p_n(r)) \phi_j^m = 0$$

We thus get after some algebra

$$\frac{d}{dr}(r^2 F(m,r)) = 2r \operatorname{Re} \sum_{j=1}^{2} \iint_{\Omega_j} \Big[ |\phi_j^{'m}|^2 (2m+1) + \Big(k_j^2 - \frac{a}{2r}\Big) |\phi_j^m|^2 + $$
$$+ \overline{\phi_j^{'m}} \phi_j^m (r p_n - a) - \frac{1}{r}\Big(\nabla_1 \overline{\phi_j^{'m}} \cdot \nabla_1 \phi_j^m + \overline{\phi_j^{'m}} \nabla_1^2 \phi_j^m\Big)\Big] d\Omega \qquad (3.5)$$

The last term in the integrand disappears by use of Green's theorem Eq. (2.6) and the continuity of $\overline{\phi}_j^{m'} \frac{\partial \phi_j^m}{\partial \nu}$, since on C we have

$$\overline{\phi_1^{m}}' \frac{\partial \phi_1^m}{\partial \nu} = (mr^{m-1}\overline{\phi}_1 + r^m \overline{\phi}_1')\, r^m \frac{\partial \phi_1}{\partial \nu} = (mr^{m-1}\overline{\phi}_2 + r^m \overline{\phi}_2')\, r^m \frac{\partial \phi_2}{\partial \nu} = \overline{\phi_2^{m}}' \frac{\partial \phi_2^m}{\partial \nu}$$

Furthermore, from Hölder's inequality for integrals on the hypersphere we have the following estimate

$$\mathrm{Re} \sum_{j=1}^{2} (rp_n - a) \iint_{\Omega_j} \overline{\phi_j^{m}}' \phi_j^m \, d\Omega \geq -(r|p_n| + a) \sum_{j=1}^{2} \iint_{\Omega_j} |\phi_j^{m'}| |\phi_j^m| \, d\Omega \geq$$

$$\geq -(r|p_n| + a) \sum_{j=1}^{2} \sqrt{\iint_{\Omega_j} |\phi_j^{m'}|^2 d\Omega} \sqrt{\iint_{\Omega_j} |\phi_j^m|^2 d\Omega}$$

Thus we can write Eq. (3.5) as

$$\frac{d}{dr}(r^2 F(m,r)) = 2r \sum_{j=1}^{2} \iint_{\Omega_j} [\,|\phi_j^{m'}|^2 (2m+1) + (k_j^2 - \tfrac{a}{2r}) |\phi_j^m|^2 + \mathrm{Re}\{(rp_n - a)\, \overline{\phi_j^{m}}' \phi_j^m\}]\, d\Omega \geq$$

$$\geq 2r \sum_{j=1}^{2} \Big\{ \iint_{\Omega_j} [\,|\phi_j^{m'}|^2 (2m+1) + (k_j^2 - \tfrac{a}{2r}) |\phi_j^m|^2]\, d\Omega - (r|p_n| + a) \sqrt{\iint_{\Omega_j} |\phi_j^{m'}|^2 d\Omega} \sqrt{\iint_{\Omega_j} |\phi_j^m|^2 d\Omega} \Big\}$$

The right hand side is a quadratic form which is greater than zero if

$$(2m+1)(k_j^2 - \tfrac{a}{2r}) \geq \tfrac{1}{4}(r|p_n| + a)^2 = \tfrac{1}{4}\Big(\frac{|n-1||n-3|}{4r} + R_o \kappa^2\Big)^2 \tag{3.6}$$

Since $k_j^2 - \frac{a}{2r} = k_j^2 - \frac{R_o}{2r}\kappa^2 > 0$ for $r \geq R_o$ and the right hand side of Eq. (3.6) is independent of m and bounded for large r we can find $m_1$ such that Eq. (3.6) is fulfilled for all $m \geq m_1$ and $r \geq R_o$, i.e.

$$\frac{d}{dr}\left(r^2 F(m,r)\right) \geq 0$$ for all $m \geq m_1$ and all $r \geq R_o$

and the first part of the lemma is proven.

If $\phi_j \not\equiv 0$ then there exists an $r_1 \geq R_o$ such that

$$\sum_{j=1}^{2} \iint_{\Omega_j} |\phi_j|^2 d\Omega > 0 \qquad \text{for } r=r_1$$

(If this is false for every $r_1 \geq R_o$ then $\phi_j=0$ for all $r \geq R_o$ and by the properties of solutions to Helmholtz' equation $\phi_j$ is zero everywhere.) We can write $F(m,r)$ Eq. (3.4) as

$$F(m,r) = r^{2m} \sum_{j=1}^{2} \iint_{\Omega_j} \left[ \left|\frac{m}{r}\phi_j + \phi_j'\right|^2 + \left(k_j^2 - \frac{a}{r} + \frac{m(m+1)}{r^2}\right)|\phi_j|^2 - \frac{1}{r^2}|\nabla_1 \phi_j|^2 \right] d\Omega \tag{3.7}$$

For an $r_1$ chosen as above let $m_o \geq m_1$, where $m_1$ is given by the first part of this lemma, such that $F(m_o,r_1) > 0$. But according to the first part of this lemma we then have

$$F(m_o,r) > 0 \qquad \text{for all } r \geq r_1 \geq R_o$$

since $r^2 F(m_o,r)$ is a nondecreasing function in r, i.e. $F(m_o,r)$ can not change sign. The proof of lemma 2 is thus completed.

The next lemma, in which we will use the previous one, reads:

Lemma 3: Let conditions 1-4 be fulfilled and furthermore let

$$\sum_{j=1}^{2} \iint_{\Omega_j} |\phi_j|^2 d\Omega = O(1) \qquad r \to \infty$$

If $\phi_j \not\equiv 0$ then there exists an infinite sequence $\{r_\mu\}_{\mu=1}^{\infty}$ such that $r_\mu \to \infty$, $\mu \to \infty$ and $G(r_\mu) > 0$.

Proof: Define a set T such that

$$T = \{ r \geq R_0 : \frac{d}{dr} \sum_{j=1}^{2} \iint_{\Omega_j} |\phi_j|^2 d\Omega \leq 0 \}$$

T is an infinite set and furthermore contains arbitrarily big elements. This is a consequence of the assumption

$$\sum_{j=1}^{2} \iint_{\Omega_j} |\phi_j|^2 d\Omega = O(1) \qquad r \to \infty$$

For an $r_\mu \in T$ we have

$$\mathrm{Re} \sum_{j=1}^{2} \iint_{\Omega_j} \overline{\phi_j}\, \phi_j' \, d\Omega = \frac{1}{2} \frac{d}{dr} \sum_{j=1}^{2} \iint_{\Omega_j} |\phi_j|^2 d\Omega \leq 0$$

Thus we have for $r_\mu \in T$

$$\sum_{j=1}^{2} \iint_{\Omega_j} |\phi_j' + \frac{m}{r} \phi_j|^2 d\Omega \leq \sum_{j=1}^{2} \iint_{\Omega_j} \left[ |\phi_j'|^2 + \frac{m^2}{r^2} |\phi_j|^2 \right] d\Omega$$

We can now estimate $F(m, r_\mu)$ (for definitions see Eq. (3.4) or (3.7))

$$F(m, r_\mu) \leq r_\mu^{2m} \sum_{j=1}^{2} \iint_{\Omega_j} \left[ |\phi_j'|^2 + \left(k_j^2 - \frac{a}{r_\mu} + \frac{m(2m+1)}{r_\mu^2}\right) |\phi_j|^2 - \frac{1}{r_\mu^2} |\nabla_1 \phi_j|^2 \right] d\Omega$$

Let $m=m_o$, where $m_o$ is chosen according to lemma 2 b)
For all $r_\mu \in T$ such that $r_\mu \geq r_1 \geq R_o$ ($r_1$ given by lemma 2 b)) we have

$$0 < F(m_o, r_\mu) \leq r_\mu^{2m_o} \left\{ G(r_\mu) - \sum_{j=1}^{2} \iint_{\Omega_j} \left( \frac{a}{r_\mu} - \frac{m_o(2m_o+1)}{r_\mu^2} \right) |\phi_j|^2 \, d\Omega \right\}$$

Let $r_2 \geq r_1$ such that $\frac{a}{r_2} = \frac{R_o}{r_2} \kappa^2 \geq \frac{m_o(2m_o+1)}{r_2^2}$. Then for all $r_\mu \in T$ such that $r_\mu \geq r_2$ we have

$$0 < F(m_o, r_\mu) \leq r_\mu^{2m_o} G(r_\mu) \Rightarrow G(r_\mu) > 0$$

and the lemma is proven.

The last lemma makes explicit use of the radiation condition (condition 5 in section II) and furthermore contradicts lemma 3 as will be seen in the theorem below.

Lemma 4: Let conditions 1-5 be fulfilled. Then

$$\begin{cases} \displaystyle\iint_{\Sigma_j(R)} |\psi_j|^2 \, dS = o(1) \\ \displaystyle\iint_{\Sigma_j(R)} \left|\frac{\partial \psi_j}{\partial R}\right|^2 dS = o(1) \end{cases} \qquad R \to \infty \quad i=1,2$$

Proof: The radiation condition 5 can explicitly be written as

$$\iint_{\Sigma_j(R)} \left|\frac{\partial \psi_j}{\partial R} - ik_j\psi_j\right|^2 dS = \iint_{\Sigma_j(R)} \left[ \left|\frac{\partial \psi_j}{\partial R}\right|^2 + k_j^2|\psi_j|^2 + 2k_j \operatorname{Im}\left(\psi_j \frac{\partial \overline{\psi_j}}{\partial R}\right)\right] dS$$

Multiply both sides by $C_j/k_j$ and sum over j=1,2. We get

$$o(1) = \sum_{j=1}^{2} \frac{C_j}{k_j} \iint_{\Sigma_j(R)} \left[ \left|\frac{\partial \psi_j}{\partial R}\right|^2 + k_j^2|\psi_j|^2 \right] dS + 2\sum_{j=1}^{2} \operatorname{Im}\left[ \iint_{\Sigma_j(R)} C_j \psi_j \frac{\partial \overline{\psi_j}}{\partial R} dS \right] \tag{3.8}$$

We now apply Green's first formula in $R^n$ (this is analogous to Eq. (2.6) which holds on the hypersphere) on the field $\psi_j$ and its complex conjugate in $V_j$ and $V_j'$ j=1,2. We get for j=1,2

$$\begin{cases} \iint\limits_{\Sigma_j(R)} \psi_j \frac{\partial\bar{\psi}_j}{\partial R} dS = \pm \iint\limits_{S(R)} \psi_j \frac{\partial\bar{\psi}_j}{\partial \nu} dS + \iint\limits_{S_j} \psi_j \frac{\partial\bar{\psi}_j}{\partial \nu} dS + \iiint\limits_{V_j(R)} [|\nabla\psi_j|^2 - k_j^2|\psi_j|^2] dV \\ \iint\limits_{S_j} \psi_j' \frac{\partial\bar{\psi}_j'}{\partial \nu} dS = \iiint\limits_{V_j'} [|\nabla\psi_j'|^2 - k_j'^2|\psi_j'|^2] dV \end{cases}$$

The plus sign holds for j=1 and the minus for j=2. This is a consequence of the direction of the surface normal $\hat{\nu}$ on S.

We have so far not specified the smoothness properties assumed for the surface S, $S_1$ and $S_2$. These properties are here assumed to guarantee the finiteness of the surface integrals over S, $S_1$ and $S_2$ above. This property was not used in the preceding lemmas.

The last term in Eq. (3.8) can be rewritten as

$$\mathrm{Im}\Big[\sum_{j=1}^{2} \iint\limits_{\Sigma_j(R)} c_j\psi_j \frac{\partial\bar{\psi}_j}{\partial\nu} dS\Big] = \mathrm{Im}\Big[\sum_{j=1}^{2} \iint\limits_{S_j} c_j\psi_j \frac{\partial\bar{\psi}_j}{\partial\nu} dS\Big] + $$
$$+ \iint\limits_{S(R)} \Big[c_1\psi_1\frac{\partial\bar{\psi}_1}{\partial\nu} - c_2\psi_2\frac{\partial\bar{\psi}_2}{\partial\nu}\Big] dS = 0$$

since $c_j\psi_j\frac{\partial\bar{\psi}_j}{\partial\nu}$ is continuous over S, $S_1$ and $S_2$ by the boundary conditions 3. Thus we have from Eq. (3.8)

$$\sum_{j=1}^{2} \frac{c_j}{k_j} \iint\limits_{\Sigma_j(R)} \left[ \left|\frac{\partial \psi_j}{\partial R}\right|^2 + k_j^2 |\psi_j|^2 \right] dS = o(1) \qquad R \to \infty$$

Since each term is positive the statement of the lemma is proven.

We have now collected results enough to prove the main theorem of this paper.

Theorem: Let $\psi_j$ and $\psi_j'$ j=1,2 be fields satisfying conditions 1-5, defined and discussed in section II. The only possible solution to this problem is the solution which is zero everywhere.

Proof: Lemma 1 gives for $r \geq R_0$

$$G'(r) = \frac{2}{r^3} \sum_{j=1}^{2} \iint\limits_{\Omega_j} |\nabla_1 \phi_j|^2 d\Omega + 2p_n(r) \operatorname{Re} \sum_{j=1}^{2} \iint\limits_{\Omega_j} \phi_j \bar{\phi}_j' d\Omega$$

Furthermore we have $(\varkappa = \min_{j=1,2} k_j)$

$$0 \leq \sum_{j=1}^{2} \iint\limits_{\Omega_j} |\phi_j' \pm \varkappa \phi_j|^2 d\Omega = \sum_{j=1}^{2} \iint\limits_{\Omega_j} \left[ |\phi_j'|^2 + \varkappa^2 |\phi_j|^2 \pm 2\varkappa \operatorname{Re} \phi_j \bar{\phi}_j' \right] d\Omega$$

and we can write (if $n \neq 2$ use the plus sign, if, $n=2$ the minus sign)

$$G'(r) \geq -\frac{|p_n(r)|}{\varkappa} \sum_{j=1}^{2} \iint\limits_{\Omega_j} \left[ |\phi_j'|^2 + \varkappa^2 |\phi_j|^2 \right] d\Omega + \frac{2}{r^3} \sum_{j=1}^{2} \iint\limits_{\Omega_j} |\nabla_1 \phi_j|^2 d\Omega \geq$$

$$\geq -\frac{|p_n(r)|}{\varkappa} \sum_{j=1}^{2} \iint\limits_{\Omega_j} \left[ |\phi_j'|^2 + k_j^2 |\phi_j|^2 \right] d\Omega + \frac{2}{r^3} \sum_{j=1}^{2} \iint\limits_{\Omega_j} |\nabla_1 \phi_j|^2 d\Omega =$$

$$= -\frac{|p_n(r)|}{\varkappa} G(r) + \sum_{j=1}^{2} \iint\limits_{\Omega_j} \left( \frac{2}{r^3} - \frac{|p_n(r)|}{r^2 \varkappa} \right) |\nabla_1 \phi_j|^2 d\Omega \geq$$

$$\geq -\frac{|p_n(r)|}{\kappa} G(r) \qquad \text{for all } r \geq r_3 \geq R_0$$

where $r_3$ satisfies

$$\frac{2}{r_3^3} \geq \frac{|p_n(r)|}{r_3^2 \kappa} = \frac{|n-1||n-3|}{4 r_3^4 \kappa}$$

Thus we have the following differential inequality

$$G'(r) + f(r) G(r) \geq 0 \qquad (3.9)$$

where $f(r) = |p_n(r)|/\kappa = b_n/r^2$. The solution to Eq. (3.9) is

$$G(r) \geq G(r_0) \exp\left[-\int_{r_0}^{r} f(t)dt\right] = G(r_0) \exp\left[b_n\left(\frac{1}{r} - \frac{1}{r_0}\right)\right] \quad \text{for } r \geq r_0$$

Lemma 4 gives

$$\sum_{j=1}^{2} \iint_{\Omega_j} |\phi_j|^2 d\Omega = o(1) \qquad R \to \infty$$

Lemma 3 now states that there exists an arbitrarily large $r_0$ such that $G(r_0) > 0$ provided $\phi_j \not\equiv 0$ and we thus have $\lim_{r\to\infty} G(r) > 0$ unless $\phi_j \equiv 0$. On the other hand we have

$$\lim_{r\to\infty} G(r) \leq \lim_{r\to\infty} E(r) = \lim_{r\to\infty} \sum_{j=1}^{2} \iint_{\Omega_j} \left[|\phi_j'|^2 + k_j^2 |\phi_j|^2\right] d\Omega$$

Furthermore lemma 4 gives

$$\sum_{j=1}^{2} \iint_{\Omega_j} |\phi_j'|^2 d\Omega = \sum_{j=1}^{2} c_j \iint_{\Sigma_j(R)} \left|\frac{n-1}{2R}\psi_j + \frac{\partial \psi_j}{\partial R}\right|^2 dS = o(1) \qquad R \to \infty$$

This last step can be shown by Hölder's inequality.

We thus have

$$\lim_{r\to\infty} G(r) \leq 0$$

This contradicts $\lim_{r\to\infty} G(r) > 0$ and we have $\phi_j \equiv 0$ everywhere.

## IV Conclusions and applications

We have in this paper shown the uniqueness of the solution for Helmholtz' equation in the lossless case for a special class of infinite surfaces, namely those which eventually become conical. As a special case, if $A_j$ or $B_j$ equals zero, we have the result of Jones [9].

The proof of the theorem in several places relies on the fact that $\hat{r}\cdot\hat{\nu}=0$ for $r \geq R_0$. This property makes it possible to differentiate one of the boundary conditions in the radial direction. A uniqueness theorem, valid for a more general geometry thus must in some parts use different techniques and arguments. In ref. [12] it is stated that the case with losses (complex $k_j$) can be proven by simple boundedness conditions but this is not carried out in detail. The theorem proven in this paper can not be extended to complex values as it stands, but we expect that only slight modifications will be required in order to make this extension possible. The boundary conditions assumed in this paper are fairly general, but an interesting extension would be to investigate how more general conditions would affect the uniqueness. At present this is an unsolved problem. The volumes $V_1'$ and $V_2'$ were assumed homogenous and lossless, but these assumptions can easily be relaxed.

The uniqueness theorem for Helmholtz equation together with the derived growth properties at large distances is of great interest in many situations. One application, which recently has appeared, is the question of completeness of various systems of functions on a given surface, finite or not, see e.g.

[27] - [30]. This question is of special interest for eigenfunctions to Helmholtz equation for surfaces which are not coordinate surfaces to the eigenfunctions. The technique used by Millar [29] relies on the uniqueness results (in the use of either Dirichlet or Neumann boundary value problems) for the corresponding geometry, in the interior and the exterior case.

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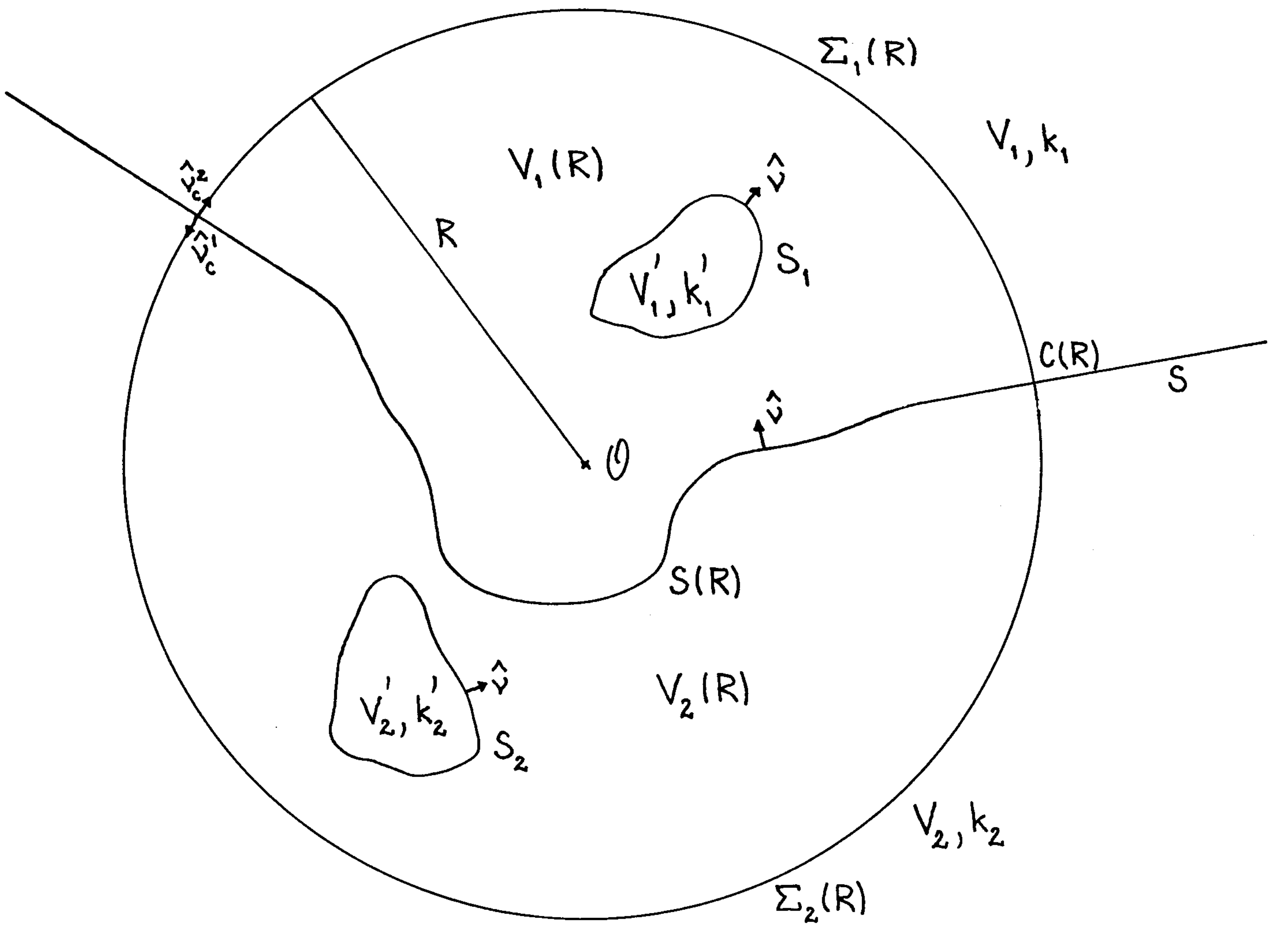
Figure


$\Sigma_1(R)$
$V_1, k_1$
$V_1(R)$
$\hat{\nu}$
$\hat{\nu}_c^2$
$\hat{\nu}_c^1$
R
$V_1', k_1'$
$S_1$
C(R)
S
$\hat{\nu}$
$\mathcal{O}$
S(R)
$\hat{\nu}$
$V_2', k_2'$
$V_2(R)$
$S_2$
$V_2, k_2$
$\Sigma_2(R)$